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The equivariant Riemann–Roch theorem and the graded Todd class

Le théorème de Riemann–Roch équivariant et la classe de Todd graduée

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ABSTRACT

Let G be a torus with Lie algebra $\mathfrak{g}$ and let M be a G-Hamiltonian manifold with Kostant line bundle $\mathcal{L}$ and proper moment map. Let $\Lambda \subset \mathfrak{g}^*$ be the weight lattice of G. We consider a parameter $k \geq 1$ and the multiplicity $m(\lambda, k)$ of the quantized representation $RR_G(M, \mathcal{L}^k)$. Define $\langle \Theta(k), f \rangle = \sum_{\lambda \in \Lambda} m(\lambda, k) f(\lambda/k)$ for f a test function on $\mathfrak{g}^*$. We prove that the distribution $\Theta(k)$ has an asymptotic development $\langle \Theta(k), f \rangle \sim k^{\dim M/2} \sum_{n=0}^{\infty} k^{-n} \langle DH_n, f \rangle$ where the distributions DH_n are the twisted Duistermaat–Heckman distributions associated with the graded equivariant Todd class of M. When M is compact, and f polynomial, the asymptotic series is finite and exact.

RÉSUMÉ

Soit G un tore d'algèbre de Lie $\mathfrak{g}$ agissant de manière hamiltonienne sur une variété M. Soit $\mathcal{L}$ un fibré de Kostant tel que l'application moment associée soit propre. Soit $\Lambda \subset \mathfrak{g}^*$ le réseau des poids de G. On considère un paramètre $k \geq 1$ et la multiplicité $m(\lambda, k)$ de la représentation quantifiée $RR_G(M, \mathcal{L}^k)$. On définit la distribution $\langle \Theta(k), f \rangle = \sum_{\lambda \in \Lambda} m(\lambda, k) f(\lambda/k)$ pour f une fonction test sur $\mathfrak{g}^*$. La distribution $\Theta(k)$ admet un développement asymptotique $\langle \Theta(k), f \rangle \sim k^{\dim M/2} \sum_{n=0}^{\infty} k^{-n} \langle DH_n, f \rangle$ où les distributions DH_n sont des distributions associées aux composantes homogènes de la classe de Todd équivariante de M. Lorsque M est compacte et f polynomiale, cette série est finie et exacte.

1. Introduction

Let G be a torus with Lie algebra $\mathfrak{g}$. Identify $\hat{G}$ to a lattice Λ of $\mathfrak{g}^*$. If $\lambda \in \Lambda$, we denote by g^λ the corresponding character of G. If $g = \exp(X)$ with $X \in \mathfrak{g}$, then $g^\lambda = e^{i\langle \lambda, X \rangle}$.

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Let M be a prequantizable G-Hamiltonian manifold with symplectic form Ω, Kostant line bundle $\mathcal{L}$, and moment map $\Phi : M \to \mathfrak{g}^*$. Assume M compact and of dimension $2d$. The Riemann–Roch quantization $RR_G(M, \mathcal{L})$ is a virtual finite dimensional representation of G, constructed as the index of a Dolbeault–Dirac operator on M. The dimension of the space $RR_G(M, \mathcal{L})$ will be called the Riemann–Roch number of $(M, \mathcal{L})$. The character of the representation of $RR_G(M, \mathcal{L})$ is a function on G, denoted by $RR_G(M, \mathcal{L})(g)$. We write

$$RR_G(M, \mathcal{L})(g) = \sum_{\lambda \in \Lambda} m_{\text{rep}}(\lambda) g^\lambda.$$

The typical example is the case where M is a projective manifold, and $\mathcal{L}$ the corresponding ample bundle. Then

$$RR_G(M, \mathcal{L})(g) = \sum_{i=0}^{d} (-1)^i \mathrm{Tr}_{H^i(M, \mathcal{O}(\mathcal{L}))}(g)$$

is the alternate sum of the traces of the action of g in the cohomology spaces of $\mathcal{L}$. In particular $\dim RR_G(M, \mathcal{L}) = \sum_{i=0}^{d} (-1)^i \dim H^i(M, \mathcal{O}(\mathcal{L}))$ *is given by the Riemann–Roch formula.*

It is natural to introduce the kth power $\mathcal{L}^k$ of the line bundle $\mathcal{L}$. Thus

$$RR_G(M, \mathcal{L}^k)(g) = \sum_{\lambda \in \Lambda} m_{\text{rep}}(\lambda, k) g^\lambda.$$

Assume $k \geq 1$. We associate with $(M, \mathcal{L})$ the distribution on $\mathfrak{g}^*$ given by

$$\langle \Theta_M(k), f \rangle = \sum_{\lambda \in \Lambda} m_{\text{rep}}(\lambda, k) f(\lambda/k)$$

where f is a test function.

Example. When M is a toric manifold associated with the Delzant polytope P, then $\dim RR_G(M, \mathcal{L})$ is the number of integral points in the convex polytope P, and $\frac{1}{k^d} \langle \Theta_M(k), f \rangle$ is the Riemann sum of the values of f on the sample points $\frac{\Lambda}{k} \cap P$.

We prove that $\Theta_M(k)$ has an asymptotic behavior when the integer k tends to ∞ of the form

$$\Theta_M(k) \sim k^d \sum_{n=0}^{\infty} k^{-n} DH_n$$

where DH_n are distributions on $\mathfrak{g}^*$ supported on $\Phi(M)$. We determine the distributions DH_n in terms of the decomposition of the equivariant Todd class $\mathrm{Todd}(M)$ of M in its homogeneous components $\mathrm{Todd}_n(M)$ in the graded equivariant cohomology ring of M. The distribution DH_0 is the Duistermaat–Heckmann measure. The asymptotics are exact when f is a polynomial. This generalizes the weighted Ehrhart polynomial for an integral polytope, and the asymptotic behavior of Riemann sums over convex integral polytopes established by Guillemin–Sternberg [8].

We then consider the case where M is a prequantizable G-Hamiltonian manifold, not necessarily compact, but with proper moment map $\Phi : M \to \mathfrak{g}^*$. The formal quantization of $(M, \mathcal{L}^k)$ [19] is defined by

$$RR_G(M, \mathcal{L}^k)(g) = \sum_{\lambda \in \Lambda} m_{\text{geo}}(\lambda, k) g^\lambda.$$

Here $m_{\text{geo}}(\lambda, k)$ is the geometric multiplicity function constructed by Guillemin–Sternberg in terms of the Riemann–Roch number of the reduced fiber $M_\lambda = \Phi^{-1}(\lambda)/G$ of the moment map. When M is compact, Meinrenken–Sjamaar [10] proved that $m_{\text{rep}}(\lambda, k) = m_{\text{geo}}(\lambda, k)$, so this purely geometric definition extends the definition of $RR_G(M, \mathcal{L}^k)$ given in terms of index theory when M is compact.

Similarly, we construct distributions DH_n on $\mathfrak{g}^*$ using the equivariant cohomology classes $\mathrm{Todd}_n(M)$ and push-forwards by the proper map Φ. The main result of this announcement is that the distribution $\Theta_M(k)$ defined by

$$\langle \Theta_M(k), f \rangle = \sum_{\lambda \in \Lambda} m_{\text{geo}}(\lambda, k) f(\lambda/k),$$

is asymptotic to $k^d \sum_{n=0}^{\infty} k^{-n} DH_n$.

A similar result holds for Dirac operators twisted by powers of a line bundle $\mathcal{L}^k$.

Recall that we introduced a truncated Todd class (of the cotangent bundle T^*M) for determining the multiplicities of the equivariant index of any transversally elliptic operator on M [17]. Here the use of the parameter k allows us to have families of such equivariant indices, and the full series $\sum_{n=0}^{\infty}\mathrm{Todd}_n(M)$ enters in the description of the asymptotic behavior. This is similar to the Euler–Maclaurin formula evaluating sums of the values of a function at integral points of an interval involving all Bernoulli numbers. We finally give some information on the piecewise polynomial behavior of the distributions DH_n.

2. Equivariant cohomology

Let N be a G-manifold and let $\mathcal{A}(N)$ be the space of differential forms on N, graded by its exterior degree. Following [3] and [20], an equivariant form is a G-invariant smooth function $\alpha : \mathfrak{g} \to \mathcal{A}(N)$, thus $\alpha(X)$ is a differential form on N depending differentiably of $X \in \mathfrak{g}$. Consider the operator

$$d_{\mathfrak{g}}\alpha(X) = d\alpha(X) - \iota(v_X)\alpha(X) \tag{2.1}$$

where $\iota(v_X)$ is the contraction by the vector field v_X generated by the action of $-X$ on N. Then $d_{\mathfrak{g}}$ is an odd operator with square 0, and the equivariant cohomology is defined to be the cohomology space of $d_{\mathfrak{g}}$. It is important to note that the dependance of α on X may be C^∞. If the dependance of α in X is polynomial, we denote by $H^*_G(N)$ the corresponding $\mathbb{Z}$-graded algebra. By definition, the grading of $P(X) \otimes \mu$, P a homogeneous polynomial and μ a differential form on N, is the exterior degree of μ plus twice the polynomial degree in X.

The Hamiltonian structure on M determines the equivariant symplectic form $\Omega(X) = \langle \Phi, X\rangle + \Omega$.

Choose a G-invariant Riemannian metric on M. This provides the tangent bundle TM with the structure of a Hermitian vector bundle. Let $J(A) = \det_{\mathbb{C}^d} \frac{\mathrm{e}^A - 1}{A}$, an invariant function of $A \in \mathrm{End}(\mathbb{C}^d)$. Then, $J(0) = 1$. Consider $\frac{1}{J(A)}$ and its Taylor expansion at 0:

$$\frac{1}{J(A)} = \det_{\mathbb{C}^d}\left(\frac{A}{\mathrm{e}^A - 1}\right) = \sum_{n=0}^{\infty} B_n(A).$$

Each function $B_n(A)$ is an invariant polynomial of degree n on $\mathrm{End}(\mathbb{C}^d)$ and by the Chern–Weil construction, B_n determines an equivariant characteristic class $\mathrm{Todd}_n(M)(X)$ on M of homogeneous degree $2n$. Remark that $\mathrm{Todd}_0(M) = 1$. We define the formal series of equivariant cohomology classes:

$$\mathrm{Todd}(M)(X) = \sum_{n=0}^{\infty} \mathrm{Todd}_n(M)(X).$$

For X small enough, the series is convergent, and $\mathrm{Todd}(M)(X)$ is the equivariant Todd class of M. In particular, $\mathrm{Todd}(M)(0)$ is the usual Todd class of M.

In the rest of this note, using the Lebesgue measure $\mathrm{d}\xi$ determined by the lattice Λ, we may identify distributions and generalized functions on $\mathfrak{g}^*$, and we may write $\langle \theta, f\rangle = \int_{\mathfrak{g}^*} \theta(\xi) f(\xi) d\xi$ for the value of a distribution θ on a test function f on $\mathfrak{g}^*$.

3. The compact case

Let M be a compact G-Hamiltonian manifold. Recall (see [2]) the "delocalized Riemann–Roch formula." For $X \in \mathfrak{g}$ sufficiently small, we have

$$RR_G(M, \mathcal{L})(\exp X) = \frac{1}{(2\mathrm{i}\pi)^d} \int_M \mathrm{e}^{\mathrm{i}\Omega(X)} \mathrm{Todd}(M)(X).$$

Here $\mathrm{i} = \sqrt{-1}$.

For each integer n, consider the analytic function on $\mathfrak{g}$ given by

$$\theta_n(X) = \frac{1}{(2\mathrm{i}\pi)^d} \int_M \mathrm{e}^{\mathrm{i}\Omega(X)} \mathrm{Todd}_n(M)(X).$$

There is a remarkable relation between the Riemann–Roch character associated with $\mathcal{L}^k$ and the dilation $X \to X/k$ on $\mathfrak{g}$.

Lemma 3.1. *When $X \in \mathfrak{g}$ is sufficiently small, then for any $k \geq 1$, one has*

$$RR_G(M, \mathcal{L}^k)(\exp(X/k)) = \sum_{n=0}^{\infty} k^{d-n} \theta_n(X).$$

Proof. When $X \in \mathfrak{g}$ is small, then $\sum_{n=0}^{\infty} \mathrm{Todd}_n(M)(X)$ is a convergent series with sum the equivariant Todd class. Thus we obtain

$$RR_G(M, \mathcal{L}^k)(\exp(X/k)) = \sum_{n=0}^{\infty} \frac{1}{(2\mathrm{i}\pi)^d} \int_M \mathrm{e}^{\mathrm{i}k\Omega + \mathrm{i}k\Phi(X/k)} \mathrm{Todd}_n(M)(X/k)$$
$$= \sum_{n,m} \frac{1}{(2\mathrm{i}\pi)^d} \int_M \mathrm{e}^{\mathrm{i}\Phi(X)} \frac{1}{m!} k^m (\mathrm{i}\Omega)^m \mathrm{Todd}_n(M)(X/k).$$

For each m, only the term of differential degree $2d - 2m$ of $\mathrm{Todd}_n(M)$ contributes to the integral, and this term is homogeneous in X of degree $n + m - d$. This implies the result. □

When $n = 0$,

$$\theta_0(X) = \frac{1}{(2\mathrm{i}\pi)^d} \int_M \mathrm{e}^{\mathrm{i}\Omega(X)}$$

is the equivariant volume of M, and the Fourier transform DH_0 of θ_0 is the Duistermaat–Heckmann measure of M, a piecewise polynomial measure on $\mathfrak{g}^*$.

Theorem 3.2. *Let DH_n be the Fourier transform of θ_n. Then DH_n is a distribution supported on $\Phi(M)$. For any polynomial function P of degree N on $\mathfrak{g}^*$, we have*

$$\sum_{\lambda \in \Lambda} m_{\mathrm{rep}}(\lambda) P(\lambda) = \sum_{n \leq N+d} \int_{\mathfrak{g}^*} DH_n(\xi) P(\xi) \mathrm{d}\xi.$$

In particular, we have the following Euler–MacLaurin formula for the Riemann–Roch number of $(M, \mathcal{L})$:

$$\dim RR_G(M, \mathcal{L}) = \sum_{\lambda \in \Lambda} m_{\mathrm{rep}}(\lambda) = \int_{\mathfrak{g}^*} \sum_{n \leq d} DH_n(\xi) \mathrm{d}\xi.$$

We now give a theorem for smooth functions.

Theorem 3.3. *When the integral parameter k tends to ∞, the distribution $\Theta_M(k)$ admits the asymptotic expansion*

$$\Theta_M(k) \sim k^d \sum_{n=0}^{\infty} k^{-n} DH_n.$$

Let us sketch the proof of Theorems 3.2 and 3.3. It is easy to see that the distributions DH_n are supported on the image $\Phi(M)$ of M by the moment map. Furthermore, it follows from the piecewise quasi-polynomial behavior of the function $m_{\mathrm{rep}}(\lambda, k)$ that for P a homogeneous polynomial of degree N, the sum $\sum_{\lambda \in \Lambda} m_{\mathrm{rep}}(\lambda, k) P(\lambda)$ is a quasi-polynomial function of $k \geq 1$ of degree less than or equal to $N + d$. Thus Theorem 3.2 will be a consequence of Theorem 3.3, which we now prove.

The Fourier transform of $\Theta_M(k)$ is

$$\sum_{\lambda \in \Lambda} m_{\mathrm{rep}}(\lambda, k) \mathrm{e}^{\mathrm{i}\langle \lambda, X/k \rangle} = RR_G(M, \mathcal{L}^k)(\exp(X/k)).$$

Against a test function ϕ of X, this is

$$\frac{1}{(2\mathrm{i}\pi)^d} \int_{\mathfrak{g}} RR_G(M, \mathcal{L}^k)(\exp(X/k)) \phi(X) dX.$$

For k large, and X in the support of ϕ, X/k is small, and we use Lemma 3.1.

Let us give an example of the asymptotic expansion.

Let $P_1(\mathbb{C})$ equipped with the torus action $g([z_1, z_2]) = [gz_1, z_2]$ of $g = \mathrm{e}^{\mathrm{i}\theta}$, in homogeneous coordinates. We consider $M = P_1(\mathbb{C}) \times P_1(\mathbb{C})$ with diagonal action, and let $\mathcal{L}$ be its Kostant line bundle. Then we have

$$RR(M, \mathcal{L}^k)(g) = \sum_{j \in \mathbb{Z}} m_{\mathrm{rep}}(j, k) g^j$$

with

$$m_{\text{rep}}(j,k)=\begin{cases}0 & \text{if } j<-2k,\\ 2k+1+j & \text{if } -2k\leq j\leq 0,\\ 2k+1-j & \text{if } 0\leq j\leq 2k,\\ 0 & \text{if } j>2k.\end{cases}$$

We have

$$\Theta(k)\sim k^2\left(DH_0+\frac{1}{k}DH_1+\frac{1}{k^2}DH_2+\frac{1}{k^3}DH_3+\cdots\right).$$

Let us give the explicit formulae for DH_0, DH_1, DH_2, DH_3.

$$\langle DH_0, f\rangle=\int_{-2}^{2} m(\xi)f(\xi)\mathrm{d}\xi$$

with

$$m(\xi)=\begin{cases}2+\xi & \text{if } -2\leq\xi\leq 0,\\ 2-\xi & \text{if } 0\leq\xi\leq 2,\end{cases}$$

$$\langle DH_1, f\rangle=\int_{-2}^{2} f(\xi)\mathrm{d}\xi,$$

$$\langle DH_2, f\rangle=\frac{5}{12}f(-2)+\frac{1}{6}f(0)+\frac{5}{12}f(2),$$

$$\langle DH_3, f\rangle=-\frac{1}{12}f'(-2)+\frac{1}{12}f'(2).$$

We now sketch another proof of Theorem 3.3, which can be extended to the non-compact case. We use Paradan's decomposition ([11,12], see also [18]) of $RR_G(M,\mathcal{L})$ in a sum of simpler characters supported on cones. Let us consider a generic value r of the moment map, and choose a scalar product on $\mathfrak{g}^*$. Then there exists a certain finite subset $\mathcal{B}(r)$ of $\mathfrak{g}^*$, and for each $\beta\in\mathcal{B}(r)$, a cone $C(\beta)$ in $\mathfrak{g}^*$ and an (infinite dimensional) representation $P_{\beta,k}$ such that

$$RR_G(M,\mathcal{L}^k)=\sum_{\beta\in\mathcal{B}(r)} P_{\beta,k}.$$

Here $P_{\beta,k}(g)=\sum_{\lambda\in\Lambda\cap kC(\beta)} m_{\text{rep},\beta}(\lambda,k)g^{\lambda}$. Thus $\Theta(k)$ is decomposed in $\sum_{\beta\in\mathcal{B}(r)}\Theta_\beta(k)$. Similarly, each distribution DH_n is decomposed as $DH_n=\sum_{\beta\in\mathcal{B}(r)} DH_{n,\beta}$ and the support of $DH_{n,\beta}$ is contained in the cone C_β. It is easily verified that, for each β, the distribution $\Theta_\beta(k)$ is asymptotic to $k^d\sum_{n=0}^{\infty}k^{-n}DH_{n,\beta}$. Here we use the explicit Euler–Maclaurin expansion on half lines, and convolutions of such distributions. The proof is entirely similar to that in the case of a polytope given in [4].

Let us return to the example of the case of $M=P_1(\mathbb{C})\times P_1(\mathbb{C})$, for $r<0$ a small negative number. Then $\mathcal{B}(r)=\{-2,r,0,2\}$. We have

$$P_{\beta=-2,k}(g)=-\sum_{j<-2k}(2k+1+j)g^j,$$

$$P_{\beta=r,k}(g)=\sum_{j=-\infty}^{j=\infty}(2k+1+j)g^j,$$

$$P_{\beta=0,k}(g)=-2\sum_{j>0} jg^j,$$

$$P_{\beta=2,k}(g)=\sum_{j>2k}(j-(2k+1))g^j.$$

Consider, for example, the asymptotic development of the distribution

$$\langle\Theta_{\beta=2}(k), f\rangle=\sum_{j>2k}(j-(2k+1))f(j/k).$$

It is easy to see that this distribution is the convolution $K(k) * K(k)$ where $K(k)$ is the distribution defined by $\langle K(k), f\rangle = \sum_{j>k} f(j/k) = \sum_{j\geq k} f(j/k) - f(1)$. We then use the explicit exact Euler–Maclaurin formula to evaluate the distribution $K(k)$, thus its convolution. In particular, the Fourier transform of $K(k) * K(k)$ coincides with the analytic function $\frac{e^{2ix}}{(1-e^{-ix/k})^2}$ for $(1-e^{-ix/k}) \neq 0$. As is natural, the asymptotic series of distributions $q^{-d}\sum_{n=0}^{\infty} q^n DH_{\beta=2,n}$ is the unique series of distributions supported on $\xi \geq 2$ and with Fourier transform, for $x \neq 0$, the Laurent series in q of $\frac{e^{2ix}}{(1-e^{-iqx})^2}$ at $q = 0$.

4. Proper moment maps

Consider the case where M is non-necessarily compact, but $\Phi : M \to \mathfrak{g}^*$ is a proper map. One can then define [19,13] the formal geometric quantification of M with respect to the line bundle $\mathcal{L}^k$ to be

$$RR_{G,\mathrm{geo}}(M, \mathcal{L}^k)(g) = \sum_{\lambda\in\Lambda} m_{\mathrm{geo}}(\lambda, k) g^{\lambda},$$

using a function $m_{\mathrm{geo}}(\xi)$ on $\mathfrak{g}^*$. The definition of the function $m_{\mathrm{geo}}(\xi)$ is due to Guillemin–Sternberg [7]. Let us recall its delicate definition ([10], see also [16]). There is a closed set $\mathcal{A}$, union of affine hyperplanes, such that if r is in the complement of $\mathcal{A}$, then either r is not in $\Phi(M)$ or r is a regular value of Φ. Consider the open subset $\mathfrak{g}^*_{\mathrm{reg}} = \mathfrak{g}^* \setminus \mathcal{A}$. When $\xi \in \mathfrak{g}^*_{\mathrm{reg}}$ but not in $\Phi(M)$, $m_{\mathrm{geo}}(\xi)$ is defined to be 0. If $\xi \in \mathfrak{g}^*_{\mathrm{reg}} \cap \Phi(M)$, the reduced fiber $M_\xi = \Phi^{-1}(\xi)/G$ is a compact symplectic orbifold, and $m_{\mathrm{geo}}(\xi)$ is defined to be a sum of integrals on the various strata of the compact orbifold M_ξ. When $\lambda \in \mathfrak{g}^*_{\mathrm{reg}} \cap \Lambda \cap \Phi(M)$, then M_λ is a prequantizable compact symplectic orbifold and $m_{\mathrm{geo}}(\lambda)$ is the Riemann–Roch number of M_λ equipped with its Kostant orbifold line bundle. Let $\lambda \in \Lambda$ be any point in $\Phi(M)$. Choose a vector ϵ such that $\lambda + t\epsilon$ is in $\Phi(M) \cap \mathfrak{g}^*_{\mathrm{reg}}$ for any $t > 0$ and sufficiently small. It can be proved, using the wall crossing formulae of Paradan [15], that $(\lim_\epsilon m_{\mathrm{geo}})(\lambda) = \lim_{t>0,t\to 0} m_{\mathrm{geo}}(\lambda + t\epsilon)$ is independent of the choice of such an ϵ. This allows us to define $m_{\mathrm{geo}}(\lambda)$ by “continuity on $\Phi(M)$” for any $\lambda \in \Lambda$.

The $[Q, R] = 0$ theorem [10,9,14] asserts that $RR_{G,\mathrm{geo}}(M, \mathcal{L})$ coincides with a representation of G defined using index theory. In particular, $RR_{G,\mathrm{geo}}(M, \mathcal{L})$ coincides with $RR_G(M, \mathcal{L})$ when M is compact. However, in the rest of this note, we only use the geometric definition of $RR_{G,\mathrm{geo}}(M, \mathcal{L})$.

Replacing $\mathcal{L}$ by $\mathcal{L}^k$, and the moment map Φ by $k\Phi$, define the distribution, with parameter k,

$$\langle \Theta_M(k), f\rangle = \sum_{\lambda\in\Lambda} m_{\mathrm{geo}}(\lambda, k) f(\lambda/k).$$

As in the compact case, the asymptotic behavior of $\Theta_M(k)$ is determined by the graded Todd class, using push-forwards by the proper map Φ. Indeed if α is an equivariant cohomology class with polynomial coefficients, then the Duistermaat–Heckman twisted distribution $DH(M, \Phi, \alpha)$ is well defined by the formula

$$\langle DH(M, \Phi, \alpha), f\rangle = \frac{1}{(2i\pi)^d} \int_{M\times\mathfrak{g}} e^{i\Omega(X)} \alpha(X) \hat{f}(X) dX$$

where $\hat{f}(X) = \int_{\mathfrak{g}^*} e^{i\langle\xi,X\rangle} f(\xi) d\xi$ is the Fourier transform of the test function $f(\xi)$ (see [6]). It is a distribution supported on $\Phi(M)$.

Definition 4.1. We define DH_n to be the distribution on $\mathfrak{g}^*$ associated with the equivariant cohomology class $\mathrm{Todd}_n(M)$:

$$\langle DH_n, f\rangle = \frac{1}{(2i\pi)^d} \int_{M\times\mathfrak{g}} e^{i\Omega(X)} \mathrm{Todd}_n(M)(X) \hat{f}(X) dX.$$

The distribution DH_0 is the Duistermaat–Heckman measure, a locally polynomial function.

The distribution DH_n is given by a polynomial function on each connected component of the open set $\mathfrak{g}^*_{\mathrm{reg}}$. Its restriction to $\mathfrak{g}^*_{\mathrm{reg}}$ vanishes when $n > d - \dim G$. Furthermore, if all stabilizers of points of M are connected, it follows from Witten non-Abelian localization theorem that

$$m_{\mathrm{geo}}(\lambda, k) = k^d \sum_{n=0}^{\infty} k^{-n} DH_n(\lambda/k)$$

when λ/k is a regular value of Φ. Otherwise, it can be defined by the limit of the function $m_{\mathrm{geo}}(\xi, k) = k^d \sum_{n=0}^{\infty} k^{-n} DH_n(\xi/k)$ along $\xi = \lambda + t\epsilon_\lambda$ and $t > 0, t \to 0$, where the direction ϵ_λ is chosen to be arbitrary if λ does not belong to $k\Phi(M)$, or in such a way that $\lambda + t\epsilon_\lambda$ stays in $k\Phi(M)$ if $\lambda \in k\Phi(M)$. Similar formulae can be given without assumption on connected stabilizers.

We can see that, for any n, the distributions DH_n can be expressed (but not uniquely) as derivatives of locally polynomial functions associated with symplectic submanifolds M^T where T are subtori of G.

The main result of this note is the following theorem.

Theorem 4.2. *When the integer k tends to ∞,*

$$\Theta_M(k) \sim k^d \sum_{n=0}^{\infty} k^{-n} DH_n.$$

Let us sketch the proof of this theorem, in the case where each stabilizer is connected. We use Paradan's decomposition formula [12,11]. We choose r a generic element of $\mathfrak{g}^*_{\mathrm{reg}}$. As in the compact case, there is a locally finite set $\mathcal{B}(r) \subset \Phi(M)$, cones C_β, and decompositions

$$DH_n = \sum_{\beta \in \mathcal{B}(r)} DH_{n,\beta}$$

where $DH_{n,\beta}$ are supported on C_β. The functions $DH_{n,\beta}$ are given by polynomial functions on each connected component of $\mathfrak{g}^*_{\mathrm{reg}}$ and vanishes on $\mathfrak{g}^*_{\mathrm{reg}}$ when $n > d - \dim G$. Thus the locally polynomial function $A_\beta(\xi, k) = k^d \sum_{n=0}^{\infty} k^{-n} DH_{n,\beta}(\xi/k)$ is well defined when $\xi/k \in \mathfrak{g}^*_{\mathrm{reg}}$. For each $\beta \in \mathcal{B}(r)$, choose a direction ϵ_β such that $\beta + t\epsilon_\beta$ is in $\Phi(M) \cap \mathfrak{g}^*_{\mathrm{reg}}$ for $t > 0$ small. Then $w_\beta(\lambda, k) = \lim_{t>0, t\to 0} A_\beta(\lambda + t\epsilon_\beta, k)$ is well defined. Define

$$P_{\beta,k}(g) = \sum_{\lambda \in \Lambda} w_\beta(\lambda, k) g^\lambda$$

and

$$\langle \Theta_{\beta,\mathrm{geo}}(k), f \rangle = \sum_{\lambda \in \Lambda} w_\beta(\lambda, k) f(\lambda/k).$$

As before, it is easy to see that $\Theta_{\beta,\mathrm{geo}}(k) \sim k^d \sum_{n=0}^{\infty} k^{-n} DH_{n,\beta}$. Here we use the following "continuity" result on partition function (see for example [5]). Let Δ be a unimodular list of non-zero vectors in Λ, and $\gamma \in \mathfrak{g}$ generic. There is a unique function K (the Kostant partition function) on Λ supported on the half space $\langle \xi, \gamma \rangle \geq 0$ and such that $\sum_{\lambda \in \Lambda} K(\lambda) g^\lambda = \prod_{\alpha \in \Delta} \frac{1}{1 - g^\alpha}$ for g in the open set $\prod_{\alpha \in \Delta}(1 - g^\alpha) \neq 0$. Let $d = |\Delta|$. Consider the Laurent series expansion in q

$$\prod_{\alpha \in \Delta} \frac{1}{1 - \mathrm{e}^{q\langle \alpha, X \rangle}} = q^{-d} \sum_{n=0}^{\infty} q^n U_n(X)$$

and the distributions D_n on $\mathfrak{g}^*$ supported on the half space $\langle \xi, \gamma \rangle \geq 0$, such that

$$\int_{\mathfrak{g}^*} D_n(\xi) \mathrm{e}^{\mathrm{i}\langle \xi, X \rangle} = U_n(X)$$

when $\prod_\alpha \langle \alpha, X \rangle \neq 0$. Define $T(\xi) = \sum_{n=0}^{\infty} D_n(\xi)$, which is well defined outside a system of hyperplanes. Then for any $\lambda \in \Lambda$, and ϵ_Δ generic and belonging to the cone $\mathrm{Cone}(\Delta)$ generated by Δ, we have $K(\lambda) = \lim_{t>0, t\to 0} T(\lambda + t\epsilon_\Delta)$.

Define $P_{r,k} = \sum_{\beta \in \mathcal{B}(r)} P_{\beta,k}$. It remains to see that $P_{r,k} = RR_{G,\mathrm{geo}}(M, \mathcal{L}^k)$. This is not immediate, since we do not have a global representation theoretic object for describing $RR_{G,\mathrm{geo}}(M, \mathcal{L}^k)$. Each coefficient $m_{\mathrm{geo}}(\lambda, k)$ is defined using a limit direction depending on λ, while each $w_\beta(\lambda, k)$ is defined using the same limit direction (depending on β) for any λ. So additivity is not clear. However, we can prove that $P_{r,k}$ is independent of r, using [15]. This is very similar to the technique used in [1] to establish decompositions *à la Paradan* of characteristic functions of polyhedra. It then follows that $P_{r,k} = RR_{G,\mathrm{geo}}(M, \mathcal{L}^k)$. Indeed for each connected component $\mathfrak{c}$ of $\mathfrak{g}^*_{\mathrm{reg}}$ contained in $\Phi(M)$, we choose r in $\mathfrak{c}$. In the decomposition $P_{r,k} = \sum_{\beta \in \mathcal{B}(r)} P_{\beta,k}$, the term $w_\beta(\lambda, k)$ for $\beta = r \in \mathcal{B}(r)$ is the polynomial function coinciding with $m_{\mathrm{geo}}(\lambda, k)$ for $\lambda \in k\bar{\mathfrak{c}}$. The other terms $w_\beta(\lambda, k)$ for $\beta \in \mathcal{B}(r)$ and $\beta \neq r$ vanishes when $\lambda \in k\bar{\mathfrak{c}}$ ([12], see also [18]).

A quicker route, but less instructive, for determining asymptotics of $\Theta_{M,\mathrm{geo}}$ would be to take a test function with small support around a point $r \in \mathfrak{g}^*$. Then we can choose ϵ_λ coinciding with ϵ_β for all $\beta \in \mathcal{B}(r)$ and in the support of the test function f. The additivity is immediate on those β.

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