C. R. Acad. Sci. Paris, Ser. I 355 (2017) 170–175

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

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Partial differential equations

Large deviations of a velocity jump process with a Hamilton–Jacobi approach

Grandes déviations pour un processus à sauts de vitesse avec une approche de Hamilton–Jacobi

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ARTICLE INFO

Article history:
Received 15 February 2016
Accepted 5 January 2017
Available online 4 February 2017

Presented by the Editorial Board

ABSTRACT

We study a random process on $\mathbb{R}^n$ moving in straight lines and changing randomly its velocity at random exponential times. We focus more precisely on the Kolmogorov equation in the hyperbolic scale $(t, x, v) \to \left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, v\right)$, with $\varepsilon > 0$, before proceeding to a Hopf–Cole transform, which gives a kinetic equation on a potential. We show convergence as $\varepsilon \to 0$ of the potential towards the viscosity solution to a Hamilton–Jacobi equation $\partial_t \varphi + H(\nabla_x \varphi) = 0$ where the Hamiltonian may lack $\mathcal{C}^1$ regularity, which is quite unseen in this type of studies.

RÉSUMÉ

Nous nous intéressons à un processus aléatoire sur $\mathbb{R}^n$ qui alterne des phases de mouvements rectilignes uniformes et change de vitesse à des temps exponentiels. Nous étudions plus précisément l'équation de Kolmogorov après rééchelonnement hyperbolique $(t, x, v) \to \left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, v\right)$, $\varepsilon > 0$, puis nous effectuons une transformée de Hopf–Cole qui nous donne une équation cinétique suivie par un potentiel. Nous montrons la convergence pour $\varepsilon \to 0$ de ce potentiel vers la solution de viscosités d'une équation de Hamilton–Jacobi $\partial_t \varphi + H(\nabla_x \varphi) = 0$ où le hamiltonien peut présenter une singularité $\mathcal{C}^1$, ce qui est assez inédit dans ce type d'études.

Version française abrégée

Nous nous donnons une densité de probabilité $M \in L^1(\mathbb{R}^n)$ et nous notons V son support. Nous supposons que V est compact et que 0 appartient à l'intérieur de l'enveloppe convexe de V, que l'on note $\mathrm{Conv}(V)$. Pour $p \in \mathbb{R}^n$, nous

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http://dx.doi.org/10.1016/j.crma.2016.12.011

notons $\mu(p) = \max\{v \cdot p \mid v \in \mathrm{Conv}(V)\}$. Nous étudions le mouvement de particules dans $\mathbb{R}^n$ suivant le processus de Markov déterministe par morceaux défini comme suit : une particule donnée se déplace de manière rectiligne uniforme avec une vitesse $v \in V$ tirée aléatoirement en suivant la loi de probabilité $M(v')\,\mathrm{d}v'$. À des temps exponentiels de paramètre 1, la particule change de direction en tirant une nouvelle vitesse tirée selon la loi $M(v')\,\mathrm{d}v'$. Afin d'étudier des résultats de larges déviations du processus similairement aux techniques développées dans [3–8], nous nous intéressons à l'équation de Chapman–Kolmogorov forward suivie par la densité de particules après un rééchelonement hyperbolique $(t,x,v) \to \left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, v\right)$, $\varepsilon > 0$:

$$\partial_t f^\varepsilon + v \cdot \nabla_x f^\varepsilon = \frac{1}{\varepsilon}\left(M(v)\,\rho^\varepsilon - f^\varepsilon\right), \quad (t,x,v) \in \mathbb{R}_+ \times \mathbb{R}^n \times V.$$

Nous étudions plus particulièrement l'équation vérifiée par un potentiel φ^ε obtenu après passage par une transformée de Hopf–Cole : $f^\varepsilon(t,x,v) = M(v)\,\mathrm{e}^{-\frac{\varphi^\varepsilon(t,x,v)}{\varepsilon}}$. Nous cherchons alors une éventuelle limite pour φ^ε. Nous procédons à un développement WKB : $\varphi^\varepsilon = \varphi + \varepsilon\eta$, ce qui amène, en posant $p = \nabla_x\varphi$ et $H = -\partial_t\varphi$, à la résolution d'un problème spectral dans l'espace des mesures positives : chercher (H, Q) un couple valeur/vecteur propres associé à l'opérateur $Q \mapsto (v \cdot p - 1)\,Q + \int_V M'Q'\,\mathrm{d}v'$. On obtient une équation de Hamilton–Jacobi $\partial_t\varphi + H(\nabla_x\varphi) = 0$. Pour $n = 1$ et $M \geq \delta > 0$ sur son support, le vecteur propre Q a une densité et conduit à un hamiltonien H défini par l'équation implicite

$$\int_V \frac{M(v)}{1 + H(p) - v \cdot p}\,\mathrm{d}v = 1.$$

La positivité de Q garantit que $H(p) \geq \mu(p) - 1$. En dimension supérieure toutefois, et même si $M \geq \delta > 0$, cette équation peut ne pas avoir de solution $H(p)$ lorsque p devient grand. Cela se manifeste pour le vecteur propre par une concentration de la mesure Q autour des valeurs v qui annulent $1 + H(p) - v \cdot p$, ce qui force $H(p) = \mu(p) - 1$. Cette transition entraîne une singularité $\mathcal{C}^1$ du hamiltonien.

Nous démontrons la convergence de φ^ε vers φ, où φ est solution de viscosité [5] de l'équation de Hamilton–Jacobi en utilisant la méthode de la fonction test perturbée [7].

1. Introduction

We continue the work initiated in [1,2]. Let $M \in L^1(\mathbb{R}^n)$ be a probability density function. We suppose that the support of M, which we denote V, is compact and that 0 belongs to the interior of $\mathrm{Conv}(V)$, the convex hull of V. We denote by $|\cdot|$ the Euclidian norm in $\mathbb{R}^n$ and by $\cdot$ the canonical scalar product. For $p \in \mathbb{R}^n$, we define

$$\mu(p) := \max\{v \cdot p \mid v \in \mathrm{Conv}(V)\}, \tag{1}$$

$\mathrm{Arg}\mu(p) := \{v \in \mathrm{Conv}(V) \mid v \cdot p = \mu(p)\}$ and $\mathrm{Sing}(M) := \left\{p \in \mathbb{R}^n, \quad \int_V \frac{M(v)}{\mu(p) - v \cdot p}\mathrm{d}v \leq 1\right\}$.

We focus on the motion dynamics in $\mathbb{R}^n$ of particles given by the following piecewise deterministic Markov process: a particle moves successively in straight lines with velocity v, chosen randomly with probability distribution $M(v')\,\mathrm{d}v'$. At random exponential times (with parameter 1), the particle changes its velocity, choosing randomly a new velocity with distribution $M(v')\,\mathrm{d}v'$. The Chapman–Kolmogorov forward equation associated with the probability density function $f(t,x,v)$ of this process is given by:

$$\partial_t f + v \cdot \nabla_x f = M\rho - f, \quad (t,x,v) \in \mathbb{R}_+ \times \mathbb{R}^n \times V, \tag{2}$$

where $\rho(t,x) = \int_V f(t,x,v)\,\mathrm{d}v$. In order to investigate large deviation principles for the process, one can study the large scale hyperbolic limit $(t,x) \to \left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right)$ with $\varepsilon > 0$. In this scale, the kinetic equation (2) reads:

$$\partial_t f^\varepsilon + v \cdot \nabla_x f^\varepsilon = \frac{1}{\varepsilon}\left(M\rho^\varepsilon - f^\varepsilon\right), \quad (t,x,v) \in \mathbb{R}_+ \times \mathbb{R}^n \times V. \tag{3}$$

Then, we perform the following Hopf–Cole transformation: $f^\varepsilon(t,x,v) = M(v)\,\mathrm{e}^{-\frac{\varphi^\varepsilon(t,x,v)}{\varepsilon}}$, where we expect the potential φ^ε to become independent of v as $\varepsilon \to 0$. Such techniques have already been studied for a more general case of Markov process with a finite discrete set of states in [3] and, from a probabilistic point of view, in [8].

Here, assume that the initial condition is well-prepared, i.e. it does not depend on v: $\varphi^\varepsilon(0,x,v) = \varphi_0(x)$. We believe that the conclusion of this paper is not dramatically modified if $\varphi^\varepsilon(0,x,v) = \varphi_0(x,v)$. Indeed, the only expected change concerns the initial condition of the Hamilton–Jacobi equation, which should be independent of v. This is left for future work. The equation satisfied by φ^ε reads

$$\partial_t\varphi^\varepsilon + v \cdot \nabla_x\varphi^\varepsilon = \int_V M(v')\left(1 - \mathrm{e}^{\frac{\varphi^\varepsilon - \varphi'^\varepsilon}{\varepsilon}}\right)\mathrm{d}v', \quad (t,x,v) \in \mathbb{R}_+ \times \mathbb{R}^n \times V. \tag{4}$$

As in [9], the limit potential satisfies a Hamilton–Jacobi equation. Surprisingly enough, our Hamiltonian may lack $\mathcal{C}^1$ regularity as we will show in Proposition 2.

Theorem 1. *Under the previous assumptions, φ^ε converges locally uniformly on $\mathbb{R}_+ \times \mathbb{R}^n \times V$ toward φ, where φ does not depend on v. Moreover, φ is the viscosity solution to the following Hamilton–Jacobi equation:*

$$\partial_t \varphi(t,x) + H(\nabla_x \varphi(t,x)) = 0, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n, \tag{5}$$

with initial condition $\varphi(0,\cdot) = \varphi_0$ and a Hamiltonian H given as follows: if $p \in \mathrm{Sing}(M)$, then $H(p) = \mu(p) - 1$. Else, $H(p)$ is uniquely determined by the following formula:

$$\int_V \frac{M(v)}{1 + H(p) - v \cdot p} \mathrm{d}v = 1. \tag{6}$$

A corollary to Theorem 1 is that Theorem 1.1 from [2] is only correct when $\mathrm{Sing}(M) = \emptyset$, since there is no solution to (6) when $p \in \mathrm{Sing}(M)$. The present result establishes the appropriate statement in the case $\mathrm{Sing}(M) \neq \emptyset$. Interestingly enough, the proof appears quite different due to the apparition of Dirac masses in the velocity variable in the expression of the corrector.

2. Identification of the Hamiltonian

In order to identify the limit $\varphi := \lim_{\varepsilon \to 0} \varphi^\varepsilon$, we perform the formal expansion $f^\varepsilon = MQ\mathrm{e}^{-\frac{\varphi}{\varepsilon}}$ where Q is to be determined. Plugging this ansatz into the kinetic formulation (3) and writing $p = \nabla_x \varphi$ and $H = -\partial_t \varphi$, we get (at the formal limit $\varepsilon \to 0$) the following spectral problem:

$$(1 + H - v \cdot p)\, Q = \int_V M(v') Q(v') \mathrm{d}v'. \tag{7}$$

A similar spectral problem has been studied in [4] in a more general case. The positivity of Q yields $H \geq v \cdot p - 1$ for all $v \in V$ hence $H \geq \mu(p) - 1$. Suppose $H > \mu(p) - 1$. Then, $1 + H - vp > 0$ for all $v \in V$ and $Q(v) = \dfrac{\int_V M(v') Q(v') \mathrm{d}v'}{1 + H - v \cdot p}$. Integrating against M with respect to v, we obtain the following problem: find H such that $\int_V \frac{M(v)}{1+H-v\cdot p} \mathrm{d}v = 1$. If $p \in \mathrm{Sing}(M)^c$, by monotonicity, such H exists and is unique. Equation (6), however, does not have an L^1 solution for $p \in \mathrm{Sing}(M)$. Similarly to [4], we look for solutions in a larger set, namely the set of positive measures. Then, a solution to the spectral problem is the eigenvalue $H = \mu(p) - 1$ associated with the positive measure $Q = \frac{\mathrm{d}v}{\mu(p) - v \cdot p} + \alpha(p)\delta_w$, where $\alpha(p) = 1 - \int_V \frac{M(v)}{\mu(p) - vp} \mathrm{d}v \geq 0$ and δ_w is the Dirac measure centered in $w \in \mathrm{Arg}\mu(p) \cap V$. Here is an example where $\mathrm{Sing}(M) \neq \emptyset$:

Example 1. *Let $n > 1$ and $M = \omega_n^{-1}.\mathbb{1}_{\overline{B(0,1)}}$ where ω_n is the Lebesgue measure of the n-dimensional unit ball. Then, $\mathrm{Sing}(M) = B\left(0, \frac{n}{n-1}\right)^c$. Indeed, for $p = |p| \cdot e_1$, we have $\mu(p) = |p|$ and $v \cdot p = |p| v_1$ hence*

$$\int_V \frac{M(v)}{\mu(p) - v \cdot p} \mathrm{d}v = \frac{1}{|p|\omega_n} \int_{B(0,1)} \frac{1}{1 - v_1} \mathrm{d}v = \frac{\omega_{n-1}}{|p|\omega_n} \int_{-1}^{1} \frac{\left(1 - v_1^2\right)^{\frac{n-1}{2}}}{1 - v_1} \mathrm{d}v_1 = \frac{1}{|p|} \times \frac{n}{n-1}.$$

By rotational invariance, we conclude that $\mathrm{Sing}(M) = B\left(0, \frac{n}{n-1}\right)^c$. The Fig. 1 gives illustrations of the Hamiltonian and μ as functions of the radius of p, in the cases $n = 1$ and $n = 3$. In the cases $n = 3$ we can see the $\mathcal{C}^1$ singularity where $|p| = \frac{3}{2}$.

Proposition 2. *The following properties hold:*

(i) *The set $\mathrm{Sing}(M)^c$ is convex.*
(ii) *The function H is continuous and convex.*
(iii) *If $\mathrm{Sing}(M) \neq \emptyset$, then H is not $\mathcal{C}^1$. More precisely, ∇H has a jump discontinuity at $\partial \mathrm{Sing} M$.*

Proof. Let us first notice that μ is positively 1-homogeneous. Moreover, it is convex since it is a supremum of linear functions.

(i) Let $p, q \in \mathrm{Sing}(M)^c$ with $p \neq q$. Since μ is convex, we have for all $\tau \in [0,1]$

$$I(\tau) := \int_V \frac{M(v)}{\mu(p) - v \cdot p + \tau(\mu(q) - \mu(p) - v \cdot (q - p))} \mathrm{d}v \leq \int_V \frac{M(v)}{\mu((1-\tau)p + \tau q) - v \cdot ((1-\tau)p + \tau q)} \mathrm{d}v.$$

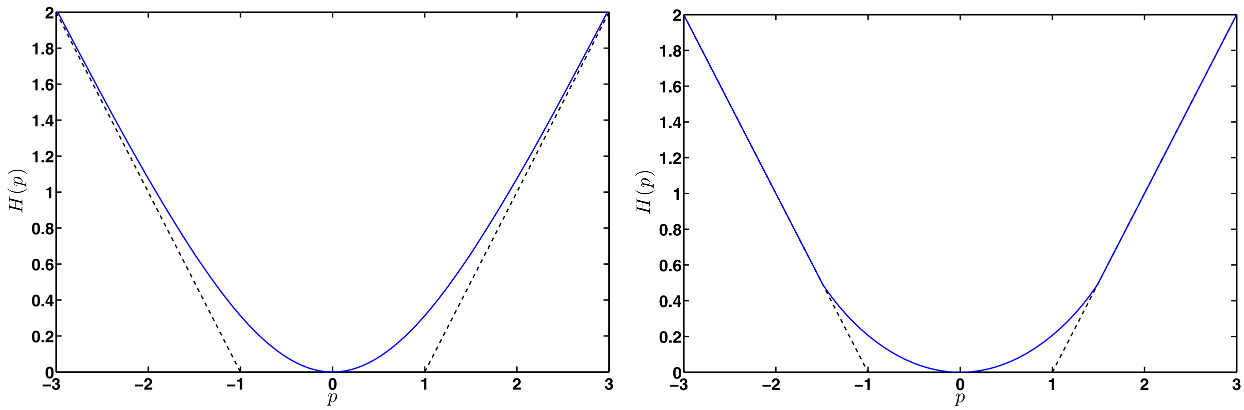

Fig. 1. Blue plain lines: Hamiltonian for $n=1,3$ and $M=\omega_n^{-1}.\mathbb{1}_{\overline{B(0,1)}}$. Black dotted lines: $|p|\mapsto\mu(p)-1$. Lignes pleines bleues : Hamiltonien pour $n=1,3$ et $M=\omega_n^{-1}.\mathbb{1}_{\overline{B(0,1)}}$. Lignes noires en pointillés : $|p|\mapsto\mu(p)-1$.

Moreover, $I(0), I(1)>1$ and I is differentiable on $[0,1]$ with

$$\partial_\tau I(\tau)=\int_V \frac{M(v)}{(\mu(p)-v\cdot p+\tau(\mu(q)-\mu(p)-v\cdot(q-p)))^2}(\mu(p)-\mu(q)-v\cdot(p-q))\,\mathrm{d}v.$$

It is clear that the sign of $\partial_\tau I$ does not change hence $I(\tau)>1$, which proves (i).

(ii) We refer to [2], section 1, to prove that H is twice differentiable and strictly convex on $\mathrm{Sing}(M)^c$ and that

$$\int_V \frac{M(v)}{(1+H(q)-v\cdot q)^2}(\nabla H(q)-v)\,\mathrm{d}v=0,\quad \forall q\in\mathrm{Sing}(M)^c. \tag{8}$$

In particular, $\nabla H(q)\in\mathrm{Conv}(V)$ for all $q\in\mathrm{Sing}(M)^c$. It is easy to see that H is continuous in the interior of $\mathrm{Sing}(M)$. To show continuity of H on $\partial\mathrm{Sing}(M)$, let $(p_m)_m$ converge to $p\in\partial\mathrm{Sing}(M)\subset\mathrm{Sing}(M)$. If we can extract a subsequence $\left(p_{m_l}\right)_l\subset\mathrm{Sing}(M)$, then $H\left(p_{m_l}\right)=\mu\left(p_{m_l}\right)-1\underset{l\to\infty}{\longrightarrow}\mu(p)-1=H(p)$. If not, then $p_m\in\mathrm{Sing}(M)^c$ for m large enough and $1=\int_V\frac{M(v)}{1+H(p_m)-v\cdot p_m}\mathrm{d}v<\int_V\frac{M(v)}{\mu(p_m)-v\cdot p_m}\mathrm{d}v$. Taking the limit, we get by dominated convergence $1=\int_V\lim\limits_{m\to\infty}\frac{M(v)}{1+H(p_m)-v\cdot p_m}\mathrm{d}v\le\int_V\frac{M(v)}{\mu(p)-v\cdot p}\mathrm{d}v\le 1$ hence $H(p_m)\underset{m\to\infty}{\longrightarrow}\mu(p)-1=H(p)$.

We now show that H is convex by proving that it is a maximum of convex functions:

$$H(p)=\max\left(\sup\left\{\nabla H(q)\cdot(p-q)+H(q)\mid q\in\mathrm{Sing}(M)^c\right\},\mu(p)-1\right),\quad\forall p\in\mathbb{R}^n. \tag{9}$$

In $\mathrm{Sing}(M)^c$, (9) holds by convexity of H and $H(p)>\mu(p)-1$. Let $p\in\mathrm{Sing}(M)$ and $q\in\mathrm{Sing}(M)^c$. By convexity of $\mathrm{Sing}(M)^c\ni 0$, there exists a unique $\lambda\in(0,1]$ such that $\lambda p\in\partial\mathrm{Sing}(M)$. For all $\tau\in[0,1]$, we set $\omega_1(\tau):=\mu(\tau p)-1=\tau\mu(p)-1$ and $\omega_2(\tau):=\nabla H(q)\cdot(\tau p-q)+H(q)$. By continuity of H, $\mu(\lambda p)-1=H(\lambda p)\ge\nabla H(q)(\lambda p-q)+H(q)$ hence $\omega_1(\lambda)\ge\omega_2(\lambda)$. Moreover, ω_1 and ω_2 are both differentiable and $\partial_\tau\omega_1(\tau)=\mu(p)\ge\nabla H(q)\cdot p=\partial_\tau\omega_2(\tau)$ since $\nabla H(q)\in\mathrm{Conv}(V)$. Hence, $\omega_1(1)\ge\omega_2(1)$, which ends the proof of (ii).

(iii) Suppose $\mathrm{Sing}(M)\ne\emptyset$ and H is $\mathcal{C}^1$. Since $H+1=\mu$ is positive homogeneous of degree 1 on $\mathrm{Sing}(M)$ and since $\lambda p\in\mathrm{Sing}(M)$ for all $\lambda\ge1$ and $p\in\mathrm{Sing}(M)$, we know that $\nabla H(p)\cdot p=H(p)+1=\mu(p)$ for all $p\in\mathrm{Sing}(M)\subset\mathrm{Sing}(M)$ hence $p\cdot(\nabla H(p)-v)\ge0$, for all $v\in V$, the inequality being strict on a neighborhood of 0. Then,

$$p\cdot\int_V\frac{M(v)}{(1+H(p)-v\cdot p)^2}(\nabla H(p)-v)\,\mathrm{d}v>0,\quad\forall p\in\partial\mathrm{Sing}(M). \tag{10}$$

By continuity, equations (8) and (10) are contradictory. □

3. Proof of Theorem 1

Let $\varphi_0\in W^{1,\infty}(\mathbb{R}^n)$. We refer to Proposition 2.1 in [2] to prove that the Cauchy Problem (4) with initial condition φ_0 has a unique solution $\varphi^\varepsilon\in W^{1,\infty}$ which is locally (in t) uniformly (in ε, x and v) bounded in norm $W^{1,\infty}$. In particular, let us mention that

$$0\le\varphi^\varepsilon(t,\cdot,\cdot)\le\|\varphi_0\|_\infty,\quad \left\|\nabla_v\varphi^\varepsilon(t,\cdot,\cdot)\right\|_\infty\le t\|\nabla_x\varphi_0\|_\infty. \tag{11}$$

Using the Arzelá–Ascoli theorem, we extract a locally uniformly converging subsequence. We denote by φ the limit. The function φ does not depend on v since $\int_V M(v)\, e^{\frac{\varphi^\varepsilon - \varphi'^\varepsilon}{\varepsilon}} \mathrm{d}v$ is uniformly bounded on $[0,T]\times\mathbb{R}^n\times V$ for all $T>0$. We use the perturbed test function method [7] to show that φ is a viscosity solution to (5). Theorem 1 will follow by uniqueness of the solution [6], thanks to the properties of H (see Proposition 2).

3.1. Subsolution procedure

Let $\psi \in \mathcal{C}^1(\mathbb{R}_+\times\mathbb{R}^n)$ be a test function such that $\varphi-\psi$ has a local strict maximum at (t^0,x^0). We want to show that ψ is a subsolution to (5). If $\nabla_x\psi(t^0,x^0)\in \mathrm{Sing}(M)^c$, then we refer to [2], section 2, step 2.

Suppose now that $\nabla_x\psi(t^0,x^0)\in\mathrm{Sing}(M)$. Let $w\in \mathrm{Arg}\mu(\nabla_x\psi(t^0,x^0))\cap V$. Then, $w\cdot\nabla_x\psi(t^0,x^0)=\mu(\nabla_x\psi(t^0,x^0))$. The uniform convergence of φ^ε toward φ ensures that the function $(t,x)\mapsto\varphi^\varepsilon(t,x,w)-\psi(t,x)$ has a local maximum at a point $(t^\varepsilon,x^\varepsilon)$ satisfying $(t^\varepsilon,x^\varepsilon)\to(t^0,x^0)$, as $\varepsilon\to0$. We then have:

$$\partial_t\psi(t^\varepsilon,x^\varepsilon)+w\cdot\nabla_x\psi(t^\varepsilon,x^\varepsilon)=\partial_t\varphi^\varepsilon(t^\varepsilon,x^\varepsilon)+w\cdot\nabla_x\varphi^\varepsilon(t^\varepsilon,x^\varepsilon)=1-\int_V M(v')\,e^{\frac{\varphi^\varepsilon(t^\varepsilon,x^\varepsilon,w)-\varphi^\varepsilon(t^\varepsilon,x^\varepsilon,v')}{\varepsilon}}\mathrm{d}v'\le 1.$$

Passing to the limit $\varepsilon\to0$, we get $\partial_t\psi(t^0,x^0)+\mu(\nabla_x\psi(t^0,x^0))\le1$. We conclude that φ is a viscosity subsolution to (5).

3.2. Supersolution procedure

Let $\psi \in \mathcal{C}^1(\mathbb{R}_+\times\mathbb{R}^n)$ be a test function such that $\varphi-\psi$ has a local strict minimum at (t^0,x^0). We want to show that ψ is a supersolution to (5). If $\nabla_x\psi(t^0,x^0)\in \mathrm{Sing}(M)^c$, then we refer to [2], section 2, step 2.

Suppose now that $\nabla_x\psi(t^0,x^0)\in\mathrm{Sing}(M)$. Then, $\nabla_x\psi(t^0,x^0)\neq0$ because $0\in\mathrm{Sing}(M)^c$. We suppose without loss of generality that the minimum of $\varphi-\psi$ is global and that $\varphi(t^0,x^0)-\psi(t^0,x^0)=0$. Let $\psi^\varepsilon:=\psi-C(t-t^0)^2+\varepsilon\eta$ with $C>0$ yet to be determined and

$$\eta(v):=\ln\left(\mu\left(\nabla_x\psi\left(t^0,x^0\right)\right)-v\cdot\nabla_x\psi\left(t^0,x^0\right)\right).$$

Then, η is a continuous function on $D(\eta)=V\setminus\mathrm{Arg}\mu(\nabla_x\psi(t^0,x^0))$ and, for all $w\in\mathrm{Arg}\mu(\nabla_x\psi(t^0,x^0))\cap V$, we have $\lim_{v\to w}\eta(v)=-\infty$. Moreover, η is bounded from below on all compact sets yielding the uniform convergence $\psi^\varepsilon\to\psi$ on all compact sets of $D(\eta)$. Finally, $\int_V M(v')\,\mathrm{e}^{-\eta(v')}\mathrm{d}v'\le1$ since $\nabla_x\psi(t^0,x^0)\in\mathrm{Sing}(M)$.

The function $\varphi-\left(\psi-C(t-t^0)^2\right)$ has a global strict minimum at (t^0,x^0). The first inequality (11) ensures that the function $\varphi^\varepsilon-\psi^\varepsilon$ has a local minimum at a point $(t^\varepsilon,x^\varepsilon,v^\varepsilon)\in\mathbb{R}_+\times\mathbb{R}^n\times D(\eta)$. As V compact, we can extract a subsequence $(v^\varepsilon)_\varepsilon$, without relabeling, such that $v^\varepsilon\to v^0$, as $\varepsilon\to0$.

If $v^0\in V\setminus\mathrm{Arg}\mu(p)$, then there exists a compact $A\subset D(\eta)$ such that $v^0\in A$ and the uniform convergence of ψ^ε towards ψ on A guarantees that $(t^\varepsilon,x^\varepsilon)\to(t^0,x^0)$, as $\varepsilon\to0$. We then get at point $(t^\varepsilon,x^\varepsilon,v^\varepsilon)$,

$$\begin{aligned}\partial_t\psi-2C\left(t^\varepsilon-t^0\right)+v^\varepsilon\cdot\nabla_x\psi=\partial_t\psi^\varepsilon+v^\varepsilon\cdot\nabla_x\psi^\varepsilon=\partial_t\varphi^\varepsilon+v^\varepsilon\cdot\nabla_x\varphi^\varepsilon&=1-\int_V M'\mathrm{e}^{\frac{\varphi^\varepsilon-\varphi'^\varepsilon}{\varepsilon}}\mathrm{d}v'\\&\ge1-\int_V M(v')\,\mathrm{e}^{\eta(v^\varepsilon)-\eta(v')}\mathrm{d}v'.\end{aligned}$$

We take the limit $\varepsilon\to0$:

$$\partial_t\psi\left(t^0,x^0\right)+v^0\cdot\nabla_x\psi\left(t^0,x^0\right)\ge1-\mathrm{e}^{\eta(v^0)}\int_V M(v')\,\mathrm{e}^{-\eta(v')}\mathrm{d}v'\ge1-\mathrm{e}^{\eta(v^0)}.$$

By construction, for all $v,v'\in D(\eta)$, we have $\mathrm{e}^{\eta(v)}-\mathrm{e}^{\eta(v')}=(v'-v)\cdot\nabla_x\psi(t^0,x^0)$ hence, for all $v\in D(\eta)$,we have $\partial_t\psi(t^0,x^0)+v\cdot\nabla_x\psi(t^0,x^0)\ge1-\mathrm{e}^{\eta(v)}$. Let $w\in V\cap\mathrm{Arg}\mu(\nabla_x\psi(t^0,x^0))$. Since $\mathrm{Arg}\mu(\nabla_x\psi(t^0,x^0))$ is a null-set, V is dense in $\mathrm{Arg}\mu(\nabla_x\psi(t^0,x^0))$. Taking the limit $v\to w$, we get: $\partial_t\psi(t^0,x^0)+\mu(\nabla_x\psi(t^0,x^0))\ge1$.

If $v^0\in V\cap\mathrm{Arg}\mu(p)$, we still have $(t^\varepsilon,x^\varepsilon)\underset{\varepsilon\to0}{\longrightarrow}(t^0,x^0)$ thanks to the following lemma:

Lemma 3. *For $C=4\|\varphi_0\|_\infty$, we have $\lim_{\varepsilon\to0}\varepsilon\eta(v^\varepsilon)=0$.*

Proof of Lemma 3. We have $\varphi^{\varepsilon}(t,x,v)-\varphi(t,x)\geq -2\|\varphi_0\|_{\infty}$ by (11) and $\varphi(t,x)-\psi(t,x)\geq 0$ hence

$$\varphi^{\varepsilon}(t,x,v)-\psi^{\varepsilon}(t,x,v)\geq -2\|\varphi_0\|_{\infty}+C\left(t-t^0\right)^2-\varepsilon\eta(v),\quad \forall\varepsilon>0.$$

Moreover,

$$\varphi^{\varepsilon}\left(t^0,x^0,v\right)-\psi^{\varepsilon}\left(t^0,x^0,v\right)=\varphi^{\varepsilon}\left(t^0,x^0,v\right)-\varphi\left(t^0,x^0\right)-\varepsilon\eta(v)\leq 2\|\varphi_0\|_{\infty}-\varepsilon\eta(v).$$

Since $C=4\|\varphi_0\|_{\infty}$, we have $\varphi^{\varepsilon}(t,x,v)-\psi^{\varepsilon}(t,x,v)>\varphi^{\varepsilon}\left(t^0,x^0,v\right)-\psi^{\varepsilon}\left(t^0,x^0,v\right)$ for all $t>t^0+1$ and, thus, the minimum of $\varphi^{\varepsilon}-\psi^{\varepsilon}$ cannot be attained for $t>t^0+1$ hence $t^{\varepsilon}\leq t^0+1$ for all $\varepsilon>0$. At point $\left(t^{\varepsilon},x^{\varepsilon},v^{\varepsilon}\right)$ we have:

$$\nabla_v\varphi^{\varepsilon}\left(t^{\varepsilon},x^{\varepsilon},v^{\varepsilon}\right)=\nabla_v\psi^{\varepsilon}\left(t^{\varepsilon},x^{\varepsilon},v^{\varepsilon}\right)=\varepsilon\nabla_v\eta\left(v^{\varepsilon}\right)=-\frac{\varepsilon\nabla_x\psi\left(t^0,x^0\right)}{\mu\left(\nabla_x\psi\left(t^0,x^0\right)\right)-v^{\varepsilon}\cdot\nabla_x\psi\left(t^0,x^0\right)}.$$

The second estimation (11) yields $\left\|\nabla_v\varphi^{\varepsilon}\left(t^{\varepsilon},\cdot,\cdot\right)\right\|_{\infty}\leq t^{\varepsilon}\|\nabla_x\varphi_0\|_{\infty}\leq\left(t^0+1\right)\|\nabla_x\varphi_0\|_{\infty}$ hence

$$\frac{\varepsilon}{\left(t^0+1\right)\|\nabla_x\varphi_0\|_{\infty}}\left|\nabla_x\psi\left(t^0,x^0\right)\right|\leq\mu\left(\nabla_x\psi\left(t^0,x^0\right)\right)-v^{\varepsilon}\cdot\nabla_x\psi\left(t^0,x^0\right),$$

$$\Longrightarrow\ \varepsilon K\geq\varepsilon\eta\left(v^{\varepsilon}\right)\geq\varepsilon\ln\left(\frac{\varepsilon}{\left(t^0+1\right)\|\nabla_x\varphi_0\|_{\infty}}\left|\nabla_x\psi\left(t^0,x^0\right)\right|\right),$$

and $\varepsilon\eta\left(v^{\varepsilon}\right)\to 0$ as $\varepsilon\to 0$. □

Thanks to Lemma 3, the function $(t,x)\mapsto\psi^{\varepsilon}\left(t,x,v^{\varepsilon}\right)=\psi(t,x)-4\|\varphi_0\|_{\infty}\left(t-t^0\right)^2+\varepsilon\eta\left(v^{\varepsilon}\right)$ converges uniformly towards $(t,x)\mapsto\psi(t,x)-4\|\varphi_0\|_{\infty}\left(t-t^0\right)^2$ and has a local minimum at $\left(t^{\varepsilon},x^{\varepsilon}\right)$ satisfying $\left(t^{\varepsilon},x^{\varepsilon}\right)\to\left(t^0,x^0\right)$, as $\varepsilon\to 0$. At point $\left(t^{\varepsilon},x^{\varepsilon},v^{\varepsilon}\right)$, we have:

$$\partial_t\psi^{\varepsilon}+v^{\varepsilon}\cdot\nabla_x\psi^{\varepsilon}=\partial_t\varphi^{\varepsilon}+v^{\varepsilon}\cdot\nabla_x\varphi^{\varepsilon}=1-\int_V M\left(v'\right)e^{\frac{\varphi^{\varepsilon}\left(t^{\varepsilon},x^{\varepsilon},v^{\varepsilon}\right)-\varphi^{\varepsilon}\left(t^{\varepsilon},x^{\varepsilon},v'\right)}{\varepsilon}}\mathrm{d}v'.$$

The minimal property of $\left(t^{\varepsilon},x^{\varepsilon},v^{\varepsilon}\right)$ implies at this point:

$$\begin{aligned}\partial_t\psi\left(t^{\varepsilon},x^{\varepsilon}\right)-8\|\varphi_0\|_{\infty}\left(t^{\varepsilon}-t^0\right)+v^{\varepsilon}\cdot\nabla_x\psi\left(t^{\varepsilon},x^{\varepsilon}\right)=\partial_t\psi^{\varepsilon}+v^{\varepsilon}\cdot\nabla_x\psi^{\varepsilon}&\geq 1-\int_V M\left(v'\right)e^{\eta\left(v^{\varepsilon}\right)-\eta\left(v'\right)}\mathrm{d}v'\\&\geq 1-e^{\eta\left(v^{\varepsilon}\right)}.\end{aligned}$$

Passing to the limit $\varepsilon\to 0$, we get $\partial_t\psi\left(t^0,x^0\right)+\mu\left(\nabla_x\psi\left(t^0,x^0\right)\right)\geq 1$. We conclude that φ is a viscosity supersolution to (5). □

Acknowledgements

This project has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No 639638).

The author also wishes to thank Vincent Calvez and Julien Vovelle for their kind help.

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