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r-Bell polynomials in combinatorial Hopf algebras

Polynomes de r-Bell dans les algèbres de Hopf combinatoires

Ali Chouria, Jean-Gabriel Luque

Laboratoire d'informatique, de traitement de l'information et des systèmes, Normandie Université, UNIROUEN, UNIHAVRE, INSA Rouen, LITIS, 76000 Rouen, France

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ABSTRACT

We introduce partial r-Bell polynomials in three combinatorial Hopf algebras. We prove a factorization formula for the generating functions which is a consequence of the Zassenhauss formula.

RÉSUMÉ

Nous définissons des polynômes r-Bell partiels dans trois algèbres de Hopf combinatoires. Nous prouvons une formule de factorisation pour les fonctions génératrices, qui est une conséquence de la formule de Zassenhauss.

1. Introduction

Partial multivariate Bell polynomials have been defined by E.T. Bell [2] in 1934. Their applications in Combinatorics, Analysis, Algebra, Probabilities *etc.* are numerous (see, *e.g.*, [8]). They are usually defined on an infinite set of commuting variables $\{a_1, a_2, \ldots\}$ by the following generating function:

$$\sum_{n\geqslant 0} B_{n,k}(a_1,\ldots,a_p,\ldots)\frac{x^n}{n!}t^k = \exp\left\{\sum_{m\geqslant 1} a_m\frac{x^m}{m!}t\right\}. \tag{1}$$

The partial Bell polynomials are related to several combinatorial sequences. Let $\left\{ n \atop k \right\}$ denotes the Stirling number of second kind, which counts the number of ways to partition a set of n objects into k nonempty subsets, and let $\left[n \atop k \right]$ denote the Stirling number of first kind, which counts the number of permutations according to their number of cycles. We have, $B_{n,k}(1,1,\ldots) = \left\{ n \atop k \right\}$ and $B_{n,k}(0!,1!,2!,\ldots) = \left[n \atop k \right]$.

The connection between the Bell polynomials and the combinatorial Hopf algebras has been investigated by one of the authors in [3].

E-mail addresses: ali.chouria@yahoo.fr (A. Chouria), jean-gabriel.luque@univ-rouen.fr (J.-G. Luque).

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Aiming to generalize these polynomials, *Mihoubi et al.* [9] defined partial r-Bell polynomials by setting

$$B^r_{n+r,k+r}(a_1,a_2,\cdots;b_1,b_2,\cdots)=\sum_{n'+n''=n+r}\sum_{\substack{\lambda'_1+\cdots+\lambda'_r=n'\\ \lambda''_1+\cdots+\lambda''_k=n''}}\alpha^r_{\lambda',\lambda''}a_{\lambda'_1}\cdots a_{\lambda'_r}b_{\lambda''_1}\cdots b_{\lambda''_k}, \tag{2}$$

where the second sum runs over pairs of (integer) partitions (λ',λ''), $\alpha^r_{\lambda',\lambda''}$ is the number of set partitions $\pi=\{\pi'_1,\pi'_2,\cdots,\pi'_r,\pi''_1,\pi''_2,\cdots,\pi''_k\}$ of $\{1,2,\cdots,n\}$ such that $\#\pi'_1=\lambda'_1,\cdots,\#\pi'_r=\lambda'_r$, $\#\pi''_1=\lambda''_1,\cdots,\#\pi''_k=\lambda''_k$ and $1\in\pi'_1, 2\in\pi'_2,\cdots,r\in\pi'_r$, and $\#\pi_i$ denotes the cardinality of π_i. Comparing our notation to those of [9], the roles of the variables a_i and b_i have been switched. The generating function of the r-Bell polynomials is known to be

$$\sum_{n\geqslant k}B^r_{n+r,k+r}(a_1,a_2,\cdots;b_1,b_2,\cdots)\frac{x^n}{n!}\frac{y^r}{r!}t^k=\exp\left(\sum_{j\geqslant 0}a_{j+1}\frac{x^j}{j!}y\right)\exp\left(\sum_{j\geqslant 1}b_j\frac{x^jt}{j!}\right), \tag{3}$$

where $(a_n;n\geqslant 1)$ and $(b_n;n\geqslant 1)$ are two sequences of nonnegative integers.

The aim of our paper is to show that we can define three versions of the r-Bell polynomials in three combinatorial Hopf algebras in the same way. The first algebra is $\mathrm{Sym}^{(2)}$, the algebra of bisymmetric functions (or symmetric functions of level 2). The r-Bell polynomials as defined by Mihoubi belong to this algebra. The second algebra is $\mathbf{NCSF}^{(2)}$, the algebra of noncommutative bisymmetric functions. In this algebra, we define non-commutative analogues of r-Bell polynomials that generalize the Munthe-Kaas polynomials. The third algebra is $\mathbf{WSym}^{(2)}:=\mathbf{CWSym}(2,2,\cdots)$, the algebra of 2-colored word symmetric functions. In this algebra, we define word analogues of r-Bell polynomials. The common feature of the three constructions is that they are based on the same algorithm, which generates 2-colored set partitions without redundance. Our main result is a factorization formula for the generating function which holds in the three algebras and which is a consequence of the Zassenhauss formula.

2. Bi-colorations of partitions, compositions and set partitions

A bicolored partition λ of n is a multiset $\{(\lambda_1,j_1),\ldots,(\lambda_k,j_k)\}$ such that $\lambda_1+\cdots+\lambda_k=n$ and $j_1,\ldots,j_k\in\{1,2\}$. We set $\lambda\vdash n$, $\omega(\lambda)=n$ and $\ell(\lambda)=k$. A bicolored composition I of n is a list $I=[(i_1,j_1),\ldots,(i_k,j_k)]$ with $i_1+\cdots+i_k=n$ and $j_1,\ldots,j_k\in\{1,2\}$. We set $I\vDash n$, $\omega(I)=n$ and $\ell(I)=k$. A bicolored set partition is a set $\pi=\{(\pi_1,j_1),\ldots,(\pi_k,j_k)\}$ such that $\{\pi_1,\ldots,\pi_k\}$ is a set partition of size n and $j_1,\ldots,j_k\in\{1,2\}$. We set $\pi\Vdash n$, $\omega(\pi)=n$ and $\ell(\pi)=k$.

We define

$$S^r_{n+r,k+r}=\left\{\pi=\{(\pi_1,1),\cdots,(\pi_r,1),(\pi_{r+1},2),\cdots,(\pi_{k+r},2)\}:\pi\Vdash(n+r),1\in\pi_1,\cdots,r\in\pi_r\right\}. \tag{4}$$

We have $S^r_{r,r}=\{\{(\{1\},1),(\{2\},1),\cdots,(\{r\},1)\}\}$ and

$$\begin{aligned}S^r_{n+1+r,k+r}=&\left\{\pi\cup\{(n+1,2)\}:\pi\in S^r_{n+r,r+k-1}\right\}\cup\\ &\left\{\pi\setminus\{(\pi_\ell,j_\ell)\}\cup\{(\pi_\ell\cup\{n+1\},j_\ell)\}:\pi=\{(\pi_1,j_1),\cdots,(\pi_{r+k},j_{r+k}),1\le\ell\le r+k\}\in S^{r+k}_{n+r,k}\right\}.\end{aligned} \tag{5}$$

The underlying recursive algorithm generates one and only one times each element of $S^r_{n+1+r,k+r}$.

We consider also two applications: $c(\pi)=[(\#\pi_1,j_1),\ldots,(\#\pi_k,j_k)]$ if $\pi=\{(\pi_1,j_1),\ldots,(\pi_k,j_k)\}$ with $\min\{\pi_1\}<\cdots<\min\{\pi_k\}$ and $\lambda(\pi)=\{(\#\pi_1,j_1),\ldots,(\#\pi_k,j_k)\}$. We define

$$f^r_{n+r,k+r}(I)=\#\{\pi\in S^r_{n+1+r,k+r}:c(\pi)=I\}\text{ and }g^r_{n+r,k+r}(\lambda)=\#\{\pi\in S^r_{n+1+r,k+r}:\lambda(\pi)=\lambda\}.$$

3. Three combinatorial Hopf algebras

3.1. Algebras of symmetric functions of level 2

In this section, we define three combinatorial Hopf algebras indexed by bicolored objects. The model of construction is the algebra $\mathrm{Sym}^{(l)}$, which is the representation ring of a wreath product $(\Gamma\wr\mathfrak{S}_n)_{n\geq 0}$, Γ being a group with l conjugacy classes [6]. Let us recall briefly its definition for $l=2$. The combinatorial Hopf algebra $\mathrm{Sym}^{(2)}$ [6] is naturally realized as symmetric functions in 2 independent sets of variables $\mathrm{Sym}^{(2)}:=\mathrm{Sym}(\mathbb{X}^{(1)};\mathbb{X}^{(2)})$. It is the free commutative algebra generated by two sequences of formal symbols $p_1(\mathbb{X}^{(1)}),p_2(\mathbb{X}^{(1)}),\ldots$ and $p_1(\mathbb{X}^{(2)}),p_2(\mathbb{X}^{(2)}),\ldots$, named power sums, which are primitive for its coproduct. The set of the polynomials $p^\lambda:=p_{\lambda_1}(\mathbb{X}^{(i_1)})\cdots p_{\lambda_k}(\mathbb{X}^{(i_k)})$, where $\lambda=\{(\lambda_1,i_1),\ldots,(\lambda_k,i_{i_k})\}$ is a bicolored partition, is a basis of $\mathrm{Sym}^{(2)}$.

The Hopf algebra **NCSF** of formal noncommutative symmetric functions [5] is the free associative algebra $\mathbb{C}\langle\Psi_1,\Psi_2,\cdots\rangle$ generated by an infinite sequence of primitive formal variables $(\Psi_i)_{i\geqslant 1}$. Its level l is analogous to that described in [11] as the free algebra generated by level-l complete homogeneous functions $S_{\mathbf{n}}$. Here we set $l=2$ and we use another basis. We recall that the level-2 complete homogeneous function $S_{\mathbf{n}}$, for $\mathbf{n}\in\mathbb{N}^2$, is defined as a free quasi-symmetric function of level 2 as $S_{\mathbf{n}} = \sum_{|u_i|=n_i} \mathbf{G}_{1\dots n,u}$, where $\mathbf{G}_{\sigma,u}$ denotes the dual free l-quasi-ribbon labeled by the colored permutation (σ, u) [11]. Notice that $\mathbf{G}_{\sigma,u}$ is realized as a polynomial in $\mathbb{C}\langle\mathbb{A}^{(1)}\cup\mathbb{A}^{(2)}\rangle$, where $\mathbb{A}^{(i)}$ denotes two disjoint copies of the same alphabet $\mathbb{A}$ as $\mathbf{G}_{\sigma,u} = \sum_{\substack{w\in(\mathbb{A}^{(1)}\cup\mathbb{A}^{(1)})^n \\ \mathrm{std}(w)=\sigma, w_i\in\mathbb{A}^{(i)}}} w$, where std is the usual standardization applied after identifying the two alphabets $\mathbb{A}^{(1)}$ and $\mathbb{A}^{(2)}$. Alternatively, for dimensional reasons, $\mathbf{NCSF}^{(2)}$ is the minimal sub (free) algebra of $\mathbb{C}\langle\mathbb{A}^{(1)}\cup\mathbb{A}^{(2)}\rangle$ containing both $\mathbf{NCSF}(\mathbb{A}^{(1)})$ and $\mathbf{NCSF}(\mathbb{A}^{(2)})$ as subalgebras. Hence, it is freely generated by the (primitive) power sums $\Psi_i(\mathbb{A}^{(j)})$. If $I = [(i_1, j_1), \dots, (i_k, j_k)]$ denotes a bi-colored composition, then the set of the polynomials $\Psi^I = \Psi_{i_1}(\mathbb{A}^{(j_1)})\cdots\Psi_{i_k}(\mathbb{A}^{(j_k)})$ is a basis of the space $\mathbf{NCSF}^{(2)}$.

The last algebra, $\mathbf{WSym}^{(2)}$, is a level 2 analogue of the algebra of word symmetric functions introduced by *M.C. Wolf* [12] in 1936. We construct it as a special case of the Hopf algebras $\mathbf{CWSym}(a)$ of colored set partitions introduced in [1] for $a = (2, 2, \dots, 2, \dots)$. As a space $\mathbf{CWSym}(a)$ is generated by the set Φ^π where π denotes a bicolored set partition. Its product is defined by

$$\Phi^\pi\Phi^{\pi'} = \Phi^{\pi\hat{\cup}\pi'}, \tag{6}$$

where $\hat{\cup}$ denotes the shifted union obtained by shifting first the elements of π' by the weight of π and hence compute the union, and its coproduct is

$$\Delta(\Phi^\pi) = \sum_{\substack{\hat{\pi}_1\cup\hat{\pi}_2=\pi \\ \hat{\pi}_1\cap\hat{\pi}_2=\emptyset}} \Phi^{\mathrm{std}(\hat{\pi}_1)} \otimes \Phi^{\mathrm{std}(\hat{\pi}_2)}, \tag{7}$$

where the *standardized* $\mathrm{std}(\pi)$ of π is defined as the unique colored set partition obtained by replacing the ith smallest integer in the π_j by i.

The algebra $\mathrm{Sym}^{(2)}$ (resp. $\mathbf{NCSF}^{(2)}$, $\mathbf{WSym}^{(2)}$) is naturally bigradued $\mathrm{Sym}^{(2)} = \bigoplus_{n,k}\mathrm{Sym}^{(2)}_{n,k}$ (resp. $\mathbf{NCSF}^{(2)} = \bigoplus_{n,k}\mathbf{NCSF}^{(2)}_{n,k}$, $\mathbf{WSym}^{(2)} = \bigoplus_{n,k}\mathbf{WSym}^{(2)}_{n,k}$) where $\mathrm{Sym}^{(2)}_{n,k} = \mathrm{span}\{p^\lambda : \ell(\lambda) = k, \omega(\lambda) = n\}$ (resp. $\mathbf{NCSF}^{(2)}_{n,k} = \mathrm{span}\{\Psi^I : \ell(I) = k, \omega(I) = n\}$, $\mathbf{WSym}^{(2)}_{n,k} = \mathrm{span}\{\Phi^\pi : \ell(\pi) = k, \omega(\pi) = n\}$). We denote by $\mathbb{R}$ the subalgebra of $\mathrm{Sym}^{(2)}$ (resp. $\mathbf{NCSF}^{(2)}$, $\mathbf{WSym}^{(2)}$) spanned by the polynomials $p^{\{(\lambda_1,2),\dots,(\lambda_k,2)\}}$ (resp. $\Psi^{[(i_1,2),\dots,(i_k,2)]}$, $\Phi^{\{(\pi_1,2),\dots,(\pi_k,2)\}}$), which is isomorphic to Sym (resp. $\mathbf{NCSF}$, $\mathbf{WSym}$). Notice also that $\mathbb{R} = \bigoplus_{n,k}\mathbb{R}_{n,k}$ is naturally bigraded.

In the rest of the paper, when there is no ambiguity, we use a_i to refer to $p_i(\mathbb{X}^{(1)})$, $\Psi_i(\mathbb{A}^{(1)})$ or $\Phi^{\{(\{1,\dots,n\},1)\}}$ and b_i to refer to $p_i(\mathbb{X}^{(2)})$, $\Psi_i(\mathbb{A}^{(2)})$ or $\Phi^{\{(\{1,\dots,n\},2)\}}$. Notice that with this notation all the a_i and the b_i are primitive elements. We define the natural linear maps $\Xi : \mathbf{WSym}^{(2)} \to \mathbf{NCSF}^{(2)}$ and $\xi : \mathbf{WSym}^{(2)} \to \mathrm{Sym}^{(2)}$ by $\Xi(\Phi^\pi) = \Psi^{c(\pi)}$ and $\xi(\Phi^\pi) = p^{\lambda(\pi)}$. Notice that these maps are morphisms of Hopf algebras.

3.2. *r*-Bell polynomials and (commutative/noncommutative/word) symmetric functions

In $\mathrm{Sym}^{(2)}$ and $\mathbf{NCSF}^{(2)}$, we define the operator ∂ as the unique derivation acting on the right and satisfying $a_i\partial = a_{i+1}$ and $b_i\partial = b_{i+1}$. In $\mathbf{WSym}^{(2)}$, we define ∂ as the operator acting linearly on the right by $1\partial = 0$ and

$$\Phi^{\{[\pi_1,i_1],\dots,[\pi_k,i_k]\}}\partial = \sum_{j=1}^{k} \Phi^{(\{[\pi_1,i_1],\dots,[\pi_k,i_k]\}\setminus[\pi_j,i_j])\cup\{[\pi_j\cup\{n+1\},i_j]\}}. \tag{8}$$

In the three algebras, we define r-Bell polynomials in a similar way to Ebrahimi-Fard et al., who defined Munthe-Kaas polynomials, that is by the use of the operator ∂. More precisely, the polynomial $B^r_{n+r,k+r}$ is the coefficient of t^k in $a_1^r(tb_1 + \partial)^n$. In $\mathbf{WSym}^{(2)}$, from (5), we have

$$B^r_{n+r,k+r} = \sum_{\pi\in S^r_{n+r,k+r}} \Phi^\pi. \tag{9}$$

Hence, using the maps Ξ and ξ, we obtain

$$B^r_{n+r,k+r} = \sum_{\pi\in S^r_{n+r,k+r}} p^{\lambda(\pi)} = \sum_\lambda g^r_{n+r,k+r}(\lambda)p^\lambda \tag{10}$$

in $\mathrm{Sym}^{(2)}$ and

$$B^r_{n+r,k+r} = \sum_{\pi\in S^r_{n+r,k+r}} \Psi^{\lambda(\pi)} = \sum_I f^r_{n+r,k+r}(I)\Psi^I \tag{11}$$

in $\mathbf{NCSF}^{(2)}$. Notice that in $\mathrm{Sym}^{(2)}$, $B^r_{n+r,k+r}$ is nothing but the classical r-Bell polynomial and in $\mathbf{NCSF}^{(2)}$, it is a r-version of the Munthe-Kaas polynomial [4,10].

Example 1. In $\mathbf{WSym}^{(2)}$, we have

$$B^2_{4,3} = \Phi^{\{(\{1,3\},1),(\{2\},1),(\{4\},2)\}} + \Phi^{\{(\{1,4\},1),(\{2\},1),(\{3\},2)\}} + \Phi^{\{(\{1\},1),(\{2,3\},1),(\{4\},2)\}} + \Phi^{\{(\{1\},1),(\{2,4\},1),(\{3\},2)\}} + \Phi^{\{(\{1\},1),(\{2\},1),(\{3,4\},2)\}}.$$

In $\mathbf{NCSF}^{(2)}$, we have

$$B^2_{4,3} = 2\Psi^{[(2,1),(1,1),(1,2)]} + 2\Psi^{[(1,1),(2,1),(1,2)]} + \Psi^{[(1,1),(1,1),(2,2)]} = 2a_2a_1b_2 + 2a_1a_2b_1 + a_1a_1b_2.$$

In $\mathrm{Sym}^{(2)}$, $B^2_{4,3} = 4p^{\{(2,1),(1,1),(1,2)\}} + p^{\{(1,1),(1,1),(2,2)\}} = 4a_2a_1b_2 + a_1a_1b_2$.

We consider also the polynomials $\tilde{B}^r_{n+k+r,k+r} = a_1^r b_1^k \partial^n$. Notice that in $\mathbf{WSym}^{(2)}$, we have

$$\tilde{B}^r_{n+k+r,k+r} = \sum_{\substack{\{(\pi_1,1),\dots,(\pi_{k+r},1)\}\in S^{k+r}_{n+k+r,k+r}\\ 1\in\pi_1,\dots,r\in\pi_r}} \Phi^{\{(\pi_1,1),\dots,(\pi_r,1),(\pi_{r+1},2),\dots,(\pi_{r+k},2)\}}. \tag{12}$$

4. Generating functions

We consider the following generating functions:

$$S(t,x,y) = \sum_{n,r,k} B^r_{n+r,k+r} \frac{x^n}{n!}\frac{y^r}{r!} t^k = \exp(a_1 y)\exp(x(tb_1+\partial)), \tag{13}$$

$$S^\circ(t,x) = S(t,x,0) = \sum_{n,k} B_{n,k}\frac{x^n}{n!}t^k = 1.\exp(x(tb_1+\partial)), \tag{14}$$

$$S^\bullet(t,x,y) = \sum_{n,r,k} \tilde{B}^r_{n+k+r,k+r}\frac{x^n}{n!}\frac{y^r}{r!}\frac{t^k}{k!} = \exp(a_1 y)\exp(tb_1)\exp(x\partial), \tag{15}$$

and

$$S^*(x,y) = \sum_{n,r} B^r_{n+r,r}\frac{x^n}{n!}\frac{y^r}{r!} = \exp(yb_1)\exp(x\partial). \tag{16}$$

Theorem 4.1. *The generating functions $S(t,x,y)$ and $S^\circ(t,x)$ satisfy the following factorization*

$$S(t,x,y) = S^\bullet(xt,x,y)Z(x,t) \text{ and } S^\circ(t,x) = S^*(x,xt)Z(x,t), \tag{17}$$

where $Z(x,t) = \prod_{n\geqslant 2}\exp(x^n \sum_k t^k C_{n,k})$, $C_{n,k} = \frac{(-1)^{n+1}}{n}\frac{1}{k!(n-k-1)!}ad_\partial^{n-k-1}ad^k_{b_1}\partial$, and ad_x is the derivation $ad_x P = [x,P] = xP - Px$. In $\mathrm{Sym}^{(2)}$ and $\mathbf{NCSF}^{(2)}$ the operator $C_{n,k}$ is the multiplication by a primitive polynomial belonging to the subalgebra $\mathbb{R}_{n,k}$.

Proof. Equalities (17) are obtained from (13) and (14) by using *Zassenhaus* formula [7]. In $\mathrm{Sym}^{(2)}$ and $\mathbf{NCSF}^{(2)}$, since ∂ is a derivation, $ad^i_\partial ad^j_{b_1}\partial$ is primitive. Furthermore, remarking that $[b_i,\partial] = b_{i+1}$, we prove that $ad^i_\partial ad^j_{b_1}\partial \in \mathbb{R}_{n,k}$. □

Example 2. In $\mathbf{NCSF}^{(2)}$, consider the coefficient of $\frac{x^3}{3!}\frac{y^2}{2!}t$ in the left equality of (17). In the left-hand side, we find $B^2_{5,3} = 3a_2a_1b_1^2 + 3a_1a_2b_1^2 + 2a_1^2b_2b_1 + a_1^2b_1b_2$. The same coefficient in the right-hand sides is $3\tilde{B}^2_{5,4} - 3\tilde{B}^2_{3,3}C_{2,1} + 3!\tilde{B}^2_{2,2}C_{3,2}$. Since $\tilde{B}^2_{5,4} = a_2a_1b_1^2 + a_1a_2b_1^2 + a_1^2b_2b_1 + a_1^2b_1b_2$, $\tilde{B}^2_{3,3} = a_1^2b_1$, $\tilde{B}^2_{2,2} = a_1^2$, $C_{2,1} = -\frac{1}{2}b_2$, and $C_{3,2} = \frac{1}{3!}[b_1,b_2]$, we check that $3\tilde{B}^2_{5,4} - 3\tilde{B}^2_{3,3}C_{2,1} + 3!\tilde{B}^2_{2,2}C_{3,2} = B^2_{5,3}$ as expected by Theorem 4.1.

In $\mathbf{NCSF}^{(2)}$, we compute explicitly the polynomial $C_{n,k}$

$$C_{n,k} = \frac{(-1)^k}{n}\frac{1}{k!(n-k-1)!}\sum_{i_1,\dots,i_k}\binom{n-k-1}{i_1-1,\dots,i_{k-1}-1,i_k-2}[b_{i_1},[b_{i_2},\cdots,[b_{i_{k-1}},b_{i_{k-1}}]\cdots]]. \tag{18}$$

Example 3. Consider for instance the polynomial $C_{5,2}$ in $\mathbf{NCSF}^{(2)}$

$$\begin{aligned}C_{5,2} &= -\frac{1}{48} ad_\partial^4 ad_{b_1}^2 \partial = -\frac{1}{48} ad_\partial^4 [b_1, b_2] \\ &= -\frac{1}{48} [[[[[b_1, b_2], \partial], \partial], \partial], \partial] \\ &= -\frac{1}{48} (2[b_3, b_4] + 3[b_2, b_5] + [b_1, b_6]) \\ &= -\frac{1}{48} ([b_5, b_2] + 4[b_4, b_3] + 6[b_3, b_4] + 4[b_2, b_5] + [b_1, b_6]).\end{aligned}$$

Remark 1. If we set $a_i = b_i$ for each i, then we have $S^\bullet(t, x, y) = S^*(y + t, x)$, and so $S(t, x, y) = S^*(y + xt, x)Z(x, t)$.

In $\mathrm{Sym}^{(2)}$, the series $Z(x, t)$ has a nice closed form

$$Z(x, t) = \exp\left(-\sum_{i \geqslant 2} \frac{(i-1)}{i!} b_i t^i\right). \tag{19}$$

Indeed, since the algebra is commutative $ad_\partial^i ad_{b_1}^j \partial$ is nonzero only if $j = 1$ and when $j = 1$ formula (18) gives $[\partial, b_i] = -b_{i+1}$.

As a consequence, using equality (19) together with Theorem 4.1 and Formula (3), we find

$$S^\bullet(xt, x, y) = \exp\left(\sum_{j \geqslant 0} a_{j+1} \frac{x^j}{j!} y\right) \exp\left(\sum_{j \geqslant 1} j b_j \frac{x^j t}{j!}\right). \tag{20}$$

In other words, equating the coefficients in the left- and the right-hand sides of (20), we find

$$\tilde{B}^r_{n+k+r,k+r} = \binom{n+k}{n}^{-1} B^r_{n+k+r,k+r}(a_1, a_2, \ldots; b_1, 2b_2, 3b_3, \ldots). \tag{21}$$

In the case where $r = 0$, we obtain

$$\tilde{B}^0_{n+k,k}(a_1, a_2, \ldots; b_1, \ldots) = B^k_{n+k,k}(b_1, b_2, \ldots; b_1, b_2, \ldots) = \binom{n+k}{n}^{-1} B_{n+k,k}(b_1, 2b_2, 3b_3, \ldots). \tag{22}$$

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