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Dispersive estimates for the wave equation inside cylindrical convex domains: A model case ☆

Estimation de dispersion pour les ondes dans un convexe : le cas modèle

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A R T I C L E   I N F O

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A B S T R A C T

In this work, we will establish local in time dispersive estimates for solutions to the model-case Dirichlet wave equation inside a cylindrical convex domain $\Omega \subset \mathbb{R}^3$ with a smooth boundary $\partial\Omega \neq \emptyset$. Let us recall that dispersive estimates are key ingredients to prove Strichartz estimates. Nonoptimal Strichartz estimates for waves inside an arbitrary domain Ω have been proved by Blair–Smith–Sogge [1,2]. Better estimates in strictly convex domains have been obtained in [4]. Our case of cylindrical domains is an extension of the result of [4] in the case where the curvature radius ≥ 0 depends on the incident angle and vanishes in some directions.

R É S U M É

Dans ce travail, nous allons établir des estimations de dispersion locales en temps pour les solutions de l'équation des ondes dans un domaine cylindrique convexe $\Omega \subset \mathbb{R}^3$ à bord C^∞ $\partial\Omega \neq \emptyset$. Les estimations de dispersion sont classiquement utilisées pour prouver les estimations de Strichartz. Dans un domaine Ω général, des estimations de Strichartz non optimales ont été démontrées par Blair–Smith–Sogge [1,2]. De meilleures estimations ont été prouvées dans [4] lorsque Ω est strictement convexe. Le cas des domaines cylindriques que nous considérons ici généralise les resultats de [4] dans le cas où la courbure ≥ 0 dépend de l'angle d'incidence et s'annule dans certaines directions.

Version française abrégée

Soit $\Omega = \{x \geq 0, (y, z) \in \mathbb{R}^2\} \subset \mathbb{R}^3$. Nous établissons des estimations de dispersion en temps petit pour l'équation des ondes dans Ω avec condition de Dirichlet.

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$$Pu = 0, \quad u_{|t=0} = \delta_a, \quad \partial_t u_{|t=0} = 0, \quad u_{|x=0} = 0$$

avec $P(t, x, y, z, \partial_t, \partial_x, \partial_y, \partial_z) = \partial_t^2 - (\partial_x^2 + (1+x)\partial_y^2 + \partial_z^2)$, $u = u(t, x, y, z)$, et pour $a \in \Omega$, $\delta_a = \delta_{x=a, y=0, z=0}$. Le problème est local près de chaque point du bord. Les phénomènes nouveaux apparaissent pour $0 < a \le 1$ petit. Nous utilisons les notations $\tau = \frac{h}{i}\partial_t$, $\eta = \frac{h}{i}\partial_y$, $\xi = \frac{h}{i}\partial_x$, $\zeta = \frac{h}{i}\partial_z$.

Résultats

On note $G_a(t, x, y, z)$ la fonction de Green associée à (1). La fonction χ appartient à $C_0^\infty(]0, \infty[)$ et est égale à 1 sur l'intervalle $[1, 2]$. Nous prouvons le théorème suivant.

Théorème 0.1. *Il existe $C > 0$ tel que, pour tout $a \in]0, 1]$, $h \in]0, 1]$, $t \in [-1, 1] \setminus \{0\}$, on a*

$$\|\chi(hD_t)G_a(t, x, y, z)\|_{L^\infty} \le Ch^{-3} \min\left(1, \left(\frac{h}{|t|}\right)^{1/2} \gamma\big(t, h, \sup(x, a)\big)\right),$$

avec

$$\gamma(t, h, b) = \begin{cases} \left(\frac{h}{|t|}\right)^{1/2} + b^{1/8}h^{1/4} & \text{si } b \ge h^{2/3-\epsilon}, \text{ pour tout } \epsilon > 0 \text{ petit}, \\ h^{1/4} + \left(\frac{h}{|t|}\right)^{1/3} & \text{si } b \le h^{1/2}. \end{cases}$$

1. Introduction

We will study the following model case. Let $\Omega = \{x \ge 0, (y, z) \in \mathbb{R}^2\} \subset \mathbb{R}^3$. Our goal is to establish local in time dispersive estimates for solutions to the linear Dirichlet wave equation inside Ω with smooth boundary $\partial\Omega$:

$$Pu = 0, \quad u_{|t=0} = \delta_a, \quad \partial_t u_{|t=0} = 0, \quad u_{|x=0} = 0 \tag{1}$$

with $P(t, x, y, z, \partial_t, \partial_x, \partial_y, \partial_z) = \partial_t^2 - (\partial_x^2 + (1+x)\partial_y^2 + \partial_z^2)$, $u = u(t, x, y, z)$, and for $a \in \Omega$, $\delta_a = \delta_{x=a, y=0, z=0}$ (the Dirac distribution). The problem is local near any points on the boundary. We are interested only in highly reflected waves and near points where the order of tangency is infinity whose source points are located at a small distance $0 < a \le 1$ to the boundary. This gives us interesting phenomena such as caustics near the boundary for such domains. We will use the notation $\tau = \frac{h}{i}\partial_t$, $\eta = \frac{h}{i}\partial_y$, $\xi = \frac{h}{i}\partial_x$, $\zeta = \frac{h}{i}\partial_z$. We recall that the dispersive estimates for the free wave in $\mathbb{R}^3$ read as follows:

$$\|\chi(hD_t)e^{\pm it\sqrt{\Delta}}(\delta_a)\|_{L^\infty} \le Ch^{-3} \min\left(1, \frac{h}{|t|}\right).$$

1.1. Main results

In the following, we prove dispersive estimates in cylindrical domains. The main results are obtained for appropriately localized Green function in each region corresponding to different values of η and we prove that the estimates are worse than that of the free case. To be more precise, the following theorem holds true.

Theorem 1.1. *There exists C such that for every $a \in]0, 1]$, every $h \in]0, 1]$, every $t \in [-1, 1] \setminus 0$, the following holds:*

$$\|\chi(hD_t)G_a(t, x, y, z)\|_{L^\infty} \le Ch^{-3} \min\left(1, \left(\frac{h}{|t|}\right)^{1/2} \gamma\big(t, h, \sup(x, a)\big)\right), \tag{2}$$

with

$$\gamma(t, h, b) = \begin{cases} \left(\frac{h}{|t|}\right)^{1/2} + b^{1/8}h^{1/4} & \text{if } b \ge h^{2/3-\epsilon}, \text{ for any } \epsilon > 0 \text{ small}, \\ h^{1/4} + \left(\frac{h}{|t|}\right)^{1/3} & \text{if } b \le h^{1/2}. \end{cases}$$

2. Idea of the proof

The main idea to prove the theorem is to construct a suitable local parametrix together with the Airy–Poisson summation formula. In this way, the parametrix is represented as a sum over eigenmodes, which is used to prove the estimates for $a \le h^{1/2}$. On the other hand, the parametrix is represented as a sum over multiple reflections, which is used to prove the estimates for $a \ge h^{2/3-\epsilon}$.

2.1. Main ingredients

(1) Airy function. Let $z > 0$. The Airy function is defined by

$$Ai(-z) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i(s^3/3 - sz)} ds.$$

It satisfies the Airy equation $Ai''(z) - zAi(z) = 0$. We rewrite as

$$Ai(-z) = e^{-i\pi/3} Ai(e^{-i\pi/3} z) + e^{i\pi/3} Ai(e^{i\pi/3} z) = A_+(z) + A_-(z), \tag{3}$$

where we have defined $A_\pm(z) = e^{\mp 2i\pi/3} Ai(e^{\mp 2i\pi/3} z)$. Notice that $A_-(z) = \overline{A_+(\bar{z})}$. Moreover, we have

$$\frac{A_-(z)}{A_+(z)} = i\, e^{-\frac{4}{3} i z^{3/2}} e^{iB(z^{3/2})},$$

where $B(z) \sim \sum_{j\ge 1} b_j z^{-j}$ for $z \to +\infty$ and $b_1 \neq 0$.

(2) Green function. Let

$$e_k(x, \eta) = f_k \frac{|\eta|^{1/3}}{k^{1/6}} Ai(|\eta|^{2/3} x - \omega_k),$$

where f_k are constants such that $||e_k(., \eta)||_{L^2([0,\infty[)} = 1$ for every $k \ge 1$. We have that $\chi(hD_t)G_a$ is represented microlocally near tangential directions by

$$\chi(hD_t)G_a(t, x, y, z) = \frac{1}{4\pi^2 h^2} \sum_{k \ge 1} \int e^{\frac{i}{h}(y\eta + z\zeta)} e^{i\frac{t}{h}(\eta^2 + \zeta^2 + \omega_k h^{2/3} |\eta|^{4/3})^{1/2}} e_k(x, \eta/h) e_k(a, \eta/h) \times \chi_0(\eta^2 + \zeta^2) \chi_1(\omega_k h^{2/3} |\eta|^{4/3})\, d\eta\, d\zeta. \tag{4}$$

Here $\chi_0 \in C_0^\infty$, $0 \le \chi_0 \le 1$, χ_0 is supported in the neighborhood of 1 and $\chi_1 \in C_0^\infty$, $0 \le \chi_1 \le 1$, χ_1 is supported in $(-\infty, \frac{1}{10}]$, $\chi_1 = 1$ on $(-\infty, \frac{1}{20}]$.

(3) Airy–Poisson summation formula. For $\omega \in \mathbb{R}$, set

$$L(\omega) = \pi + i \log\left(\frac{A_-(\omega)}{A_+(\omega)}\right).$$

The function L is analytic, strictly increasing and from the asymptotic expansion of $A_\pm$ we have the basic properties of the function L. It satisfies

$$L(0) = \pi/3, \quad \lim_{\omega \to -\infty} L(\omega) = 0, \quad L(\omega) \underset{\omega \to +\infty}{\sim} \frac{4}{3}\omega^{3/2},$$

and for all $k \ge 1$, we have

$$L(\omega_k) = 2\pi k \Leftrightarrow Ai(-\omega_k) = 0, \quad L'(\omega_k) = 2\pi \int_0^\infty Ai^2(x - \omega_k)\, dx.$$

Lemma 2.1 *(Airy–Poisson summation formula (see [5])). The following equality holds true in $\mathcal{D}'(\mathbb{R}_\omega)$,*

$$\sum_{N \in \mathbb{Z}} e^{-iNL(\omega)} = 2\pi \sum_{k \in \mathbb{N}^*} \frac{1}{L'(\omega_k)} \delta_{\omega = \omega_k}.$$

(4) Oscillatory integrals. To evaluate various oscillatory integrals, we use the following lemma.

Lemma 2.2 *(Lemma 2.20 [4]). Let $K \subset \mathbb{R}$, and let $a(\xi, \lambda)$ be a classical symbol of degree 0 in $\lambda \ge 1$ with $a(\xi, \lambda) = 0$ for $\xi \notin K$. Let $k \ge 2$, $c_0 > 0$ and $\Phi(\xi)$ be a phase function such that*

$$\sum_{2 \le j \le k} |\Phi^{(j)}(\xi)| \ge c_0, \quad \xi \in K.$$

Then there exists C such that

$$\left| \int e^{i\lambda\Phi(\xi)} a(\xi, \lambda) d\xi \right| \le C\lambda^{-1/k}, \quad \forall \lambda \ge 1.$$

Moreover, the constant C depends only on c_0 and on an upper bound of a finite number of derivatives of order ≥ 2 of Φ, a in a neighborhood of K. In particular case $k = 2$, the usual stationary phase holds [see [3]].

2.2. Outline of the proof

These ingredients play a crucial role in obtaining dispersive estimates (2). Our local parametrix is represented as a sum of Fourier Integral Operators in two different ways. On the one hand, it is given as in (4) as the sum of gallery modes. On the other hand, applying Lemma 2.1 to (4), we can write the parametrix as a sum over N corresponding to the number of reflections on the boundary as follows:

$$G_a = \sum_{N\in\mathbb{Z}} G_{a,N} = \sum_{N\in\mathbb{Z}} \frac{(-i)^N a^2}{(2\pi)^5 h^4} \int e^{\frac{i}{h}\Phi_{a,N,h}} |\eta|^3 \chi_0\chi_1 \,\mathrm{d}s\,\mathrm{d}\sigma\,\mathrm{d}\omega\,\mathrm{d}\zeta\,\mathrm{d}\eta, \tag{5}$$

with the phase function

$$\Phi_{a,N,h}(t,x,y,z;s,\sigma,\omega,\zeta,\eta) = y\eta + |\eta|z\zeta + |\eta|t(1+\zeta^2+a\omega)^{1/2} + a^{3/2}|\eta|\left(\frac{s^3}{3} + s(X-\omega) + \frac{\sigma^3}{3} + \sigma(1-\omega) - \frac{4}{3}N\omega^{3/2} + \frac{h}{a^{3/2}|\eta|}NB\big(\omega^{3/2}a^{3/2}|\eta|/h\big)\right)$$

which generates a Lagrangian submanifold parametrized by (s,σ,η). Then we can apply stationary phase method for ζ-integration and decompose $G_{a,N} = G_{a,N,1} + G_{a,N,2}$, where $G_{a,N,2}$ is defined by introducing a cutoff function $\chi_2(\omega) \in C_0^\infty(]1/2,3/2[)$, $0 \le \chi_2 \le 1$, $\chi_2 = 1$ on $]\frac{3}{4},\frac{5}{4}[$ in the integral (5). This $G_{a,N,2}$ corresponds to the regime of swallowtails. Its counterpart $G_{a,N,1}$ is defined by introducing $1-\chi_2$ in (5). Then the following results hold.

Proposition 2.3. *Let $\alpha < 2/3$ and $a_0 \le 1$ be fixed. There exists C such that for all $h \in]0,1]$, all $a \in [h^\alpha, a_0]$, all $x \in [0,a]$, all $t \in]0,1]$, all $y \in \mathbb{R}$, all $z \in \mathbb{R}$, the following holds:*

$$\left|\sum_{2\le N\le C_0 a^{-1/2}} G_{a,N,1}(t,x,y,z;h)\right| \le Ch^{-3}\left(\frac{h}{t}\right)^{1/2} h^{1/3}.$$

Proposition 2.4. *Let $\alpha < 2/3$ and $a_0 \le 1$ be fixed. There exists C such that for all $h \in]0,1]$, all $a \in [h^\alpha, a_0]$, all $x \in [0,a]$, all $t \in]0,1]$, all $y \in \mathbb{R}$, all $z \in \mathbb{R}$, the following holds:*

$$\left|\sum_{2\le N\le C_0 a^{-1/2}} G_{a,N,2}(t,x,y,z;h)\right| \le Ch^{-3}\left(\frac{h}{t}\right)^{1/2} a^{1/8}h^{1/4}.$$

Proposition 2.5. *Let $\alpha < 2/3$ and $a_0 \le 1$ be fixed. There exists C such that for all $h \in]0,1]$, all $a \in [h^\alpha, a_0]$, all $x \in [0,a]$, all $t \in]0,1]$, all $y \in \mathbb{R}$, all $z \in \mathbb{R}$, the following holds:*

$$\left|G_{a,1}(t,x,y,z;h)\right| \le Ch^{-3}\left(\frac{h}{t}\right)^{1/2}\left(\left(\frac{h}{t}\right)^{1/2} + a^{1/8}h^{1/4}\right).$$

Putting together above results yields the following theorem which implies Theorem 1.1.

Theorem 2.6. *Let $\alpha < 2/3$ and $a_0 \le 1$ be fixed. There exists C such that for all $h \in]0,1]$, all $a \in [h^\alpha, a_0]$, all $x \in [0,a]$, all $t \in]0,1]$, all $y \in \mathbb{R}$, all $z \in \mathbb{R}$, the following holds:*

$$\left|\sum_{0\le N\le C_0 a^{-1/2}} G_{a,N}(t,x,y,z;h)\right| \le Ch^{-3}\left(\frac{h}{t}\right)^{1/2}\left(\left(\frac{h}{t}\right)^{1/2} + a^{1/8}h^{1/4}\right).$$

Next, we use (4) to obtain the dispersive estimates for $a \le h^{1/2}$. To do that we first apply the stationary phase method for ζ-integration then for some values of k with $1 \le k \le L$ for appropriate choice of L, we use the following lemma.

Lemma 2.7. *(Lemma 3.5 [4]) There exists C_0 such that for $L \ge 1$, the following holds:*

$$\sup_{b\in\mathbb{R}}\left(\sum_{1\le k\le L} k^{-1/3} Ai^2(b-\omega_k)\right) \le C_0 L^{1/3}.$$

For the values of k with $L \le k \le \frac{\varepsilon}{h}$, we use (3) together with the asymptotic expansion of $A_\pm$ to obtain oscillatory integrals to which we apply Lemma 2.2. We have the following result.

Proposition 2.8. *For small ε, there exists a constant C independent of $a \in]0, h^{1/2}]$, $t \in [h, 1[$, $x \in [0, a]$, $y \in \mathbb{R}$, $z \in \mathbb{R}$ and $k \in [L, \frac{\varepsilon}{h}]$ such that the following holds true:*

$$\left| \int e^{i\lambda \psi_k^{\pm,\pm}} \sigma_k^{\pm,\pm} d\eta \right| \leq C(hk)^{-2/3} \lambda^{-1/3}$$

where

$$h\lambda\psi_k^{\pm,\pm}(t, x, y, z, \eta) = y\eta + |\eta| t\sqrt{1-\tilde{z}^2}(1+\gamma)^{1/2} \pm \frac{2}{3}|\eta|(\gamma - x)^{3/2} \pm \frac{2}{3}|\eta|(\gamma - a)^{3/2},$$

and $\sigma_k^{\pm,\pm}$ are symbols obtained from the Airy expansion. Here $z = t\tilde{z}$ with $\tilde{z} < 1$ and $\gamma = h^{2/3}\omega_k|\eta|^{-2/3}$. For $|z| \geq t$, the integral is $O_{C^\infty}(h^\infty)$ by integration by parts.

References

[1] M.D. Blair, H.F. Smith, C.D. Sogge, On Strichartz estimates for Schrödinger operators in compact manifolds with boundary, Proc. Amer. Math. Soc. 130 (2008) 247–256.

[2] M.D. Blair, H.F. Smith, C.D. Sogge, Strichartz estimates for the wave equation on manifolds with boundary, Ann. Inst. Henri Poincaré, Anal. Non Linéaire 26 (2009) 1817–1829.

[3] L. Hörmander, The Analysis of Linear Partial Differential Operators I: Distribution Theory and Fourier Analysis, Class. Math., Springer-Verlag, New York, 2003.

[4] O. Ivanovici, G. Lebeau, F. Planchon, Dispersion for the wave equation inside strictly convex domains I: the Friedlander model case, Ann. Math. (2) 180 (2014) 323–380.

[5] O. Ivanovici, R. Lascar, G. Lebeau, F. Planchon, Dispersion for the wave equation inside strictly convex domains II: the general case, arXiv:1605.08800v1, 2016.