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Algebraic geometry

The full automorphism group of $\overline{T}$

Le groupe complet des automorphismes de $\overline{T}$

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ABSTRACT

Let $\overline{G}$ be the wonderful compactification of a simple affine algebraic group G of adjoint type defined over $\mathbb{C}$. Let $\overline{T} \subset \overline{G}$ be the closure of a maximal torus $T \subset G$. We prove that the group of all automorphisms of the variety $\overline{T}$ is the semi-direct product $N_G(T) \rtimes D$, where $N_G(T)$ is the normalizer of T in G and D is the group of all automorphisms of the Dynkin diagram, if $G \neq \mathrm{PSL}(2, \mathbb{C})$. Note that if $G = \mathrm{PSL}(2, \mathbb{C})$, then $\overline{T} = \mathbb{CP}^1$ and so in this case $\mathrm{Aut}(\overline{T}) = \mathrm{PSL}(2, \mathbb{C})$.

RÉSUMÉ

Soit $\overline{G}$ la compactification magnifique d'un groupe algébrique affine simple G de type adjoint défini sur $\mathbb{C}$. Soit $\overline{T} \subset \overline{G}$ la clôture d'un tore maximal $T \subset G$. Si $G \neq \mathrm{PSL}(2, \mathbb{C})$, nous montrons que le groupe de tous les automorphismes de la variété $\overline{T}$ est le produit semi-direct $N_G(T) \rtimes D$, où $N_G(T)$ est le normalisateur de T dans G et D est le groupe de tous les automorphismes du diagramme de Dynkin. Remarquez que si $G = \mathrm{PSL}(2, \mathbb{C})$, alors $\overline{T} = \mathbb{CP}^1$ et donc dans ce cas $\mathrm{Aut}(\overline{T}) = \mathrm{PSL}(2, \mathbb{C})$.

1. Introduction

Let G be a simple affine algebraic group of adjoint type defined over the field of complex numbers. De Concini and Procesi constructed a very important compactification of G [5, p. 14, 3.1, THEOREM]; it is known as the wonderful compactification. The wonderful compactification of G will be denoted by $\overline{G}$. Fix a maximal torus T of G, and denote by $\overline{T}$ the closure of the variety T in the wonderful compactification $\overline{G}$ [2, §1]. Let $\mathrm{Aut}(\overline{T})$ denote the group of all holomorphic

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automorphisms of $\overline{T}$. For $G \neq \mathrm{PSL}(2,\mathbb{C})$, the connected component of $\mathrm{Aut}(\overline{T})$ containing the identity element coincides with T acting on $\overline{T}$ by translations [1, Theorem 3.1]. Our aim here is to compute the full automorphism group $\mathrm{Aut}(\overline{T})$.

It may be noted that $\overline{T}$ is stable under the conjugation of the normalizer $N_G(T)$ of T in G. This indicates that $\mathrm{Aut}(\overline{T})$ needs not be connected.

For G different from $\mathrm{PSL}(2,\mathbb{C})$, we prove that $\mathrm{Aut}(\overline{T})$ is the semi-direct product $N_G(T) \rtimes D$, where $N_G(T)$ is the normalizer of T in G, and D is the group of all automorphisms of the Dynkin diagram (see Theorem 3.1).

2. Lie algebra and algebraic groups

We recall the set-up of [1]. Throughout this Note G will denote an affine algebraic group over $\mathbb{C}$ such that G is simple and of adjoint type (equivalently, the center of the simple group is trivial). We will always assume that $G \neq \mathrm{PSL}(2,\mathbb{C})$.

Fix a maximal torus T of G. The group of all characters of T will be denoted by $X(T)$. The Weyl group of G with respect to T is defined to be $W := N_G(T)/T$, where $N_G(T)$ is the normalizer of T in G. Let

$$R \subset X(T) \tag{1}$$

be the root system of G with respect to T. For a Borel subgroup B of G containing the maximal torus T, let $R^+(B)$ denote the set of positive roots determined by T and B. Let

$$S = \{\alpha_1, \cdots, \alpha_n\}$$

be the set of simple roots in $R^+(B)$, where n is the rank of G. Let B^- denote the opposite Borel subgroup of G determined by B and T. So in particular $B \bigcap B^- = T$. For any $\alpha \in R^+(B)$, let $s_\alpha \in W$ be the reflection corresponding to α.

The Lie algebras of G, T and B will be denoted by $\mathfrak{g}$, $\mathfrak{t}$ and $\mathfrak{b}$ respectively. The dual of the real form $\mathfrak{t}_{\mathbb{R}}$ of $\mathfrak{t}$ is $X(T) \otimes \mathbb{R} = \mathrm{Hom}_{\mathbb{R}}(\mathfrak{t}_{\mathbb{R}}, \mathbb{R})$.

Now, let σ be the involution of $G \times G$ defined by $\sigma(x, y) = (y, x)$. We note that the diagonal subgroup $\Delta(G)$ of $G \times G$ is the subgroup of fixed points of σ. The subgroup $T \times T \subset G \times G$ is a σ-stable maximal torus of $G \times G$, while $B \times B^-$ is a Borel subgroup of $G \times G$; this Borel subgroup $B \times B^-$ has the property that $\sigma(\alpha) \in -R^+(B \times B^-)$ for every $\alpha \in R^+(B \times B^-)$.

The group G is identified with the symmetric space $(G \times G)/\Delta(G)$. Let $\overline{G}$ denote the corresponding wonderful compactification of G (see [5, p. 14, 3.1, THEOREM]). In particular $G \times G$ acts on $\overline{G}$. Let $\overline{T}$ be the closure of T in $\overline{G}$. The action of the subgroup $N_G(T) \subset G = \Delta(G)$ on $\overline{G}$ preserves $\overline{T}$.

3. The automorphism group of $\overline{T}$

Let $\mathrm{Aut}(\overline{T})$ denote the group of all holomorphic automorphisms of $\overline{T}$; any holomorphic automorphism is algebraic. Let $\mathrm{Aut}^0(\overline{T}) \subset \mathrm{Aut}(\overline{T})$ be the connected component containing the identity element. The translation action of T on itself produces an isomorphism

$$\rho : T \longrightarrow \mathrm{Aut}^0(\overline{T}) \tag{2}$$

[1, p. 786, Theorem 3.1].

Theorem 3.1. *The automorphism group $\mathrm{Aut}(\overline{T})$ is the semi-direct product $N_G(T) \rtimes D$, where $N_G(T)$ is the normalizer of T in G, and D is the group of all automorphisms of the Dynkin diagram of G.*

Proof. For notational convenience denote

$$A = \mathrm{Aut}(\overline{T}).$$

Note that $\overline{T}$ is stable under the conjugation action of $N_G(T)$ on $\overline{G}$. Let

$$\widetilde{\Delta} \subset \mathfrak{t}_{\mathbb{R}} \tag{3}$$

be the fan of the toric variety $\overline{T}$. This $\widetilde{\Delta}$ consists of cones associated with the Weyl chambers (see [3, p. 187, 6.1.6, Lemma]). Note that any automorphism σ of the Dynkin Diagram associated with the set $S \subset R$ of simple roots with respect to (T, B) preserves the fan $\widetilde{\Delta}$. Therefore, we have [4, p. 47]

$$N_G(T) \rtimes D \subset A.$$

Next we will show that $N_G(T) \rtimes D = A$.

Since ρ in (2) is an isomorphism, it follows immediately that T is a normal subgroup of A. Therefore, the intersection $T \bigcap g(T)$ is a T stable open dense subset of $\overline{T}$ for every element $g \in A$. Consequently, the open subset $T \subset \overline{T}$ is preserved by the natural action of A on $\overline{T}$. Consequently, every automorphism $g \in A$ can be expressed as

$$g = l_{t_0} h, \tag{4}$$

where l_{t_0} is the left translation by some $t_0 \in T$, and $h \in A$ satisfies the condition that $h(1) = 1$, with 1 being the identity element of T.

By a result of Rosenlicht, the action of the h (in (4)) on T is by group automorphism (see [7, p. 986, Theorem 3]). Therefore, h gives an automorphism of $X(T)$, and hence h gives an automorphism of $\mathfrak{t}_{\mathbb{R}}$. Since T is left invariant under the action of h, the toric variety data of $\overline{T}$ is preserved by h. Hence we see that the automorphism of $\mathfrak{t}_{\mathbb{R}}$ given by h preserves the fan $\widetilde{\Delta}$ in (3). Since $\widetilde{\Delta}$ is given by the Weyl chambers and its faces, we see that the induced action of h on $X(T)$ leaves the root system R of G in (1) invariant. Consequently, h produces an automorphism of the root system R.

On the other hand, the automorphism group $\mathrm{Aut}(R)$ of the root system R is precisely

$$N_G(T)/T \rtimes D = W \rtimes D$$

(see [6, p. 231, (A.8)]). □

Corollary 3.2. *The quotient group* $\mathrm{Aut}(\overline{T})/\mathrm{Aut}^0(\overline{T})$ *is isomorphic to* $\mathrm{Aut}(R) = W \rtimes D$.

Remark 3.3. The automorphism group D is trivial except for the types A_ℓ with $\ell \geq 2$, D_ℓ and E_6 (see [6, p. 231, (A.8)]).

Remark 3.4. We note that the structure of the automorphism group of a complete simplicial toric variety is described by D.A. Cox (see [4, p. 48, Corollary 4.7]).

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