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Algebraic geometry

On the kernel of the regulator map

Sur le noyau de l'application régulateur

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ABSTRACT

By using the infinitesimal methods due to Bloch, Green, and Griffiths in [1,4], we construct an infinitesimal form of the regulator map and verify that its kernel is $\Omega^1_{\mathbb{C}/\mathbb{Q}}$, which suggests that Question 1.1 seems reasonable at the infinitesimal level.

RÉSUMÉ

Utilisant les méthodes infinitésimales dues à Bloch, Green et Griffiths [1,4], nous construisons une forme infinitésimale de l'application régulateur. Nous vérifions que son noyau est $\Omega^1_{\mathbb{C}/\mathbb{Q}}$, ce qui suggère une version infinitésimale valide de la Question 1.1 formulée dans le texte.

1. Background and question

Let X be a smooth projective curve over the complex number field $\mathbb{C}$. In the 1970s, Bloch constructed the regulator map R: $K_2(X) \to H^1(X, \mathbb{C}^*)$ in several ways. Later, Deligne found a different construction by considering $H^1(X, \mathbb{C}^*)$ as the group of line bundles with connections. We recall his construction very briefly as follows.

For x a point on X, we use i_x to denote the inclusion $x \to X$. The flasque BGQ resolution of $K_2(O_X)$

$$0 \to K_2(O_X) \to K_2(\mathbb{C}(X)) \to \bigoplus_{x \in X^{(1)}} i_{x,*}K_1(\mathbb{C}(x)) \to 0$$

shows that $H^0(K_2(O_X))$ can be computed as $\mathrm{Ker}\{K_2(\mathbb{C}(X)) \to \bigoplus_{x \in X^{(1)}} K_1(\mathbb{C}(x))\}$. So we have the exact sequence of groups

$$0 \to H^0(K_2(O_X)) \to K_2(\mathbb{C}(X)) \to \bigoplus_{x \in X^{(1)}} K_1(\mathbb{C}(x)).$$

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It is known that there exists the following Gysin exact sequence in topology,

$$0 \to H^1(X, \mathbb{C}^*) \to H^1(\mathbb{C}(X), \mathbb{C}^*) \to \bigoplus_{x \in X^{(1)}} \mathbb{C}^*,$$

where $H^1(\mathbb{C}(X), \mathbb{C}^*) = \varinjlim H^1(X - S, \mathbb{C}^*)$ and S is finite points on X.

The main ingredient to construct the regulator map R: $H^0(K_2(O_X)) \to H^1(X, \mathbb{C}^*)$ is the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & H^0(K_2(O_X)) \arrow[r] \arrow[d, "\mathrm{R}"'] & K_2(\mathbb{C}(X)) \arrow[r] \arrow[d, "\mathrm{R}"'] & \bigoplus_{x \in X^{(1)}} K_1(\mathbb{C}(x)) \arrow[d, "\cong"'] \\
0 \arrow[r] & H^1(X, \mathbb{C}^*) \arrow[r] & H^1(\mathbb{C}(X), \mathbb{C}^*) \arrow[r] & \bigoplus_{x \in X^{(1)}} \mathbb{C}^*.
\end{tikzcd} \tag{1.1}$$

That is, one constructs a map R: $K_2(\mathbb{C}(X)) \to H^1(\mathbb{C}(X), \mathbb{C}^*)$ and uses it to deduce the regulator map R: $H^0(K_2(O_X)) \to H^1(X, \mathbb{C}^*)$. We refer the readers to [1] and Section 6 in [5] for more details.

This regulator map has nice motivic features and is related with a general program of Bloch–Beilinson conjecture. In this short note, we focus on the following question, see Section 2 in [3] for a related discussion. To fix notations, for any Abelian group M, $M_{\mathbb{Q}}$ denotes the image of M in $M \otimes_{\mathbb{Z}} \mathbb{Q}$ in the following.

Question 1.1 *(Conjecture 2.4 in [3]). Let* R: $H^0(K_2(O_X)) \to H^1(X, \mathbb{C}^*)$ *be the regulator map, then* $\mathrm{Ker}(\mathrm{R})_{\mathbb{Q}} = K_2(\mathbb{C})_{\mathbb{Q}}$.

This question is very difficult to approach, though it has a very simple form. For $X = \mathrm{P}^1$, this conjecture has been verified by Kerr [6].

2. Main results

In this section, we shall define an infinitesimal form of the regulator map R: $H^0(K_2(O_X)) \to H^1(X, \mathbb{C}^*)$ and verify that its kernel is $\Omega^1_{\mathbb{C}/\mathbb{Q}}$. Our approach is inspired by the following result due to Green and Griffiths.

Theorem 2.1 *(Page 74 and page 125 in [4]). Let X be a smooth projective curve over $\mathbb{C}$, the Cousin flasque resolution of $\Omega^1_{X/\mathbb{Q}}$*

$$0 \to \Omega^1_{X/\mathbb{Q}} \to \Omega^1_{\mathbb{C}(X)/\mathbb{Q}} \xrightarrow{\rho} \bigoplus_{x \in X^{(1)}} i_{x,*} H^1_x(\Omega^1_{X/\mathbb{Q}}) \to 0,$$

is the tangent sequence to BGQ flasque resolution of the sheaf $K_2(O_X)$

$$0 \to K_2(O_X) \to K_2(\mathbb{C}(X)) \to \bigoplus_{x \in X^{(1)}} i_{x,*} K_1(\mathbb{C}(x)) \to 0,$$

where the map ρ is known to take principal parts.

It follows that $H^0(\Omega^1_{X/\mathbb{Q}})$ can be computed as $\mathrm{Ker}\{\Omega^1_{\mathbb{C}(X)/\mathbb{Q}} \xrightarrow{\rho} \bigoplus_{x \in X^{(1)}} H^1_x(\Omega^1_{X/\mathbb{Q}})\}$. So we have the exact sequence of groups

$$0 \to H^0(\Omega^1_{X/\mathbb{Q}}) \to \Omega^1_{\mathbb{C}(X)/\mathbb{Q}} \xrightarrow{\rho} \bigoplus_{x \in X^{(1)}} H^1_x(\Omega^1_{X/\mathbb{Q}}).$$

Definition 2.2 *(Page 71 and page 125 in [4]).* For X a smooth projective curve over $\mathbb{C}$ and x a point on X, there exists a residue map

$$\mathrm{Res}: H^1_x(\Omega^1_{X/\mathbb{Q}}) \to \mathbb{C},$$

which is defined as follows.

Using $\Omega^1_{O_{X,x}/\mathbb{Q}}(nx)$ to denote the absolute 1-forms with poles of order at most n at x, we define Res_x as the following composition:

$$\Omega^1_{O_{X,x}/\mathbb{Q}}(nx) \longrightarrow \Omega^1_{O_{X,x}/\mathbb{C}}(nx) \xrightarrow{\mathrm{Res}} \mathbb{C}.$$

If ξ is the local uniformizer centered at x, an element of $H^1_x(\Omega^1_{X/\mathbb{Q}})$ is represented by the following diagram

$$\begin{cases} O_{X,x} \xrightarrow{\xi^k} O_{X,x} \longrightarrow O_{X,x}/(\xi^k) \longrightarrow 0 \\ O_{X,x} \xrightarrow{\psi} \Omega^1_{O_{X,x}/\mathbb{Q}}. \end{cases} \tag{2.1}$$

For such an element, we define $\mathrm{Res}_x(\frac{\psi}{\xi^k}) \in \mathbb{C}$.

It is known that the tangent space to $\mathbb{C}^*$, which is defined to be the kernel of the natural projection:

$$\mathbb{C}[\varepsilon]^* \xrightarrow{\varepsilon=0} \mathbb{C}^*,$$

can be identified with $\mathbb{C}$ and the tangent map tan: $\mathbb{C}[\varepsilon]^* \to \mathbb{C}$ is given by $z_0 + z_1\varepsilon \to \dfrac{z_1}{z_0}$. This tangent map further induces a map between cohomology groups tan: $H^1(X, \mathbb{C}[\varepsilon]^*) \to H^1(X, \mathbb{C})$. With this interpretation, one can consider $H^1(X, \mathbb{C})$ as the tangent space to $H^1(X, \mathbb{C}^*)$ (this is used in [1]).

There exists the following Gysin exact sequence in topology:

$$0 \to H^1(X, \mathbb{C}) \to H^1(\mathbb{C}(X), \mathbb{C}) \to \bigoplus_{x \in X^{(1)}} \mathbb{C},$$

e.g., see pages 54–55 in [2]. The boundary map $H^1(\mathbb{C}(X), \mathbb{C}) \to \bigoplus_{x\in X^{(1)}} \mathbb{C}$ can be described via Hodge theory as follows. Let $D = \{p_1, \cdots, p_n\}$ be finite points on X and let U be the open complement, $U = X - D$. Let $i_D : D \to X$ denote the inclusion, the residue map Res: $\Omega^\bullet_X(\log D) \to i_{D,*}\Omega^{\bullet-1}_D$ induces Res: $\mathbb{H}^1(\Omega^\bullet_X(\log D)) \to \mathbb{H}^0(\Omega^\bullet_D)$. This gives the map Res: $H^1(U, \mathbb{C}) \to \bigoplus_{i=1,\cdots,n} \mathbb{C}$, by using the identifications $\mathbb{H}^1(\Omega^\bullet_X(\log D)) \cong H^1(U, \mathbb{C})$ and $\mathbb{H}^0(\Omega^\bullet_D) = H^0(D, \mathbb{C}) \cong \bigoplus_{i=1,\cdots,n} \mathbb{C}$.

The following theorem is an infinitesimal form of diagram (1.1):

Theorem 2.3. *There exists the following commutative diagram*

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & H^0(\Omega^1_{X/\mathbb{Q}}) \arrow[r] \arrow[d, "\mathrm{R}'"'] & \Omega^1_{\mathbb{C}(X)/\mathbb{Q}} \arrow[r, "\rho"] \arrow[d, "\mathrm{R}'"'] & \bigoplus_{x\in X^{(1)}} H^1_x(\Omega^1_{X/\mathbb{Q}}) \arrow[d, "\mathrm{Res}"'] \\ 0 \arrow[r] & H^1(X, \mathbb{C}) \arrow[r] & H^1(\mathbb{C}(X), \mathbb{C}) \arrow[r, "\mathrm{Res}"] & \bigoplus_{x\in X^{(1)}} \mathbb{C}, \end{tikzcd} \tag{2.2}$$

where the map R′*'s are the natural maps sending* $d_{/\mathbb{Q}}f$ *to* $d_{/\mathbb{C}}f$.

Proof. The map R': $\Omega^1_{\mathbb{C}(X)/\mathbb{Q}} \to H^1(\mathbb{C}(X), \mathbb{C})$ can be described as follows. Let U be open affine in X, $H^1(U, \mathbb{C})$ can be computed as $\Gamma(U, \Omega_{U/\mathbb{C}})/d_{/\mathbb{C}}\Gamma(U, O_U)$. Given any element $\alpha \in \Omega^1_{U/\mathbb{Q}}$, its image $[\alpha]$ in $\Omega^1_{U/\mathbb{C}}$ defines an element in $H^1(U, \mathbb{C})$.

To check the commutativity of the right square, working locally in a Zariski open affine neighborhood U, we can write an element $\beta \in \Omega^1_{\mathbb{C}(X)/\mathbb{Q}}$ as

$$\beta = \frac{h\, d_{/\mathbb{Q}}g}{f_1^{l_1} \dots f_k^{l_k}},$$

where $f_1, \dots, f_k, h \in \Gamma(U, O_U)$ are relatively prime and f_i's are irreducible.

The following diagram is commutative:

$$\begin{tikzcd} \dfrac{h\, d_{/\mathbb{Q}}g}{f_1^{l_1} \dots f_k^{l_k}} \arrow[r, "\rho"] \arrow[d, "\mathrm{R}'"'] & \sum_i \dfrac{h\, d_{/\mathbb{Q}}g}{f_1^{l_1} \dots \hat{f}_i^{l_i} \dots f_k^{l_k}} \arrow[d, "\mathrm{Res}"'] \\ \dfrac{h\, d_{/\mathbb{C}}g}{f_1^{l_1} \dots f_k^{l_k}} \arrow[r, "\mathrm{Res}"] & \sum_i \mathrm{Res}_{x_i}(\dfrac{h\, d_{/\mathbb{C}}g}{f_1^{l_1} \dots f_k^{l_k}}), \end{tikzcd}$$

where $x_i = \{f_i = 0\}$ and $\hat{f}_i^{l_i}$ means to omit the ith term.

The map R': $\Omega^1_{\mathbb{C}(X)/\mathbb{Q}} \to H^1(\mathbb{C}(X), \mathbb{C})$ induces R': $H^0(\Omega^1_{X/\mathbb{Q}}) \to H^1(X, \mathbb{C})$. □

Let $\{f_0, g_0\} \in H^0(K_2(O_X))$ and let $(N, \triangledown)$ denote the bundle with connection $\triangledown$, as recalled on page 4 in [1]. There exists the following commutative diagram:

$$\begin{tikzcd}
\{f_0, g_0\} \arrow[d, "\mathrm{R}"'] & \{f_0+\varepsilon f_1, g_0+\varepsilon g_1\} \arrow[l, "\varepsilon=0"'] \arrow[r, "\tan"] \arrow[d] & \dfrac{f_1}{f_0}\dfrac{d_{/\mathbb{Q}}g_0}{g_0} - \dfrac{g_1}{g_0}\dfrac{d_{/\mathbb{Q}}f_0}{f_0} \arrow[d, "\mathrm{R}'"'] \\
\{f_0, g_0\}^*(N, \triangledown) & \{f_0+\varepsilon f_1, g_0+\varepsilon g_1\}^*(N, \triangledown) \arrow[l, "\varepsilon=0"] \arrow[r, "\tan"] & \dfrac{f_1}{f_0}\dfrac{d_{/\mathbb{C}}g_0}{g_0} - \dfrac{g_1}{g_0}\dfrac{d_{/\mathbb{C}}f_0}{f_0}.
\end{tikzcd}$$

The commutativity of left square is trivial. To check the right one, since $\{f_0+\varepsilon f_1, g_0+\varepsilon g_1\} = \{f_0, g_0\}\{f_0, 1+\varepsilon\frac{g_1}{g_0}\}\{1+\varepsilon\frac{f_1}{f_0}, g_0\}\{1+\varepsilon\frac{f_1}{f_0}, 1+\varepsilon\frac{g_1}{g_0}\}$, we reduce to considering $\{1+\varepsilon f_1, g_0\}$, which is obvious:

$$\begin{tikzcd}
\{1+\varepsilon f_1, g_0\} \arrow[r, "\tan"] \arrow[d] & f_1\dfrac{d_{/\mathbb{Q}}g_0}{g_0} \arrow[d, "\mathrm{R}'"'] \\
\{1+\varepsilon f_1, g_0\}^*(N, \triangledown) \arrow[r, "\tan"] & f_1\dfrac{d_{/\mathbb{C}}g_0}{g_0},
\end{tikzcd}$$

where the up tan map is well known and the down tan map is the formula (2.12) on page 14 in [1].

In this sense, we consider the map R': $H^0(\Omega^1_{X/\mathbb{Q}}) \to H^1(X, \mathbb{C})$ as the infinitesimal form of the regulator map R: $H^0(K_2(O_X)) \to H^1(X, \mathbb{C}^*)$ and compute the kernel of R'.

Since $H^1(X, \mathbb{C})$ has Hodge decomposition $H^1(X, \mathbb{C}) \cong H^0(\Omega^1_{X/\mathbb{C}}) \oplus H^1(O_X)$ and the map R': $H^0(\Omega^1_{X/\mathbb{Q}}) \to H^1(X, \mathbb{C})$ naturally maps $d_{/\mathbb{Q}}f$ to $d_{/\mathbb{C}}f$, so R' is the composition $H^0(\Omega^1_{X/\mathbb{Q}}) \to H^0(\Omega^1_{X/\mathbb{C}}) \hookrightarrow H^1(X, \mathbb{C})$. Hence $\mathrm{Ker}(\mathrm{R}') = \mathrm{Ker}\{H^0(\Omega^1_{X/\mathbb{Q}}) \to H^0(\Omega^1_{X/\mathbb{C}})\}$.

Theorem 2.4. $\mathrm{Ker}(\mathrm{R}') = \Omega^1_{\mathbb{C}/\mathbb{Q}}$.

Proof. There is a natural short exact sequence of sheaves

$$0 \to \Omega^1_{\mathbb{C}/\mathbb{Q}} \otimes_{\mathbb{C}} O_X \to \Omega^1_{X/\mathbb{Q}} \to \Omega^1_{X/\mathbb{C}} \to 0.$$

The associated long exact sequence is of the form

$$0 \to H^0(\Omega^1_{\mathbb{C}/\mathbb{Q}} \otimes_{\mathbb{C}} O_X) \to H^0(\Omega^1_{X/\mathbb{Q}}) \to H^0(\Omega^1_{X/\mathbb{C}}) \to H^1(\Omega^1_{\mathbb{C}/\mathbb{Q}} \otimes_{\mathbb{C}} O_X) \to \cdots.$$

So the kernel of $H^0(\Omega^1_{X/\mathbb{Q}}) \to H^0(\Omega^1_{X/\mathbb{C}})$ can be identified with $H^0(\Omega^1_{\mathbb{C}/\mathbb{Q}} \otimes_{\mathbb{C}} O_X) \cong H^0(O_X) \otimes \Omega^1_{\mathbb{C}/\mathbb{Q}} \cong \mathbb{C} \otimes \Omega^1_{\mathbb{C}/\mathbb{Q}} \cong \Omega^1_{\mathbb{C}/\mathbb{Q}}$. $\square$

Since the tangent space to $K_2(\mathbb{C})$ is $\Omega^1_{\mathbb{C}/\mathbb{Q}}$, this theorem suggests hat Question 1.1 seems reasonable at the infinitesimal level.

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