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Observability estimates for the wave equation with rough coefficients ☆

Estimées d'observabilité pour l'équation des ondes avec des coefficients continus

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ABSTRACT

The goal of this note is to prove observability estimates for the wave equation with a density which is only continuous in the domain, and satisfies some multiplier-type condition only in the sense of distributions. Our main argument is that one can construct suitable approximations of such density by a sequence of smooth densities whose corresponding wave equations are uniformly observable. The end of the argument then consists in a rather standard passage to the limit.

RÉSUMÉ

Le but de cette note est de démontrer des estimées d'observabilité pour l'équation des ondes avec une densité continue dans le domaine, et qui satisfait une condition de type multiplicateur seulement au sens des distributions. Notre argument est essentiellement basé sur le fait que l'on peut alors construire des approximations convenables d'une telle fonction de densité par des fonctions régulières pour lesquelles les équations des ondes correspondantes sont uniformément observables. La preuve se termine alors par un passage à la limite relativement standard.

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1. Introduction

1.1. Setting and main result

The goal of this note is to study observability estimates for the wave equation with a density having low regularity. More precisely, we consider the wave equation

$$\begin{cases} \rho(x)\partial_t^2 u - \Delta_x u = 0 & \text{in } (0,T)\times\Omega, \\ u = 0 & \text{on } (0,T)\times\partial\Omega, \\ (u(0), \partial_t u(0)) = (u_0, u_1) & \text{in } \Omega. \end{cases} \tag{1.1}$$

Here, Ω is a smooth bounded domain of $\mathbb{R}^d$. The function $\rho = \rho(x)$ represents the density of the medium in which the wave of amplitude u propagates.

Our goal is to provide observability estimates for the wave equation (1.1) under weak regularity assumptions on the density ρ. Namely, we will assume the following regularity conditions:

- the density ρ is strictly positive and bounded in $\overline{\Omega}$: there exist $\rho_1 > 0$ and $\rho_2 > 0$ such that

$$\forall x \in \Omega, \quad 0 < \rho_1 \le \rho(x) \le \rho_2, \tag{1.2}$$

- the density ρ is continuous in $\overline{\Omega}$:

$$\rho \in C^0(\overline{\Omega}). \tag{1.3}$$

The assumptions (1.2)–(1.3) are natural as they guarantee the well-posedness of the equation (1.1) for $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$. Indeed, in this framework, the work [19, Chapter 3, Sections 8–9] (see also [13] and the paper by F. Colombini et al. [7] for an overview on this question) shows that system (1.1) admits a unique solution $u(t,x)$ in the energy space $C^0([0,+\infty[, H_0^1(\Omega)) \cap C^1([0,+\infty[, L^2(\Omega))$. Moreover, the solution u to (1.1) has a constant energy as time evolves, i.e.

$$E[u](t) := \frac{1}{2}\int_\Omega \rho(x)|\partial_t u(t,x)|^2 \mathrm{d}x + \frac{1}{2}\int_\Omega |\nabla_x u(t,x)|^2 \mathrm{d}x \tag{1.4}$$

is independent from the time t and satisfies

$$\forall t \in [0,T], \quad E[u](t) = E[u](0). \tag{1.5}$$

We will now consider an observability problem from an open subset ω which is an open neighborhood (in $\overline{\Omega}$) of an open subset Γ of the boundary satisfying the celebrated multiplier condition [18,17,14]. Namely, we assume that Γ satisfies the following condition:

$$\{x \in \partial\Omega, \text{ such that } x\cdot n_x > 0\} \subset \Gamma, \tag{1.6}$$

where for $x \in \partial\Omega$, n_x denotes the outward normal to the boundary $\partial\Omega$ at the point x and

$$\omega \text{ is an open neighborhood in } \overline{\Omega} \text{ of } \Gamma. \tag{1.7}$$

In order to simplify the presentation of our work, we will assume that the density ρ is defined on a smooth domain Ω_1 containing $\overline{\Omega}$ and satisfies assumptions (1.2)–(1.3) in Ω_1. Otherwise, one should assume that there exists a suitable extension of ρ to Ω_1 satisfying (1.2)–(1.3) and the appropriate conditions given afterwards.

We then assume that the density ρ satisfies the following condition:

$$\exists \alpha \in (0,2], \text{ such that } \quad x\cdot\nabla\rho(x) + (2-\alpha)\rho(x) \ge 0 \text{ in the sense of } \mathscr{D}'(\Omega_1). \tag{1.8}$$

We emphasize that condition (1.8) does not require any new regularity condition on ρ as the inequality in (1.8) only holds *in the sense of distributions*, meaning that:

$$\forall \varphi \in \mathscr{D}(\Omega_1), \text{ with } \varphi \ge 0, \quad \int_{\mathbb{R}^d} \rho(x)\left(-\mathrm{div}\,(x\varphi(x)) + (2-\alpha)\varphi(x)\right) \mathrm{d}x \ge 0. \tag{1.9}$$

Let us also note that this condition (1.8) describes some monotony condition on the density in the direction of the multiplier x, while it does not require any additional assumption in the tangential directions, see Section 1.3 for a more extensive discussion of this fact.

We are now in position to state our main result.

Theorem 1.1. *Let ω be an open subset of Ω as in (1.6)–(1.7), Ω_1 be a smooth domain of $\mathbb{R}^d$ containing $\overline{\Omega}$, and let ρ satisfy the assumptions (1.2)–(1.3) in Ω_1 and (1.8) for some $\alpha \in (0, 2]$ if $d \geq 2$ or $\alpha = 2$ if $d = 1$.*

We further assume that one of the two following conditions is satisfied:

$$\begin{cases} 0 \notin \overline{\Omega_1}, \\ \text{or} \\ \rho \text{ is } C^1 \text{ in a neighborhood of } 0. \end{cases} \tag{1.10}$$

Then, if we set

$$R = \sup\{|x|,\ x \in \Omega\},$$

for all T satisfying

$$\alpha T > 4R\sqrt{\rho_2}, \tag{1.11}$$

there exists a constant $C > 0$ such that the observability estimate

$$E[u](0) \leq C \int_0^T \int_\omega |\partial_t u(t, x)|^2 \mathrm{d}x\mathrm{d}t \tag{1.12}$$

holds true for every solution u of (1.1) with initial data in $H_0^1(\Omega) \times L^2(\Omega)$.

The proof of Theorem 1.1 will be done using classical multiplier techniques for suitable smooth approximations of the density. Indeed, classical multiplier techniques apply only when the density ρ belongs to $C^1(\overline{\Omega})$ under the condition (1.8). Here, we do not assume that the density ρ has such regularity, so we approximate it by suitable smooth regularizations ρ_ε satisfying condition (1.8) uniformly with respect to the regularization parameter ε.

In fact, the main novelty of Theorem 1.1 lies in the regularity of the density ρ, which is only assumed to be continuous on $\overline{\Omega}$, and to satisfy the “radial” monotony condition (1.8) (let us recall and emphasize that (1.8) only involves the derivative of ρ in the sense of distributions in the radial direction).

Let us also briefly comment the choice of α in Theorem 1.1. In particular, it might seem surprising that the conditions on α are more restrictive in dimension $d = 1$ than in dimension $d \geq 2$. But in fact, as we will note in Section 4.1, Theorem 1.1 applies in any dimension and for any $\alpha > 0$, but the proof would then require a slightly more subtle argument that we present briefly in Section 4.1. We made the choice of stating Theorem 1.1 as above only to make the arguments easier.

As explained hereafter in Section 1.2, the condition $\alpha > 0$ is nonetheless rather natural, as otherwise one can construct rays of Geometric Optics that violate the observability property, even for smooth densities.

Let us finally note that the time of observability given by (1.11) involves α explicitly. One can for instance remark that, when $\rho(x) = \rho_0$ is a positive constant, one can take $\rho_1 = \rho_2 = \rho_0$ in (1.2), and $\alpha = 2$ in (1.8), so that the time condition (1.11) simply reads $T > 2\sqrt{\rho_0}\sup\{|x|,\ x \in \Omega\}$, which coincides with the time given by the multiplier approach for the wave equation with constant coefficients. Also note that condition (1.11) imposes to choose a larger time when α gets smaller. This is rather natural as the bicharacteristics of the Hamiltonian of the wave operator then escape from the domain with more difficulty, see Section 1.2 and especially (1.14) afterwards.

1.2. Some insights on the condition (1.8)

Let us now give some insights on the condition (1.8). In this section, we discuss this condition under the following assumptions:

- ρ is smooth ($C^2(\overline{\Omega})$ is sufficient),
- condition (1.8) is satisfied pointwise,
- ω is a neighborhood of the whole boundary $\partial\Omega$.

In such case, one can define bicharacteristic rays associated with the wave operator (1.1) of symbol $p(t, x; \tau, \xi) = -\rho(x)\tau^2 + |\xi|^2$ as follows. Away from the boundary, a bicharacteristic ray γ issued from $(t_0, x_0; \tau_0, \xi_0)$ satisfying $p(t_0, x_0; \tau_0, \xi_0) = 0$ and $x_0 \in \Omega$ is given by the ODE

$$\begin{cases} \dfrac{\mathrm{d}t}{\mathrm{d}s} = -2\tau\rho(x), & \dfrac{\mathrm{d}\tau}{\mathrm{d}s} = 0, \\ \dfrac{\mathrm{d}x}{\mathrm{d}s} = 2\xi, & \dfrac{\mathrm{d}\xi}{\mathrm{d}s} = \tau^2 \nabla\rho(x). \end{cases} \tag{1.13}$$

Using then that along the flow, $p(t(s), x(s); \tau(s), \xi(s)) = 0$ for all s, one easily computes

$$\frac{d^2}{ds^2}\left(|x(s)|^2\right) = 4\tau^2\left(x\cdot\nabla\rho(x) + 2\rho(x)\right) \geq 4\tau^2\alpha\rho_1, \tag{1.14}$$

where the last inequality comes from condition (1.8) assumed to be satisfied pointwise, and (1.2).

This strict convexity of $s \mapsto |x(s)|^2$ implies in particular that any bicharacteristic ray has to enter in ω (assumed here to be a neighborhood of the whole boundary $\partial\Omega$) in finite time. Thus the celebrated geometric control condition of [1,2] is satisfied and observability holds.

It is also remarkable to note that if there exists a sphere $S(r_0)$ of radius r_0 included in $\Omega \setminus \overline{\omega}$ in which ρ satisfies[1]

$$\forall x \in S(r_0), \quad x\cdot\nabla\rho(x) + 2\rho(x) = 0, \tag{1.15}$$

then, taking $x_0 \in S(r_0)$, $\xi_0 \in \mathbb{R}^d$ such that $x_0 \cdot \xi_0 = 0$, and $\tau_0 = |\xi_0|/\sqrt{\rho(x_0)}$, from (1.14) the ray given by (1.13) satisfies

$$\frac{d^2}{ds^2}\left(|x(s)|^2\right) = 0 \text{ and } \frac{d}{ds}\left(|x(s)|^2\right)\Big|_{s=0} = 0,$$

so that the corresponding bicharacteristic ray stays in $S(r_0)$ for all s and never enters the observation set ω when the sphere $S(r_0)$ does not meet ω. In particular, if the sphere $S(r_0)$ is included in $\Omega \setminus \overline{\omega}$, the geometric control condition is not satisfied. Namely, one can construct a sequence of solutions to (1.1) that concentrates on the sphere $S(r_0)$ and makes the observability inequality (1.12) from ω blow up, whatever the time $T > 0$ is. This computation underlines the geometric relevance of the condition $\alpha > 0$ in (1.8), even if it is naturally stronger than the celebrated geometric control condition of [1,2].

1.3. *Scientific context*

The question of observability estimates for the wave equation was intensively studied in the literature mainly because of its deep connection with the problem of exact controllability. Boundary or internal observability estimates were first obtained by classical multiplier techniques (see, for instance, [11,17,14]) under global geometric conditions similar to (1.8), e.g., the Γ-condition of J.-L. Lions. Later, at the beginning of the 1990s, C. Bardos, G. Lebeau and J. Rauch introduced in their papers [1,2] the so-called Geometric Control Condition (GCC), which turned out to be almost equivalent to observability (see [4]). For internal observation, this condition essentially asserts that the projection of each bicharacteristics of the Hamiltonian of the wave equation in the physical space enters the observation zone in a uniform time.

Proving observability estimates by the first techniques (i.e. multipliers) requires some regularity for the coefficients of the wave operator: in general, they are asked to be at least of class C^1 to handle commutations with vector fields (multipliers) and integration by parts. Proofs under GCC are of micro local nature and essentially deal with pseudo differential calculus; they are based on some micro local ingredients such as wave front sets and micro local defect measures. Therefore, these proofs are only efficient for smooth coefficients, and work in general in the C^∞ framework (see [4] and [3] for cases of low regularity under these assumptions).

From this point of view, it is natural to address the question of observability estimates for the wave equation with non-smooth coefficients. This problem has already received some answers by E. Zuazua and his collaborators, in [5,6], and more recently in [9]. More precisely, in [5,6], the authors prove a lack of observability of waves in highly heterogeneous media, i.e. when the density is of low regularity. In [9], the authors establish observability with coefficients in the Zygmund class and also observability with loss when the coefficients are log-Zygmund or log-Lipschitz. Furthermore, this result is sharp since they proved an infinite loss of derivatives in the case of a regularity worse than log-Lipschitz.

But let us remark that all these works are achieved in one space dimension. In this framework, for smooth coefficients all the light rays reach the boundary in uniform time and there cannot be any problem of captive geodesics. Secondly, the proofs are based on a specific one-dimensional technique, namely sidewise energy estimates. The underlying idea consists in exchanging the role of the time and space variables and, *in fine*, to prove hyperbolic energy estimates for waves with rough coefficients.

Unfortunately, this method does not apply any longer in space dimension greater than one. Furthermore, in this case, the bicharacteristic flow is not well defined due to the low regularity of the coefficients. Therefore micro local tools as propagation of wave front sets or supports of micro local defect measures seem out of reach so far.

The present work comes in this context. It aims to establish internal observability for the wave equation with continuous density, in general space dimension. Notice finally that assumption (1.8) (or equivalently (1.9)) prevents high oscillations of the density in the "direction of the multiplier" (here in the radial direction as we chose the center of our multiplier in $x = 0$) and thus does not contradict the counterexamples in [5,9], which are strongly oscillating and therefore do not satisfy the monotony condition (1.8).

Let us in particular mention that our approach allows very strong oscillations in the tangential directions. Let us for instance consider the 2-d case in which $\Omega = B(0, R_0) \setminus \overline{B(0, r_0)}$ and a density written in radial coordinates under the form $\rho(r, \theta) = \rho_{rad}(r)\rho_{tang}(\theta)$. In this case, we simply require the two following assumptions:

[1] One can for instance take $\rho(x) = 1/|x|^2$ in a neighborhood of $S(r_0)$.

- both tangential and radial parts are continuous and strictly positive;
- there exists $\alpha > 0$ such that

$$r\partial_r \rho_{\text{rad}} + (2-\alpha)\rho_{\text{rad}} \geq 0 \quad \text{in the sense of } \mathscr{D}'(r_0, R_0).$$

In particular, the tangential part ρ_{tang} can be very rough, as it is simply required to be continuous and strictly positive. The radial part, on the other hand, is such that $r \mapsto r^{2-\alpha}\rho_{\text{rad}}(r)$ is strictly positive, continuous, bounded and increasing. As it is classical, such function is differentiable almost everywhere, but this does not imply that such function is necessarily differentiable. Indeed, one can take for instance the sum of a positive constant and the Cantor function (also called *Devil's staircase*) or construct increasing positive continuous bounded functions that do not belong to any Hölder space by gluing functions of the form $r \mapsto r^{1/n}$ for $n \in \mathbb{N}$, for instance taking, for $r \in (r_0, R_0)$,

$$\rho_{\text{rad}}(r) = 1 + \sum_{n \geq 1} \frac{1}{2^n}\left(\left(r - r_0 - \frac{1}{2^n}\right)_+\right)^{1/n},$$

where $(f(r))_+ = \max\{f(r), 0\}$ denotes the positive part of a generic function $f = f(r)$.

Let us finally point out that, by duality (see, e.g., [17]), Theorem 1.1 immediately implies the following controllability result.

Corollary 1.2. *Under the assumptions of Theorem 1.1, for all T satisfying (1.11) and for all $(y_0, y_1) \in H_0^1(\Omega) \times L^2(\Omega)$, there exists a control function $f \in L^2((0,T) \times \omega)$ such that the solution y of*

$$\begin{cases} \rho(x)\partial_t^2 y - \Delta_x y = \mathbf{1}_\omega f & \text{in } (0,T) \times \Omega, \\ y = 0 & \text{on } (0,T) \times \partial\Omega, \\ (y(0), \partial_t y(0)) = (y_0, y_1) & \text{in } \Omega, \end{cases} \tag{1.16}$$

satisfies the controllability requirement:

$$(y(T), \partial_t y(T)) = (0,0), \quad \text{in } \Omega. \tag{1.17}$$

The proof of Corollary 1.2 simply consists in a classical duality argument (cf. [17]) and is therefore left to the reader.

1.4. Outline

This note is organized as follows. Section 2 presents several approximation results, first on the possibility of approximating ρ satisfying (1.8) by smooth densities still satisfying (1.8), and then on the convergence of the solutions to the corresponding wave equations. Section 3 presents the proof of Theorem 1.1 and in particular recalls how the multiplier argument applies in our context. Section 4 aims at discussing various improvements of Theorem 1.1 (more general α, boundary observation) and possible extensions to our work.

2. Approximation results

This section gathers several approximation results. Section 2.1 exhibits suitable approximations of densities ρ satisfying the assumption of Theorem 1.1. Section 2.2 then shows the stability of the solutions to (1.1) with respect to the density ρ.

2.1. Smooth approximations of the density ρ

The goal of this section is to prove the following approximation result:

Proposition 2.1. *Let Ω_1 be a smooth domain containing $\overline{\Omega}$ and let ρ satisfy the assumptions (1.2)–(1.3) on Ω_1 and (1.8), and further assume (1.10).*

Let η be a real valued non-negative smooth function on $\mathbb{R}^d$ supported in the unit ball $B(0,1)$ and such that

$$\int_{\mathbb{R}^d} \eta(x)\,\mathrm{d}x = 1 \text{ and } \forall i \in \{1, \cdots, d\}, \int_{\mathbb{R}^d} x_i \eta(x)\,\mathrm{d}x = 0. \tag{2.1}$$

Then there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, the family of functions $(\rho_\varepsilon)_{\varepsilon > 0}$ defined on $\overline{\Omega}$ by

$$\begin{cases} \forall x \in \overline{\Omega} \setminus \{0\}, \quad \rho_\varepsilon(x) = (\varepsilon|x|)^{-d} \displaystyle\int_{\Omega_1} \rho(y)\eta\left(\frac{x-y}{\varepsilon|x|}\right)\mathrm{d}y, \\ \text{if } 0 \in \overline{\Omega}, \quad \rho_\varepsilon(0) = \rho(0), \end{cases} \tag{2.2}$$

satisfies the following properties:

(i) for each $\varepsilon \in (0, \varepsilon_0)$, ρ_ε *satisfies conditions* (1.2)*;*
(ii) for each $\varepsilon \in (0, \varepsilon_0)$, ρ_ε *belongs to* $C^1(\overline{\Omega})$*;*
(iii) the family $(\rho_\varepsilon)_\varepsilon$ *strongly converges in* $L^\infty(\Omega)$ *to* ρ *as* $\varepsilon \to 0$*:*

$$\lim_{\varepsilon\to 0} \|\rho_\varepsilon - \rho\|_{L^\infty(\Omega)} = 0; \tag{2.3}$$

(iv) for each $\varepsilon \in (0, \varepsilon_0)$*, condition* (1.8) *is satisfied pointwise in* Ω*, i.e.*

$$\forall x \in \Omega, \quad x \cdot \nabla \rho_\varepsilon(x) + (2-\alpha)\rho_\varepsilon(x) \geq 0. \tag{2.4}$$

Before going into the proof of Proposition 2.1, let us note that there exist functions η that are real-valued, non-negative, smooth, are supported in the unit ball $B(0,1)$ and satisfy (2.1). For instance, one can take $\eta(x) = c_* \exp(-1/(1-|x|^2))$ for $|x| < 1$ and 0 otherwise, where $c_* = (\int_{B(0,1)} \exp(-1/(1-|x|^2))\,\mathrm{d}x)^{-1}$.

Proof. Let us first remark that the definition of ρ_ε in (2.2) can also be written, for $\varepsilon > 0$ small enough, as

$$\forall x \in \overline{\Omega} \setminus \{0\}, \quad \rho_\varepsilon(x) = (\varepsilon|x|)^{-d} \int_{\mathbb{R}^d} \rho(y)\eta\left(\frac{x-y}{\varepsilon|x|}\right) \mathrm{d}y \tag{2.5}$$

$$= \int_{\mathbb{R}^d} \rho(x - \varepsilon|x|z)\eta(z)\,\mathrm{d}z, \tag{2.6}$$

in which the value of ρ outside Ω_1 is irrelevant as $y \mapsto \eta((x-y)/\varepsilon|x|)$ vanishes outside Ω_1 for $x \in \overline{\Omega}$ and $\varepsilon \in (0, \varepsilon_0)$ small enough. Formula (2.5) has the advantage to underline that ρ_ε is in fact some kind of convolution, thus explaining the regularity properties of ρ_ε. Besides that, the kernel takes some mean value approximation of $\rho(x)$ over balls of center x and size $\varepsilon|x|$. This induces some radial scaling that appears in fact naturally when designing approximations ρ_ε of ρ such that ρ_ε satisfies (2.4) pointwise when ρ satisfies (1.8) in the sense of distributions.

Proof of item (i). In order to prove item (i), we simply remark that from (2.1) for all $x \in \mathbb{R}^d$,

$$\int_{\mathbb{R}^d} \eta\left(\frac{x-y}{\varepsilon|x|}\right) \mathrm{d}y = (\varepsilon|x|)^d.$$

One then easily gets that ρ_ε satisfies (1.2) on $\overline{\Omega}$.

Proof of item (ii). The proof of item (ii) is rather straightforward in $\overline{\Omega} \setminus \{0\}$, where it is an immediate consequence of the smoothness of η by using formula (2.5).

When 0 belongs to $\overline{\Omega}$, the situation is slightly more intricate. We first remark that in fact formula (2.2) actually defines ρ_ε in an open neighborhood of $\overline{\Omega}$, so that 0 can always be considered as an interior point of Ω by slightly enlarging the set Ω if needed. In order to check that ρ_ε is differentiable at 0 when 0 belongs to Ω, we use the fact that ρ is differentiable at 0, that is

$$\delta(y) = \rho(y) - \rho(0) - y \cdot \nabla\rho(0) \text{ satisfies } \delta(y) = o_{|y|\to 0}(|y|). \tag{2.7}$$

Therefore, for $x \in \overline{\Omega} \setminus \{0\}$,

$$\rho_\varepsilon(x) = (\varepsilon|x|)^{-d} \int_{\mathbb{R}^d} (\rho(0) + y \cdot \nabla\rho(0) + \delta(y))\eta\left(\frac{x-y}{\varepsilon|x|}\right) \mathrm{d}y$$

$$= \rho(0) + \nabla\rho(0)(\varepsilon|x|)^{-d} \int_{\mathbb{R}^d} y\eta\left(\frac{x-y}{\varepsilon|x|}\right) \mathrm{d}y + (\varepsilon|x|)^{-d} \int_{\mathbb{R}^d} \delta(y)\eta\left(\frac{x-y}{\varepsilon|x|}\right) \mathrm{d}y.$$

But from (2.1),

$$(\varepsilon|x|)^{-d} \int_{\mathbb{R}^d} y\eta\left(\frac{x-y}{\varepsilon|x|}\right) \mathrm{d}y = x,$$

and from (2.7),

$$\delta_\varepsilon(x) = (\varepsilon|x|)^{-d} \int_{\mathbb{R}^d} \delta(y)\eta\left(\frac{x-y}{\varepsilon|x|}\right) \mathrm{d}y \text{ satisfies } \delta_\varepsilon(x) = o_{|x|\to 0}(|x|).$$

We can thus conclude that ρ_ε is differentiable at 0 and $\nabla\rho_\varepsilon(0) = \nabla\rho(0)$.

To show that ρ_ε is $C^1(\overline{\Omega})$, we then simply have to check the continuity of $\nabla \rho_\varepsilon$ close to 0. As ρ is C^1 close to 0, there exists a small ball $B(0, r_0) \subset \Omega$ in which $\nabla \rho$ is well-defined and continuous. Therefore, there exists $r_1 \in (0, r_0)$ such that differentiating (2.6), we can write for all $x \in B(0, r_1) \setminus \{0\}$,

$$\nabla \rho_\varepsilon(x) = \int_{\mathbb{R}^d} \eta(z) \left(I - \varepsilon \frac{x}{|x|} z^t \right) \nabla_x \rho(x - \varepsilon |x| z) \, \mathrm{d}z.$$

Under this form, using (2.1), one easily checks that $\nabla \rho_\varepsilon(x)$ goes to $\nabla \rho(0)$ as $|x| \to 0$ as $\nabla_x \rho$ is continuous in a neighborhood of 0. This concludes the proof of item (ii).

Proof of item (iii). Since ρ is continuous on $\overline{\Omega_1}$, it is uniformly continuous on $\overline{\Omega_1}$: for every $\beta > 0$, there exists $\gamma(\beta) > 0$ such that

$$\forall (x, y) \in \overline{\Omega_1}^2, \text{ with } |x - y| \le \gamma(\beta), \quad |\rho(x) - \rho(y)| \le \beta. \tag{2.8}$$

We then remark that

$$\forall x \in \overline{\Omega}, \quad \rho_\varepsilon(x) - \rho(x) = (\varepsilon |x|)^{-d} \int (\rho(y) - \rho(x)) \eta \left(\frac{x - y}{\varepsilon |x|} \right) \mathrm{d}y. \tag{2.9}$$

Therefore, if we set $R_1 = \max\{|x|, \text{ for } x \in \overline{\Omega_1}\}$, for all $\beta > 0$, as soon as $\varepsilon R_1 \le \gamma(\beta)$ given above in (2.8), we obtain $|\rho_\varepsilon(x) - \rho(x)| \le \beta$ for every $x \in \overline{\Omega}$ providing $\varepsilon < \min\{\varepsilon_0, \gamma(\beta)/R_1\}$. This concludes the proof of item (iii).

Proof of item (iv). Differentiating formula (2.5), we obtain for all $x \in \overline{\Omega} \setminus \{0\}$,

$$\begin{aligned}
x \cdot \nabla \rho_\varepsilon(x) &= \sum_{i=1}^d x_i \partial_{x_i} \rho_\varepsilon(x) \\
&= (\varepsilon |x|)^{-d} \int_{\mathbb{R}^d} \rho(y) \sum_{i,j} \partial_{x_j} \eta \left(\frac{x - y}{\varepsilon |x|} \right) \left[\frac{1}{\varepsilon |x|} \left(x_i \delta_{ij} - (x_j - y_j) \frac{x_i^2}{|x|^2} \right) \right] \mathrm{d}y - d \times \rho_\varepsilon(x) \\
&= (\varepsilon |x|)^{-d-1} \int_{\mathbb{R}^d} \rho(y) y \cdot \nabla_x \eta \left(\frac{x - y}{\varepsilon |x|} \right) \mathrm{d}y - d \times \rho_\varepsilon(x).
\end{aligned}$$

We then remark that

$$\nabla_x \eta \left(\frac{x - y}{\varepsilon |x|} \right) = -\varepsilon |x| \nabla_y \left(\eta \left(\frac{x - y}{\varepsilon |x|} \right) \right).$$

We thus get, for all $x \in \overline{\Omega} \setminus \{0\}$,

$$\begin{aligned}
x \cdot \nabla \rho_\varepsilon(x) &= -(\varepsilon |x|)^{-d} \int \rho(y) y \cdot \nabla_y \left(\eta \left(\frac{x - y}{\varepsilon |x|} \right) \right) \mathrm{d}y - d \times \rho_\varepsilon(x) \\
&= -(\varepsilon |x|)^{-d} \int \rho(y) \mathrm{div}_y \left(y \, \eta \left(\frac{x - y}{\varepsilon |x|} \right) \right) \mathrm{d}y.
\end{aligned}$$

Using (1.9), we thus deduce for all $x \in \overline{\Omega} \setminus \{0\}$ that

$$x \cdot \nabla \rho_\varepsilon(x) + (2 - \alpha) \rho_\varepsilon(x) \ge 0.$$

This condition can also be checked immediately close to 0 if $0 \in \Omega$ by using that in this case ρ is C^1 in a neighborhood of 0, which implies that condition (1.8) is satisfied pointwise in a neighborhood of 0, so $(2 - \alpha)\rho(0) \ge 0$. □

2.2. Stability of the solutions to the wave equation with respect to the density

Here we discuss the convergence of solutions to the wave equation (1.1) corresponding to a family of densities converging in $C^0(\overline{\Omega})$. This will be important in the following as our arguments to prove Theorem 1.1 will be developed for the smooth approximations ρ_ε of ρ given by Proposition 2.1 and we will therefore need afterwards to pass to the limit $\varepsilon \to 0$.

Namely, we will use the following result.

Proposition 2.2. *Let $(\rho_\varepsilon)_{\varepsilon > 0}$ be a family of $C^1(\overline{\Omega})$ satisfying (1.2) uniformly with coefficients ρ_1 and ρ_2 and strongly convergent to ρ in $L^\infty(\Omega)$:*

$$\lim_{\varepsilon \to 0} \| \rho_\varepsilon - \rho \|_{L^\infty(\Omega)} = 0. \tag{2.10}$$

Let $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$*. For each* $\varepsilon > 0$*, we denote by* u_ε *the solution to*

$$\begin{cases} \rho_\varepsilon(x)\partial_t^2 u_\varepsilon - \Delta_x u_\varepsilon = 0 & \text{in } (0,T)\times\Omega, \\ u_\varepsilon = 0 & \text{on } (0,T)\times\partial\Omega \\ (u_\varepsilon(0), \partial_t u_\varepsilon(0)) = (u_0, u_1) & \text{in } \Omega, \end{cases} \tag{2.11}$$

and by u *the solution to* (1.1)*.*

We then have the strong convergence of (u_ε) *to* u *in* $L^2(0,T; H_0^1(\Omega)) \cap H^1(0,T; L^2(\Omega))$*:*

$$\lim_{\varepsilon\to 0} \|u_\varepsilon - u\|_{L^2(0,T;H_0^1(\Omega))\cap H^1(0,T;L^2(\Omega))} = 0. \tag{2.12}$$

Before going into the proof, let us first remark that the convergence (2.10) implies in particular that ρ belongs to $C^0(\overline{\Omega})$ and satisfies (1.2) with coefficients ρ_1 and ρ_2.

For convenience, we also introduce the notation E_ε for describing the energy of solutions u_ε of (2.11):

$$E_\varepsilon[u_\varepsilon](t) := \frac{1}{2}\int_\Omega \rho_\varepsilon(x)|\partial_t u_\varepsilon(t,x)|^2 \mathrm{d}x + \frac{1}{2}\int_\Omega |\nabla_x u_\varepsilon(t,x)|^2 \mathrm{d}x. \tag{2.13}$$

We can now go into the proof of Proposition 2.2.

Proof. Let us first note that, according to (2.10), for all $t \in [0,T]$,

$$|E_\varepsilon[u_\varepsilon](t=0) - E[u](t=0)| \le \|\rho_\varepsilon(x) - \rho(x)\|_{L^\infty(\Omega)} \|u_1\|^2_{L^2(\Omega)} \underset{\varepsilon\to 0}{\to} 0. \tag{2.14}$$

As $E_\varepsilon[u_\varepsilon](t) = E_\varepsilon[u_\varepsilon](t=0)$ and $E[u](t) = E[u](t=0)$, it follows that

$$\limsup_{\varepsilon\to 0} \|u_\varepsilon\|_{C^0([0,T];H_0^1(\Omega))\cap C^1([0,T];L^2(\Omega))} < \infty \tag{2.15}$$

and

$$\lim_{\varepsilon\to 0} \sup_{t\in[0,T]} |E_\varepsilon[u_\varepsilon](t) - E[u](t)| = 0. \tag{2.16}$$

These two conditions immediately imply that

$$\lim_{\varepsilon\to 0} \|u_\varepsilon\|_{L^2(0,T;H_0^1(\Omega))\cap H^1(0,T;L^2(\Omega;\rho\,\mathrm{d}x))} = \|u\|_{L^2(0,T;H_0^1(\Omega))\cap H^1(0,T;L^2(\Omega;\rho\,\mathrm{d}x))}, \tag{2.17}$$

where $L^2(\Omega;\, \rho\,\mathrm{d}x)$ corresponds to the Hilbert space constructed from the scalar product

$$\langle f, g\rangle = \int_\Omega f(x)g(x)\rho(x)\,\mathrm{d}x.$$

Therefore, if we manage to show that the sequence u_ε converges in $\mathscr{D}'(\Omega)$ to u as $\varepsilon \to 0$, then the strong convergence of u_ε to u as $\varepsilon \to 0$ in $L^2(0,T; H_0^1(\Omega)) \cap H^1(0,T; L^2(\Omega;\, \rho\,\mathrm{d}x))$ follows since (2.17) guarantees the convergence of the norms. The strong convergence in (2.3) can then be easily deduced from the fact that the topologies in $L^2(\Omega;\, \rho\,\mathrm{d}x)$ and $L^2(\Omega)$ are equivalent under the assumption (1.2).

We thus focus on the proof of the weak convergence of u_ε to u in the sense of distributions. In order to do so, we introduce, for $t \in [0,T]$ and $x \in \Omega$,

$$z_\varepsilon(t,x) = \int_0^t \big(u_\varepsilon(s,x) - u(s,x)\big)\mathrm{d}s. \tag{2.18}$$

Then z_ε solves the following system:

$$\begin{cases} \rho(x)\partial_t^2 z_\varepsilon - \Delta_x z_\varepsilon = (\rho(x) - \rho_\varepsilon(x))(\partial_t u_\varepsilon - u_1) & \text{in } (0,T)\times\Omega, \\ z_\varepsilon = 0 & \text{on } (0,T)\times\partial\Omega \\ (z_\varepsilon(0), \partial_t z_\varepsilon(0)) = (0,0) & \text{in } \Omega. \end{cases} \tag{2.19}$$

From (2.10) and (2.15), the source term belongs to $C^0([0,T]; L^2(\Omega))$ and strongly converges to 0 in that space:

$$\|(\rho(x) - \rho_\varepsilon(x))(\partial_t u_\varepsilon - u_1)\|_{L^\infty(0,T;L^2(\Omega))} \le \|\rho - \rho_\varepsilon\|_{L^\infty(\Omega)} \left(\|u_\varepsilon\|_{H^1(0,T;L^2(\Omega))} + \|u_1\|_{L^2(\Omega)}\right) \underset{\varepsilon\to 0}{\to} 0. \tag{2.20}$$

The solution z_ε of (2.19) therefore belongs to $C^0([0,T]; H_0^1(\Omega)) \cap C^1(0,T; L^2(\Omega; \rho\,\mathrm{d}x))$ and strongly converges to 0 in that space. In particular,

$$\lim_{\varepsilon \to 0} \|\partial_t z_\varepsilon\|_{L^2(0,T;L^2(\Omega;\,\rho\,\mathrm{d}x))} = 0. \tag{2.21}$$

Recalling the definition of z_ε in (2.18), we get that the sequence u_ε converges to u strongly in $L^2(0,T; L^2(\Omega;\,\rho\,\mathrm{d}x))$. As explained above, combined with the convergence of the norms in (2.17), this implies the strong convergence of u_ε to u in $L^2(0,T; H_0^1(\Omega)) \cap H^1(0,T; L^2(\Omega))$, thus concluding the proof of Proposition 2.2. □

3. Proof of Theorem 1.1

Let us emphasize that in our proof below, we will consider only real-valued functions. This concerns in particular the displacement and all test functions. Let us point out that nonetheless, all our arguments apply as well for complex-valued solutions since the density remains real-valued.

3.1. Multiplier argument for smooth densities

Here, we recall the classical multiplier argument for smooth densities, see [17]. Namely, we consider a generic density σ satisfying

$$\begin{cases} \sigma \in C^1(\overline{\Omega}), \\ \exists (\rho_1, \rho_2)\, s.t.\, \forall x \in \Omega, \quad 0 < \rho_1 \le \sigma(x) \le \rho_2, \\ \exists \alpha \in (0,2] \quad s.t.\ \forall x \in \Omega, \quad x \cdot \nabla \sigma(x) + (2-\alpha)\sigma(x) \ge 0, \\ \text{if } d = 1,\ \alpha = 2. \end{cases} \tag{3.1}$$

Of course, these assumptions are remanent from the ones obtained on the family of smooth densities ρ_ε in Proposition 2.1.

For such σ, we shall consider the corresponding wave equation

$$\begin{cases} \sigma(x)\partial_t^2 u - \Delta_x u = 0 & \text{in } (0,T) \times \Omega, \\ u = 0 & \text{on } (0,T) \times \partial\Omega \\ (u(0), \partial_t u(0)) = (u_0, u_1) & \text{in } \Omega, \end{cases} \tag{3.2}$$

for $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$. In such case, the solution u is uniquely defined, belongs to $C^0([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega))$ and has a constant energy: the quantity

$$E_\sigma[u](t) := \frac{1}{2} \int_\Omega \sigma(x) |\partial_t u(t,x)|^2 \mathrm{d}x + \frac{1}{2} \int_\Omega |\nabla_x u(t,x)|^2 \mathrm{d}x$$

is independent of the time t.

We then get the following result.

Proposition 3.1. *Let ω be an open subset of Ω satisfying assumptions (1.6)–(1.7), and let σ satisfy (3.1).*

For all T satisfying (1.11), there exists a constant $C > 0$ depending only on $(\rho_1, \rho_2, \alpha, T)$ and the geometrical setting such that, for all $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$, the solution u of (3.2) satisfies the following observability inequality:

$$E_\sigma[u](0) \le C \int_0^T \int_\omega \left(|\partial_t u(t,x)|^2 + |u(t,x)|^2 \right) \mathrm{d}x\mathrm{d}t. \tag{3.3}$$

Proof. In this proof, all the constants will be denoted by C. Their value may change from line to line, but they all depend only on the geometrical setting and the constants ρ_1, ρ_2, α in (3.1), and the time parameter $T > 0$.

Step 1. An observability estimate up to localized lower order terms at time $t = 0$ and $t = T$. We choose an open set ω_0 of $\mathbb{R}^d$ such that ω_0 is an open neighborhood in $\overline{\Omega}$ of Γ and $\omega_0 \Subset \omega$. We then introduce a smooth cut-off function $\psi = \psi(x) \in C^\infty(\Omega)$ with values in $[0,1]$ and satisfying $\psi(x) = 1$ for $x \in \Omega \setminus \omega_0$ and vanishing in a neighborhood of Γ included in ω_0, and we set

$$v(t,x) = \psi(x) u(t,x) \text{ for } (t,x) \in [0,T] \times \Omega. \tag{3.4}$$

One then easily gets that $v \in C^0([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega))$ and satisfies:

$$\begin{cases} \sigma(x)\partial_t^2 v - \Delta_x v = f & \text{in } (0,T)\times\Omega, \\ v = 0 & \text{on } (0,T)\times\partial\Omega, \\ \partial_n v = 0 & \text{on } (0,T)\times\Gamma. \\ (v(0), \partial_t v(0)) = (\psi u_0, \psi u_1) & \text{in } \Omega, \end{cases} \tag{3.5}$$

where f is given by

$$f(t,x) = [\Delta_x, \psi(x)]u(t,x) \text{ for } (t,x) \in [0,T]\times\Omega. \tag{3.6}$$

Multiplying equation (3.5)$_{(1)}$ by $x \cdot \nabla_x v + \lambda v$, where λ is some constant that will be fixed later, and integrating over the cylinder $(0,T)\times\Omega$, we get:

$$\int_0^T\int_\Omega \left(\sigma\partial_t^2 v - \Delta_x v\right)(x\cdot\nabla_x v + \lambda v)\,\mathrm{d}x\mathrm{d}t = \int_0^T\int_\Omega f\,(x\cdot\nabla_x v + \lambda v)\,\mathrm{d}x\mathrm{d}t.$$

Straightforward computations yield:

$$\int_0^T\int_\Omega \sigma\partial_t^2 v\,(x\cdot\nabla_x v + \lambda v)\,\mathrm{d}x\mathrm{d}t = \int_\Omega \sigma\partial_t v\,(x\cdot\nabla_x v + \lambda v)\,\mathrm{d}x\Big|_0^T$$
$$+\frac{1}{2}\int_0^T\int_\Omega |\partial_t v|^2\,(x\cdot\nabla\sigma)\,\mathrm{d}x\mathrm{d}t + \frac{d-2\lambda}{2}\int_0^T\int_\Omega \sigma|\partial_t v|^2\,\mathrm{d}x\mathrm{d}t,$$

and

$$\int_0^T\int_\Omega (-\Delta_x v)\,(x\cdot\nabla_x v + \lambda v)\,\mathrm{d}x\mathrm{d}t = \frac{2(\lambda+1)-d}{2}\int_0^T\int_\Omega |\nabla_x v|^2\,\mathrm{d}x\mathrm{d}t - \frac{1}{2}\int_0^T\int_{\partial\Omega} x\cdot n|\partial_n v|^2 d\Gamma\mathrm{d}t.$$

Putting these identities together and using (3.1)$_{(3)}$, we obtain:

$$\frac{\alpha-2+d-2\lambda}{2}\int_0^T\int_\Omega \sigma|\partial_t v|^2\,\mathrm{d}x\mathrm{d}t + \frac{2(\lambda+1)-d}{2}\int_0^T\int_\Omega |\nabla_x v|^2\,\mathrm{d}x\mathrm{d}t$$
$$+\int_\Omega \sigma\partial_t v\,(x\cdot\nabla_x v + \lambda v)\,\mathrm{d}x\Big|_0^T - \frac{1}{2}\int_0^T\int_{\partial\Omega} x\cdot n|\partial_n v|^2 d\Gamma\mathrm{d}t \le \int_0^T\int_\Omega f\,(x\cdot\nabla_x v + \lambda v)\,\mathrm{d}x\mathrm{d}t.$$

We thus set

$$\lambda = \frac{d}{2} + \frac{\alpha}{4} - 1, \tag{3.7}$$

so that

$$\alpha - 2 + d - 2\lambda = 2(\lambda+1) - d = \frac{\alpha}{2}.$$

Using then that $x\cdot n < 0$ on $\partial\Omega\setminus\Gamma$ and that $\partial_n v = 0$ on $(0,T)\times\Gamma$, we obtain:

$$\frac{\alpha}{4}\int_0^T\int_\Omega \sigma|\partial_t v|^2\,\mathrm{d}x\mathrm{d}t + \frac{\alpha}{4}\int_0^T\int_\Omega |\nabla_x v|^2\,\mathrm{d}x\mathrm{d}t + \int_\Omega \sigma\partial_t v\,(x\cdot\nabla_x v + \lambda v)\,\mathrm{d}x\Big|_0^T \le \int_0^T\int_\Omega f\,(x\cdot\nabla_x v + \lambda v)\,\mathrm{d}x\mathrm{d}t. \tag{3.8}$$

Recalling the definition of v, one easily checks that

$$\left|\frac{\alpha}{4}\int_0^T\int_\Omega \sigma|\partial_t v|^2\,\mathrm{d}x\mathrm{d}t + \frac{\alpha}{4}\int_0^T\int_\Omega |\nabla_x v|^2\,\mathrm{d}x\mathrm{d}t - \frac{\alpha T}{2}E_\sigma[u](0)\right| \le C\int_0^T\int_{\omega_0}\left(|\partial_t u|^2 + |\nabla_x u|^2 + |u|^2\right)\mathrm{d}x\mathrm{d}t.$$

Besides, we have

$$\left|\int_\Omega \sigma\,\partial_t v\,(x\cdot\nabla_x v+\lambda v)\,\mathrm{d}x\right| \le \left\|\sqrt{\sigma}\right\|_{L^\infty(\Omega)} \left\|\sqrt{\sigma}\,\partial_t v\right\|_{L^2(\Omega)} \|x\cdot\nabla_x v+\lambda v\|_{L^2(\Omega)}\,.$$

Therefore, using Komornik's remark [18],

$$\begin{aligned}\|x\cdot\nabla_x v+\lambda v\|^2_{L^2(\Omega)} &= \int_\Omega |x\cdot\nabla_x v|^2+\lambda\int_\Omega x\cdot\nabla_x(|v|^2)\,\mathrm{d}x+\lambda^2\int_\Omega |v|^2\,\mathrm{d}x\\ &\le R^2\int_\Omega |\nabla_x v|^2\,\mathrm{d}x+\left(\lambda^2-d\lambda\right)\int_\Omega |v|^2\,\mathrm{d}x\\ &\le R^2\,\|\nabla_x v\|^2_{L^2(\Omega)}\end{aligned} \tag{3.9}$$

as $\lambda\in[0,d]$ from (3.1)$_{(3)-(4)}$ and (3.7). Note that this is the place in which we use that $\alpha\in(0,2]$ for $d\ge 2$ or $\alpha=1$ when $d=1$. We will see in Section 4.1 that we can in fact generalize Theorem 1.1 to $\alpha>0$ for the price of doing a slightly more intricate proof, see Section 4.1 for the details.

From the last estimates, we get

$$\begin{aligned}\left|\int_\Omega \sigma\,\partial_t v\,(x\cdot\nabla_x v+\lambda v)\,\mathrm{d}x\right| &\le R\sqrt{\rho_2}\left\|\sqrt{\sigma}\,\partial_t u\right\|_{L^2(\Omega)}\left(\|\nabla_x u\|_{L^2(\Omega)}+C\,\|u\|_{L^2(\omega_0)}\right)\\ &\le R\sqrt{\rho_2}E_\sigma[u]+CR\sqrt{\rho_2}\sqrt{E_\sigma[u]}\,\|u\|_{L^2(\omega_0)}\,.\end{aligned} \tag{3.10}$$

We finally remark that f in (3.6) is supported in ω_0, so that we easily obtain

$$\left|\int_0^T\int_\Omega f\,(x\cdot\nabla_x v+\lambda v)\,\mathrm{d}x\mathrm{d}t\right| \le C\int_0^T\int_{\omega_0}\left(|\nabla_x u|^2+|u|^2\right)\mathrm{d}x\mathrm{d}t.$$

Putting all these estimates in (3.8), we obtain

$$\begin{aligned}\left(\alpha T-4R\sqrt{\rho_2}\right)E_\sigma[u](0) &\le C\int_0^T\int_{\omega_0}\left(|\partial_t u|^2+|\nabla_x u|^2+|u|^2\right)\mathrm{d}x\mathrm{d}t\\ &\quad + C\sqrt{E_\sigma[u](0)}\left(\|u_0\|_{L^2(\omega_0)}+\|u(T)\|_{L^2(\omega_0)}\right).\end{aligned}$$

Thus, for T as in (1.11), we get the existence of a constant depending on T such that

$$\begin{aligned}E_\sigma[u](0) &\le C\int_0^T\int_{\omega_0}\left(|\partial_t u(t,x)|^2+|\nabla_x u(t,x)|^2+|u(t,x)|^2\right)\mathrm{d}x\mathrm{d}t\\ &\quad + C\sqrt{E_\sigma[u](0)}\left(\|u_0\|_{L^2(\omega_0)}+\|u(T)\|_{L^2(\omega_0)}\right).\end{aligned} \tag{3.11}$$

Step 2. An observability estimate without terms localized at time $t=0$ or $t=T$. If $\tilde\omega$ is an open set of $\mathbb{R}^d$ such that $\tilde\omega$ is an open neighborhood in $\overline{\Omega}$ of Γ and $\omega_0\Subset\tilde\omega\Subset\omega$, we can show that

$$\|u_0\|^2_{L^2(\omega_0)}+\|u(T)\|^2_{L^2(\omega_0)} \le C\int_0^T\int_{\tilde\omega}\left(|\partial_t u(t,x)|^2+|\nabla_x u(t,x)|^2+|u(t,x)|^2\right)\mathrm{d}x\mathrm{d}t. \tag{3.12}$$

Indeed, we introduce a smooth cut-off function ψ_1 taking value 1 in ω_0 and vanishing in $\Omega\setminus\tilde\omega$, and we multiply the equation (3.2)$_{(1)}$ by $t(T-t)\psi_1(x)u(t,x)$ and integrate in time and space. This leads to

$$\begin{aligned}\frac{T}{2}\int_\Omega \psi_1\sigma\left(|u_0|^2+|u(T)|^2\right)\mathrm{d}x &= \int_0^T\int_\Omega t(T-t)\psi_1\sigma|\partial_t u|^2\,\mathrm{d}x\mathrm{d}t-\int_0^T\int_\Omega t(T-t)\psi_1|\nabla_x u|^2\,\mathrm{d}x\mathrm{d}t\\ &\quad+\int_0^T\int_\Omega \psi_1\sigma|u|^2\,\mathrm{d}x\mathrm{d}t+\frac{1}{2}\int_0^T\int_\Omega t(T-t)\Delta_x\psi_1|u|^2\,\mathrm{d}x\mathrm{d}t.\end{aligned}$$

The conditions on the support of ψ_1 then obviously imply (3.12).

Finally, using (3.12) in (3.11), we immediately get

$$E_\sigma[u](0) \le C \int_0^T \int_{\tilde{\omega}} \left(|\partial_t u(t,x)|^2 + |\nabla_x u(t,x)|^2 + |u(t,x)|^2 \right) \mathrm{d}x\mathrm{d}t. \tag{3.13}$$

Step 3. Removing the observation in gradient. Now, we finish the proof of (3.3) and drop the gradient term from the right-hand side of (3.13), the price to pay being a space integration over ω instead of $\tilde{\omega}$. Since the problem is invariant with respect to time translation, we first notice that estimate (3.13) can be written as

$$E_\sigma[u](0) = E_\sigma[u](\tau_0) \le C \int_{\tau_0}^{T-\tau_0} \int_{\tilde{\omega}} \left(|\partial_t u(t,x)|^2 + |\nabla_x u(t,x)|^2 + |u(t,x)|^2 \right) \mathrm{d}x\mathrm{d}t, \tag{3.14}$$

where the real number τ_0 is chosen such that $0 < 4\tau_0 < T - 4\alpha^{-1} R\sqrt{\rho_2}$.

Let then $\psi_2 = \psi_2(x)$ be a smooth non-negative cut-off function taking value 1 in $\tilde{\omega}$ and supported in ω, and $\varphi = \varphi(t)$ a time-dependent smooth non-negative cut-off function taking value 1 in $[\tau_0, T - \tau_0]$ and supported in $[0, T]$. Multiplying the equation $(3.2)_{(1)}$ by $\varphi(t)\psi_2(x)u(t,x)$ and integrating over $(0,T) \times \Omega$, we obtain:

$$\int_0^T \int_\Omega \varphi(t)\psi_2(x)|\nabla_x u|^2 \,\mathrm{d}x\mathrm{d}t = \int_0^T \int_\Omega \varphi(t)\psi_2(x)\sigma|\partial_t u|^2 \mathrm{d}x\mathrm{d}t$$
$$- \frac{1}{2} \int_0^T \int_\Omega \varphi''(t)\psi_2(x)\sigma(x)|u|^2 \,\mathrm{d}x\mathrm{d}t + \frac{1}{2} \int_0^T \int_\Omega \varphi(t)\Delta_x\psi_2|u|^2 \,\mathrm{d}x\mathrm{d}t.$$

This easily yields:

$$\int_{\tau_0}^{T-\tau_0} \int_{\tilde{\omega}} |\nabla_x u(t,x)|^2 \,\mathrm{d}x\mathrm{d}t \le \int_0^T \int_\Omega \varphi(t)\psi_2(x)|\nabla_x u(t,x)|^2 \mathrm{d}x\mathrm{d}t$$
$$\le C \int_0^T \int_\omega \left(|\partial_t u(t,x)|^2 + |u(t,x)|^2 \right) \mathrm{d}x\mathrm{d}t.$$

The observability estimate (3.3) immediately follows by plugging this last estimate into (3.14).

Now, tracking the constants appearing in (3.3), one can check that they depend only on ρ_1, ρ_2 and α in (3.1), and on the geometry through the various cut-offs appearing in the above proof, namely ψ, ψ_1, φ, and ψ_2. □

3.2. End of the proof of Theorem 1.1

Step 1. An observability inequality for (1.1) with an observation containing a lower-order term. We now finish the proof of Theorem 1.1, and consider ρ satisfying the assumptions of Theorem 1.1.

First, using Proposition 2.1, we construct a sequence ρ_ε of $C^1(\overline{\Omega})$ densities satisfying (1.2) and (2.4) uniformly with respect to ε and the strong convergence (2.3).

We can therefore apply Proposition 3.1 for each $\varepsilon > 0$ and get uniform observability estimates for (2.11). Namely, for $T > 0$ satisfying (1.11), there exists a constant $C > 0$ such that for all $\varepsilon > 0$ and all u_ε solution to (1.1) with initial data in $H_0^1(\Omega) \times L^2(\Omega)$, we have:

$$E_\varepsilon[u_\varepsilon](0) \le C \int_0^T \int_\omega \left(|\partial_t u_\varepsilon(t,x)|^2 + |u_\varepsilon(t,x)|^2 \right) \mathrm{d}x\mathrm{d}t. \tag{3.15}$$

Using then the results of Proposition 2.2, in particular the strong convergence (2.12), we can pass to the limit $\varepsilon \to 0$ in (3.15). We get that for all (u_0, u_1) in $H_0^1(\Omega) \times L^2(\Omega)$, the solution u of (1.1) satisfies:

$$E[u](0) \le C \int_0^T \int_\omega \left(|\partial_t u(t,x)|^2 + |u(t,x)|^2 \right) \mathrm{d}x\mathrm{d}t. \tag{3.16}$$

Step 2. A contradiction argument. Finally, to conclude the proof, it remains to drop from this estimate the compact term

$$\int_0^T \int_\omega |u(t,x)|^2 \, dx dt.$$

For this purpose, we argue by contradiction and assume the existence of a sequence of initial data $(u_0^k, u_1^k) \in H_0^1(\Omega) \times L^2(\Omega)$ such that the corresponding solutions $u^k(t,x)$ of (1.1) satisfy

$$\begin{cases} \lim_{k\to\infty} \int_0^T \int_\omega |\partial_t u^k(t,x)|^2 \, dx dt = 0 \\ \text{and} \\ E[u^k](0) = 1. \end{cases} \tag{3.17}$$

The sequence $u^k(t,x)$ is thus bounded in the energy space $C^0([0,T], H_0^1(\Omega)) \cap C^1([0,T], L^2(\Omega))$ and therefore, we may assume that, up to a subsequence, it weakly converges to some function u in $L^2([0,T], H_0^1(\Omega)) \cap H^1([0,T], L^2(\Omega))$, satisfying

$$\begin{cases} \rho(x)\partial_t^2 u - \Delta_x u = 0 & \text{in } (0,T)\times\Omega, \\ u = 0 & \text{on } (0,T)\times\partial\Omega \\ \partial_t u = 0 & \text{in } (0,T)\times\omega. \end{cases} \tag{3.18}$$

On the other hand, for $0 < \tau_1 < \frac{1}{2}(T - 4\alpha^{-1}R\sqrt{\rho_2})$, the observability estimate (3.16) is satisfied with $T - \tau_1$ instead of T. Consider then the function

$$v_\tau(t,x) = \frac{1}{\tau}\left(u(t+\tau,x) - u(t,x)\right), \tag{3.19}$$

where $\tau \in]0,\tau_1[$. It clearly solves the wave equation (1.1) and the third identity of (3.18) implies that $v_\tau = 0$ in $(0, T-\tau_1)\times\omega$. So (3.16) leads to $v_\tau = 0$ in $(0, T-\tau_1)\times\Omega$ for all $\tau \in]0,\tau_1[$. Consequently, passing to the limit $\tau \to 0$ we obtain $\partial_t u = 0$ in $(0, T-\tau_1)\times\Omega$, hence in $(0,T)\times\Omega$ as τ_1 can be arbitrarily small. Therefore, we have $u = u(x)$ and system (3.18) yields in particular

$$\begin{cases} -\Delta_x u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.20}$$

Multiplying (3.20) by u, we easily conclude that the weak limit u vanishes identically in $(0,T)\times\Omega$. Applying then (3.16) to the sequence $(u^k)_k$, and using $(3.17)_{(1)}$ and the compact embedding of $L^2([0,T], H_0^1(\Omega)) \cap H^1([0,T], L^2(\Omega))$ into $L^2([0,T]\times\Omega)$, we see that $\lim_{k\to\infty} E[u^k](0) = 0$. This contradicts $(3.17)_{(2)}$ and finishes the proof of Theorem 1.1.

4. Comments

4.1. On the restrictions on the parameter α

Here, we worked under the condition $\alpha \in (0,2]$ in dimension $d \geq 2$ and $\alpha = 2$ in dimension 1. Let us briefly comment these assumptions.

These assumptions appear in the proof of Proposition 3.1 in (3.9). There, we use the fact that, with λ as in (3.7), $\lambda \in [0,d]$ as $\alpha \in [4-2d, 2d+4]$ (recall that we have chosen $\alpha \in (0,2]$ for $d \geq 2$, $\alpha = 2$ for $d = 1$).

But one can in fact remove that condition to the price of adding some arguments. Namely we can prove the following extension of Theorem 1.1.

Theorem 4.1. *Under the same assumptions as in Theorem 1.1 except on α that we now assume only strictly positive, system (1.1) is observable in any time T satisfying (1.11).*

We will not give the full details of the proof of Theorem 4.1, but we will rather point out the differences with the one of Theorem 1.1 and briefly explain how the new difficulties can be overcome.

Sketch of the proof of Theorem 4.1. The proof of Theorem 4.1 will mainly follow the strategy of the proof of Theorem 1.1. But we will not prove the observability estimate (3.3) obtained in Proposition 3.1 directly as the estimate (3.9) fails to be true. Instead, for σ satisfying assumption $(3.1)_{(1,2,3)}$, one can obtain an observability estimate of the form: any solution u of (3.2) with initial data $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ satisfies

$$E_\sigma[u](0) \le C \int_0^T \int_\omega \left(|\partial_t u(t,x)|^2 + |u(t,x)|^2 \right) \mathrm{d}x\mathrm{d}t + C \, \|(u_0,u_1)\|^2_{L^2(\Omega) \times H^{-1}(\Omega)}, \tag{4.1}$$

in which T satisfies (1.11) and the constant $C > 0$ only depends on $(\rho_1, \rho_2, \alpha, T)$ and on the geometrical setting.

Proof of (4.1). The proof of (4.1) mimics the proof of Proposition 3.1. However, instead of (3.9), we shall rely on the following estimate:

$$\|x \cdot \nabla_x v + \lambda v\|_{L^2(\Omega)} \le R \, \|\nabla_x v\|_{L^2(\Omega)} + |\lambda| \, \|v\|_{L^2(\Omega)} . \tag{4.2}$$

This implies that estimate (3.10) should be replaced by:

$$\begin{aligned} \left| \int_\Omega \sigma \, \partial_t v(t) \, (x \cdot \nabla_x v(t) + \lambda v(t)) \, \mathrm{d}x \right| &\le R\sqrt{\rho_2} \, \left\| \sqrt{\sigma} \, \partial_t u(t) \right\|_{L^2(\Omega)} \left(\|\nabla_x u(t)\|_{L^2(\Omega)} + C \, \|u(t)\|_{L^2(\Omega)} \right) \\ &\le R\sqrt{\rho_2} E_\sigma[u] + CR\sqrt{\rho_2}\sqrt{E_\sigma[u]} \, \|u(t)\|_{L^2(\Omega)} , \end{aligned} \tag{4.3}$$

so that we get, instead of (3.11), the following estimate:

$$\begin{aligned} E_\sigma[u](0) \le{}& C \int_0^T \int_{\omega_0} \left(|\partial_t u(t,x)|^2 + |\nabla_x u(t,x)|^2 + |u(t,x)|^2 \right) \mathrm{d}x\mathrm{d}t \\ &+ C\sqrt{E_\sigma[u](0)} \left(\|u_0\|_{L^2(\Omega)} + \|u(T)\|_{L^2(\Omega)} \right). \end{aligned} \tag{4.4}$$

To derive (4.1), we shall then use the hyperbolic energy estimate at level $L^2 \times H^{-1}$

$$\|u(T)\|_{L^2(\Omega)} \le C \, \|(u_0,u_1)\|_{L^2(\Omega)\times H^{-1}(\Omega)} ,$$

which holds for some constant $C > 0$ depending only on (ρ_1, ρ_2) according to [19, Chapter 3, Section 9]. This concludes the proof of estimate (4.1).

Based on estimate (4.1) for σ satisfying $(3.1)_{(1,2,3)}$, following the Step 1 in Section 3.2, one would prove that solutions to the wave equation (1.1) should satisfy the following relaxed observability inequality:

$$E[u](0) \le C \int_0^T \int_\omega \left(|\partial_t u(t,x)|^2 + |u(t,x)|^2 \right) \mathrm{d}x\mathrm{d}t + C \, \|(u_0,u_1)\|^2_{L^2(\Omega) \times H^{-1}(\Omega)} . \tag{4.5}$$

One should therefore adapt the Step 2 of the argument in Section 3.2. In this paragraph, we use the same notations as in Section 3.2, Step 2. The main point in order to show (1.12) for solutions to (1.1) is to show that the set of solutions $u \in L^2(0,T; H^1_0(\Omega)) \cap H^1(0,T; L^2(\Omega))$ of (1.1) such that $\partial_t u = 0$ on $(0,T) \times \omega$, called the invisible set, reduces to the zero set. Let us briefly indicate how this can be done. Let us take u in the invisible set. One can first show that using (4.5), the corresponding functions v_τ in (3.19) are bounded in $L^2(0,T-\tau_1; H^1_0(\Omega)) \cap H^1(0,T-\tau_1; L^2(\Omega))$, uniformly for $\tau \in (0,\tau_1)$. Thus, passing to the limit $\tau \to 0$, $\partial_t u$ belongs to $L^2(0,T; H^1_0(\Omega)) \cap H^1(0,T; L^2(\Omega))$. Therefore, u satisfies

$$E[\partial_t u](0) \le C \, \|(u_0,u_1)\|^2_{H^1_0(\Omega) \times L^2(\Omega)} ,$$

for the constant C in (4.5). It follows that the unit ball of the invisible set is compact for the topology of $L^2(0,T; H^1_0(\Omega)) \cap H^1(0,T; L^2(\Omega))$, so that the invisible set is of finite dimension. Besides, from the arguments above, ∂_t acts on the invisible set. Consequently, there should be an eigenvector to the operator ∂_t in the invisible set, i.e. a non-trivial function $u = u(x)$ and a constant $\gamma \in \mathbb{C}$ satisfying

$$\begin{cases} \gamma^2 \rho u - \Delta_x u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \gamma u = 0 & \text{in } \omega. \end{cases} \tag{4.6}$$

The case $\gamma = 0$ cannot happen as one would then obviously get $u = 0$ by an immediate energy estimate. In the case $\gamma \neq 0$, the classical unique continuation result for elliptic operators applies (see, e.g., [12, Chap. VIII]) and shows $u = 0$. It follows that the invisible set necessarily reduces to 0.

The contradiction arguments developed in the Step 2 of Section 3.2 can then be easily modified in order to conclude Theorem 4.1. □

4.2. The case of boundary observation

Our main result (Theorem 1.1) concerns the case of an internal observation. This choice may seem surprising as the multiplier method is much more direct when working with boundary observation (see, e.g., [17] or [14]). But the problem is that the operator corresponding to a boundary observation is not bounded, and therefore one needs to prove a hidden regularity (or *admissibility*) result for the solutions to the wave equation (1.1). Namely, one would require a result of the form: if u solves (1.1) with initial data $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$, then $\partial_n u$ belongs to $L^2((0,T) \times \partial\Omega)$ and satisfies the estimate:

$$\|\partial_n u\|_{L^2((0,T)\times\partial\Omega)} \leq C \, \|(u_0, u_1)\|_{H_0^1(\Omega)\times L^2(\Omega)} \,. \tag{4.7}$$

To our knowledge, such result is, so far, *missing* for non-smooth densities.

It turns out that the usual way to prove the hidden regularity property for solutions to the wave operator is based on a multiplier argument similar to the one developed in Proposition 3.1, see [17,14]. If one follows this approach, it will be easy to check that the admissibility property (4.7) holds for solutions u to (3.2) provided $\sigma \in C^1(\overline{\Omega})$, but the constant C will *a priori* not be uniform with respect to coefficients σ satisfying (3.1).

However, the multiplier argument to get (4.7) can be made locally close to the boundary on which the observation is done. Therefore, if ρ belongs to C^1 in a neighborhood of the boundary $\partial\Omega$, one would be able to show (4.7) for solutions u of (1.1). In this case, one can show the following result.

Theorem 4.2. *Let Γ be an open subset of $\partial\Omega$ as in (1.6), Ω_1 be a smooth domain of $\mathbb{R}^d$ containing $\overline{\Omega}$, and let ρ satisfy the assumptions (1.2)–(1.3) on Ω_1, (1.8) for some $\alpha > 0$, and assume that ρ belongs to C^1 in a neighborhood of $\partial\Omega$.*

If we further assume (1.10), for all T satisfying (1.11), there exists a constant $C > 0$ such that the observability estimate

$$E[u](0) \leq C \int_0^T \int_\Gamma |\partial_n u(t,x)|^2 \mathrm{d}x\mathrm{d}t \tag{4.8}$$

holds true for every solution u of (1.1) with initial data in $H_0^1(\Omega) \times L^2(\Omega)$.

The details of the proof are left to the reader.

4.3. A general strategy

At this point, we would like to underline the general strategy underlying this work. It is mainly based on this elementary fact: given a “rough” density ρ, if one manages to construct a family of approximate densities ρ_ε strongly converging to ρ and for which the corresponding wave equations are observable uniformly with respect to $\varepsilon > 0$, one can then pass to the limit and deduce the observability of the wave equation corresponding to ρ.

In that argument, one sees that the main point is to get uniform estimates for sequences of densities. It is therefore important there to be able to rely on explicit proofs of observability results, which are so far very low developed in the context of sharp geometric control conditions, as we also underlined in our previous work [8]. We should however quote the recent works [15,16] by C. Laurent and M. Léautaud on this topic. But even if their argument is constructive, it is not clear if it can be adapted easily in our context. So far, this is an open problem.

In fact, this strategy may probably be applied to prove Carleman estimates as in [10, Chapter 4] for wave equations with very rough coefficients. This could be of interest in the context of inverse problems. We plan to explore these questions in a near future.

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