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Harmonic analysis

Compactness for the weighted Hardy operator in variable exponent spaces

Compacité de l'opérateur de Hardy pondéré entre espaces d'exposant variable

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A R T I C L E I N F O

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A B S T R A C T

In this paper, we prove a necessary and sufficiency condition for the weighted Hardy operator

$$H_{\upsilon,\omega} f(x) = \upsilon(x) \int_0^x f(t)\,\omega(t)\,\mathrm{d}t$$

to be compactly acting from $L^{p(\cdot)}(0,\infty)$ to $L^{q(\cdot)}(0,\infty)$.

R É S U M É

Dans cette Note, nous prouvons une condition nécessaire et suffisante pour que l'opérateur de Hardy pondéré

$$H_{\upsilon,\omega} f(x) = \upsilon(x) \int_0^x f(t)\,\omega(t)\,\mathrm{d}t$$

agisse de façon compacte de $L^{p(\cdot)}(0,\infty)$ dans $L^{q(\cdot)}(0,\infty)$.

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1. Introduction

A differential equation involving a $p(x)$-growth condition arises from modeling of electrorheological fluids and has been the subject of various investigations, such as a study of boundedness problems for classical integral operators in variable Lebesgue spaces and a progress in the regularity theory of the nonlinear partial differential equations with nonstandard growth condition. In connection, we refer to the monographs [6,9,12,30,19,17] and papers like [1,27,5,35,39,4,32].

A boundedness problem for the weighted Hardy operator was studied in variable exponent Lebesgue spaces $L^{p(\cdot)}$ in [36, 37,13,15,16,18,20,21,28,33,3]. A case of general weight functions and log-regularity assumption on the exponent functions was studied in the works [7,8,2,24,22], where a necessity and sufficiency condition was obtained. Separate necessity and sufficiency conditions have been proved in [13], not applying the log-regularity condition.

In recent works [23,25,26], a study was started of a necessity and sufficiency condition for the boundedness of Hardy's operator which does not use a regularity condition on the exponent function. In particular, in [23], a monotone exponent function $p:(0,l)\to(1,\infty)$ was characterized, such that the Hardy operator be bounded in $L^{p(\cdot)}(0,l)$.

In cases different from boundedness, the compactness problem for the Hardy operator was little studied. In order to study the compactness problem for potential type integral operators in variable exponent Lebesgue spaces, we refer the reader to [29,13,12,38,11].

In this paper, we establish a necessary and sufficient condition on $\upsilon(\cdot)$, $\omega(\cdot)$ and exponent functions $p(\cdot)$, $q(\cdot)$ governing the compactness of the weighted Hardy operator

$$H_{\upsilon,\omega}f(x)=\upsilon(x)\int_0^x f(t)\,\omega(t)\,\mathrm{d}t$$

from space $L^{p(x)}(0,\infty)$ into $L^{q(x)}(0,\infty)$.

2. Auxiliary assertions, notation

To prove our main results, we need some auxiliary results. The following general assertions on compact operators are well known (see, e.g., in [10,34]).

Theorem 2.1. *Let $T\in L(X,Y)$ be a compact operator. Then T maps a weakly convergent sequence in X to the strongly convergent sequence in Y.*

Theorem 2.2. *Suppose X, Y are Banach spaces. If $\{T_n\}:X\to Y$ is a sequence of compact operators in $L(X,Y)$ and $\| T_n-T\|_{X\to Y}\to 0$ for some $T\in L(X,Y)$, then T is compact.*

We need the following assertion due to Schauder.

Theorem 2.3. *Suppose X, Y are Banach spaces. A bounded linear operator $T:X\to Y$ is compact if and only if its adjoint $T^*:Y^*\to X^*$ is compact.*

We need also the following assertion on equivalent conditions related to Hardy's operator (see, for example, [14,31]).

Theorem 2.4. *For $-\infty\le a<b\le\infty$, α, β and s positive numbers and f, g measurable functions positive a.e. in (a,b), let*

$$F(x)=\int_x^b \phi(t)\,\mathrm{d}t;\quad G(x)=\int_a^x g(t)\,\mathrm{d}t$$

and

$$B_1(x;\alpha,\beta):=F(x)^\alpha G(x)^\beta,$$

$$B_2(x;\alpha,\beta,s):=\left(\int_x^b \phi(t)G(t)^{\frac{\beta-s}{\alpha}}\,\mathrm{d}t\right)^\alpha G(x)^s,$$

$$B_3(x;\alpha,\beta,s):=\left(\int_a^x g(t)F(t)^{\frac{\alpha-s}{\beta}}\,\mathrm{d}t\right)^\beta F(x)^s,$$

$$B_4(x;\alpha,\beta,s) := \left(\int_a^x \phi(t)G(t)^{\frac{\beta+s}{\alpha}}\,\mathrm{d}t\right)^{\beta} G(x)^{-s},$$

$$B_5(x;\alpha,\beta,s) := \left(\int_x^b g(t)F(t)^{\frac{\alpha+s}{\beta}}\,\mathrm{d}t\right)^{\beta} F(x)^{-s}.$$

The numbers

$$B_1 := \sup_{a<x<b} B_1(x;\alpha,\beta) \ \textit{and}\ B_i = \sup_{a<x<b} B_i(x;\alpha,\beta,s), \quad i=2,3,4,5,$$

are mutually equivalent. The constants in the equivalence relation can depend on α, β and s.

Denote by χ_E characteristic function of the set $E \subset \mathbb{R}$.

Let $r:(0,\infty)\to(1,\infty)$ be a measurable function on the interval $(0,\infty)$. We define the space $L^{r(\cdot)}(0,\infty)$ as consisting of all measurable functions $f:(0,\infty)\to\mathbb{R}$ such that the modular

$$\rho_{r(\cdot)}(f) = \int_0^\infty |f(x)|^{r(x)}\mathrm{d}x$$

is finite. If $r^+ = \operatorname*{ess\,sup}_{x\in(0,\infty)} r(x) < \infty$, then

$$\| f \|_{L^{r(\cdot)}(0,\infty)} = \inf\left\{\lambda>0: \quad \rho_{r(\cdot)}\left(\frac{f}{\lambda}\right) < \infty\right\}$$

defines a norm on $L^{r(\cdot)}(0,\infty)$.

In the study of the compactness of the weighted Hardy operator

$$H_{\upsilon,\omega}: L^{p(x)}(0,\infty) \to L^{q(x)}(0,\infty)$$

we shall use the exponent functions $p,q:(0,\infty)\to(1,\infty)$ and the weight functions υ, ω assuming them to be measurable and to have non-negative finite values almost everywhere in $(0,\infty)$. Concerning these functions, we assume the summability properties

$$\omega(x)^{p'(x)} \in L^1(0,a), \quad \upsilon(x)^{q(x)} \in L^1(a,\infty) \tag{2.1}$$

for any $a>0$. Also these functions are assumed to verify

$$\lim_{x\to+0} V(x) = \lim_{x\to+\infty} W(x) = \infty,$$

where

$$V(x) = \int_x^\infty \upsilon(y)^{q(y)}\mathrm{d}y, \quad W(x) = \int_0^x \omega(y)^{p'(y)}\mathrm{d}y.$$

Let V, W be the above functions. Denote by Λ_0 the class of measurable functions $y:(0,\infty)\to\mathbb{R}$ such that there exists a number $y(0)\in\mathbb{R}$ and $\exists\delta>0$:

$$\sup_{x\in(0,\delta)} |y(x)-y(0)|\ln\frac{1}{W(x)} < \infty. \tag{2.2}$$

Denote by Λ_∞ the class of functions $y:(0,\infty)\to\mathbb{R}$ such that there exists a number $y(\infty)\in\mathbb{R}$ and $\exists\rho>0$:

$$\sup_{x\in(\rho,\infty)} |y(x)-y(\infty)|\ln\frac{1}{V(x)} < \infty. \tag{2.3}$$

The above introduced conditions (2.2) and (2.3) are crucial in the proof of Lemmas 2.7–2.12, which, in turn, are essentially in the proof of our main result (Theorem 3.1). Note that an estimate of the exponential term $x^{p(x)}$ through $x^{p(0)}$ from upper and below is well known (see, e.g., [8,18]). In its proof, a log condition is used. The meaning of our Lemmas is that such estimates may be valid also between different exponential terms (where not necessarily the same argument is taken).

The following result is taken from [24] (see, also [7,21]).

Theorem 2.5. *Let $p, q \in \Lambda_0 \cap \Lambda_\infty$ be measurable functions such that*

$$1 < p^-, q^-, p^+, q^+ < \infty \text{ and } q(0) \geq p(0), \ \ q(\infty) \geq p(\infty).$$

Then the inequality

$$\|H_{\upsilon,\omega} f(\cdot)\|_{L^{q(\cdot)}(0,\infty)} \leq C_1 \|f(\cdot)\|_{L^{p(\cdot)}(0,\infty)} \tag{2.4}$$

holds for every measurable function f if and only if

$$B_\delta = \sup_{0<t<\delta} V(t)^{\frac{1}{q(0)}} W(t)^{\frac{1}{p'(0)}} < \infty \tag{2.5}$$

and

$$C_\rho = \sup_{\rho<t<\infty} V(t)^{\frac{1}{q(\infty)}} W(t)^{\frac{1}{p'(\infty)}} < \infty \tag{2.6}$$

for some $0 < \delta < \rho < \infty$.

Remark 2.6. It is not difficult to see from the proof [24] that

$$C_1 = O(B_\delta + C_\rho) \text{ as } \delta \to 0, \text{ and } \rho \to \infty. \tag{2.7}$$

In this paper, we prove also the following auxiliary assertions.

Lemma 2.7. *Let $W(a) \leq 1$. Then it follows that*

$$W(t)^{-\frac{1}{p(s)}} \geq \frac{1}{C} W(t)^{-\frac{1}{p(x)}}, \ \ 0 < s < x < t < a.$$

Proof. Using the condition (2.2) for the exponent $p(\cdot)$ it follows that

$$\begin{aligned} W(t)^{\frac{1}{p(s)}} &= W(t)^{\frac{1}{p(x)}} W(t)^{\frac{1}{p(s)} - \frac{1}{p(x)}} \\ &= W(t)^{\frac{1}{p(x)}} W(t)^{\frac{1}{p(s)} - \frac{1}{p(0)}} W(t)^{\frac{1}{p(0)} - \frac{1}{p(x)}} \\ &\leq W(t)^{\frac{1}{p(x)}} \left(\frac{1}{W(t)} \right)^{\frac{C}{\ln \frac{1}{W(s)}}} \left(\frac{1}{W(t)} \right)^{\frac{C}{\ln \frac{1}{W(x)}}} \\ &\leq W(t)^{\frac{1}{p(x)}} \left(\frac{1}{W(t)} \right)^{\frac{C}{\ln \frac{1}{W(t)}}} \\ &= C_1 W(t)^{\frac{1}{p(x)}}. \end{aligned}$$

This completes the proof of Lemma 2.7. □

Lemma 2.8. *Let $W(a) \leq 1$. Then it follows that*

$$W(t)^{-\frac{q(x)}{p(x)}} \geq \frac{1}{C} W(t)^{-\frac{q(0)}{p(0)}}, \ \ 0 < x < t < a.$$

Proof. By using condition (2.2) for the functions $q(\cdot)$, $p(\cdot)$ it follows that

$$\begin{aligned} W(t)^{\frac{q(x)}{p(x)}} &= W(t)^{\frac{q(0)}{p(0)}} W(t)^{\frac{q(x)}{p(x)} - \frac{q(0)}{p(0)}} \\ &\leq W(t)^{\frac{q(0)}{p(0)}} \left(\frac{1}{W(t)} \right)^{\frac{q(0)}{p(0)} - \frac{q(x)}{p(x)}} \\ &\leq C^{-\frac{C}{\log \frac{1}{W(x)}}} W(t)^{\frac{q(0)}{p(0)}} \left(\frac{C}{W(t)} \right)^{\frac{C}{\log \frac{1}{W(x)}}} \\ &\leq C_1 W(t)^{\frac{q(0)}{p(0)}} \left(\frac{1}{W(x)} \right)^{\frac{C}{\log \frac{1}{W(x)}}} \\ &= C_2 W(t)^{\frac{q(0)}{p(0)}} \end{aligned}$$

since $W(x) \leq W(t)$.

This completes the proof of Lemma 2.8. □

Lemma 2.9. *Let $W(a) \le 1$. Then it follows that*

$$W(x)^{q(x)} \ge \frac{1}{C} W(x)^{q(0)}, \quad 0 < x < a.$$

Proof. From (2.2) for the function $q(\cdot)$, it follows that

$$\begin{aligned} W(x)^{q(x)} &= W(x)^{q(0)} W(x)^{q(x)-q(0)} \\ &\ge W(x)^{q(0)} W(x)^{\frac{C}{\log \frac{1}{W(x)}}} \\ &= \frac{1}{C} W(x)^{q(0)}. \end{aligned}$$

This completes the proof of Lemma 2.9. □

We also use the following simple assertions.

Lemma 2.10. *Let $V(b) \le 1$. Then it follows that*

$$V(t)^{-\frac{1}{q'(s)}} \ge \frac{1}{C} V(t)^{-\frac{1}{q'(x)}}, \quad b < t < x < s < \infty.$$

Proof. Using condition (2.3), it follows that

$$\begin{aligned} V(t)^{-\frac{1}{q'(s)}} &= V(t)^{\frac{1}{q'(x)}} V(t)^{\frac{1}{q'(s)} - \frac{1}{q'(x)}} \\ &= V(t)^{\frac{1}{q'(x)}} V(t)^{\frac{1}{q'(s)} - \frac{1}{q'(0)}} V(t)^{\frac{1}{q'(0)} - \frac{1}{q'(x)}} \\ &\le V(t)^{\frac{1}{q'(x)}} \left(\frac{1}{V(t)}\right)^{\frac{C}{\ln \frac{1}{V(s)}}} \left(\frac{1}{V(t)}\right)^{\frac{C}{\ln \frac{1}{V(x)}}} \\ &\le V(t)^{\frac{1}{q'(x)}} \left(\frac{1}{V(t)}\right)^{\frac{2C}{\ln \frac{1}{V(t)}}} \\ &= C_1 V(t)^{\frac{1}{q'(x)}}. \end{aligned}$$

This completes the proof of Lemma 2.10. □

Lemma 2.11. *Let $V(b) \le 1$. Then it follows that*

$$V(t)^{-\frac{p'(x)}{q'(x)}} \ge \frac{1}{C} V(t)^{-\frac{p'(\infty)}{q'(\infty)}}, \quad b < t < x < \infty. \tag{2.8}$$

Proof. From (2.3) for the functions $p(\cdot)$, $q(\cdot)$, it follows that

$$\begin{aligned} V(t)^{-\frac{p'(x)}{q'(x)}} &= V(t)^{-\frac{p'(\infty)}{q'(\infty)}} V(t)^{\frac{p'(\infty)}{q'(\infty)} - \frac{p'(x)}{q'(x)}} \\ &\ge V(t)^{-\frac{p'(\infty)}{q'(\infty)}} V(t)^{\frac{C}{\ln \frac{1}{V(x)}}} = \frac{1}{C} V(t)^{-\frac{p'(\infty)}{q'(\infty)}}. \quad \square \end{aligned} \tag{2.9}$$

Lemma 2.12. *Let $V(b) \le 1$. Then it follows that*

$$V(x)^{p'(x)} \ge \frac{1}{C} V(x)^{p'(\infty)}, \quad x > b.$$

Proof. From (2.3) it follows that

$$\begin{aligned} V(x)^{p'(x)} &= V(x)^{p'(\infty)} V(x)^{p'(x)-p'(\infty)} \\ &\ge V(x)^{p'(\infty)} V(x)^{\frac{C}{\log \frac{1}{V(x)}}} \\ &= \frac{1}{C} V(x)^{p'(\infty)}. \quad \square \end{aligned}$$

3. Main result

The main result of this paper is the following assertion.

Theorem 3.1. *Let $p, q \in \Lambda_0 \cap \Lambda_\infty$ be measurable functions such that*

$$1 < p^-, q^-, p^+, q^+ < \infty \text{ and } q(0) \geq p(0), \quad q(\infty) \geq p(\infty).$$

Then operator $H_{\upsilon,\omega}$ is compact from $L^{p(\cdot)}(0,\infty)$ to $L^{q(\cdot)}(0,\infty)$ iff

$$\lim_{a\to 0} B_a = 0, \qquad \text{where} \qquad B_a = \sup_{0<t<a} V(t)^{\frac{1}{q(0)}} W(t)^{\frac{1}{p'(0)}} \tag{3.1}$$

and

$$\lim_{b\to\infty} C_b = 0, \qquad \text{where} \qquad C_b = \sup_{b<t<\infty} V(t)^{\frac{1}{q(\infty)}} W(t)^{\frac{1}{p'(\infty)}}. \tag{3.2}$$

Proof. *Sufficiency.* Let f be a function from space $L^{p(\cdot)}(0,\infty)$. Following [13], for $0 < a < 1 < b < \infty$ set

$$P_1 f(x) = \chi_{0,a}(x)\upsilon(x) \int_0^x f(t)\,\omega(t)\,\mathrm{d}t,$$

$$P_2 f(x) = \chi_{a,b}(x)\upsilon(x) \int_0^a f(t)\,\omega(t)\,\mathrm{d}t,$$

$$P_3 f(x) = \chi_{a,b}(x)\upsilon(x) \int_a^x f(t)\,\omega(t)\,\mathrm{d}t,$$

$$P_4 f(x) = \chi_{b,\infty}(x)\upsilon(x) \int_0^b f(t)\,\omega(t)\,\mathrm{d}t,$$

$$P_5 f(x) = \chi_{b,\infty}(x)\upsilon(x) \int_b^x f(t)\,\omega(t)\,\mathrm{d}t.$$

Then

$$H_{\upsilon,\omega} f(x) = \sum_{i=1}^{5} P_i f(x).$$

Taking into account Lemma 2 from [11], we find that P_3 is a norm limit of a sequence of finite rank operators, while P_2 and P_4 are finite-rank operators. Now using Theorem 2.5, Remark 2.6, conditions (2.1), and $\lim_{a\to 0} B_a = 0$, $\lim_{b\to\infty} C_b = 0$, it follows that

$$\|P_1 f(x)\|_{L^{q(\cdot)}(0,\infty)} = \left\| \upsilon(x) \int_0^x f(t)\,\omega(t)\,\mathrm{d}t \right\|_{L^{q(\cdot)}(0,a)} \leq O(B_a)\|f\|_{L^{p(\cdot)}(0,a)} \to 0 \tag{3.3}$$

as $a \to 0$ and

$$\|P_5 f(x)\|_{L^{q(\cdot)}(0,\infty)} = \left\| \upsilon(x) \int_b^x f(t)\,\omega(t)\,\mathrm{d}t \right\|_{L^{q(\cdot)}(b,\infty)} \leq O(C_b)\|f\|_{L^{p(\cdot)}(b,\infty)} \to 0 \tag{3.4}$$

as $b \to \infty$. Hence

$$\begin{aligned} \| H_{\upsilon,\omega} f - P_2 - P_3 - P_4 \|_{L^{p(\cdot)}\to L^{q(\cdot)}} &\leq \|P_1\| + \|P_5\|_{L^{p(\cdot)}\to L^{q(\cdot)}} \\ &= O\,(B_a) + O\,(C_b) \to 0 \text{ as } a \to 0,\ b \to \infty. \end{aligned}$$

This and Theorem 2.2 complete the proof of the sufficiency part of Theorem 3.1.

Necessity. Consider the family of test functions

$$f_t(x)=\left(\int_0^t \omega(\tau)^{p'(\tau)}\mathrm{d}\tau\right)^{-\frac{1}{p(x)}} \chi_{(0,t)}(x)\,\omega(x)^{p'(x)-1},\quad t>0.$$

It follows that

$$\rho_{p(\cdot)}(f_t)=$$
$$=\int_0^\infty\left(\left(\int_0^t \omega(\tau)^{p'(\tau)}\mathrm{d}\tau\right)^{-\frac{1}{p(x)}}\chi_{(0,t)}(x)\omega(x)^{p'(x)-1}\right)^{p(x)}\mathrm{d}x$$
$$=\left(\int_0^t \omega(x)^{p'(x)}\mathrm{d}x\right)\left(\int_0^t \omega(\tau)^{p'(\tau)}\mathrm{d}\tau\right)^{-1}=1.$$

Therefore,

$$\rho_{p(\cdot)}(f_t)\le 1.$$

It follows from the elementary properties of the variable exponent norm (see, e.g., [6,9]) that

$$\|f_t(x)\|_{L^{p(\cdot)}(0,\infty)}\le 1.$$

Therefore, and by Holder's inequality, we have:

$$\left|\int_0^\infty f_t(x)\varphi(x)\mathrm{d}x\right|\le k(p)\,\|\,f_1(\cdot)\,\|_{L^{p(\cdot)}(0,\infty)}\,\|\chi_{(0,t)}(\cdot)\varphi(\cdot)\|_{L^{p'(\cdot)}(0,\infty)}\to 0$$

as $t\to 0$ for all $\varphi\in L^{p'(\cdot)}(0,\infty)$. Since $L^{p'(\cdot)}(0,\infty)$ is the conjugate space for $L^{p(\cdot)}(0,\infty)$, it follows that the sequence $\{f_t\}$ converges weakly in $L^{p(\cdot)}(0,\infty)$ to 0 as $t\to 0$.

Now, by the compactness hypothesis of the operator $H_{\upsilon,\omega}$ and Theorem 2.1, it follows that the sequence $\{H_{\upsilon,\omega}f_t\}$ converges to 0 in the norm of $L^{q(\cdot)}(0,\infty)$. Therefore,

$$\rho_{q(\cdot)}(H_{\upsilon,\omega}f_t)\to 0 \text{ as } t\to 0. \tag{3.5}$$

On the other hand,

$$\rho_{q(\cdot)}(H_{\upsilon,\omega}f_t)=$$
$$=\int_0^\infty\left(\upsilon(x)\int_0^x\left(\int_0^t \omega(\tau)^{p'(\tau)}\mathrm{d}\tau\right)^{-\frac{1}{p(s)}}\chi_{(0,t)}(s)\omega(s)^{p'(s)-1}\,\omega(s)\,\mathrm{d}s\right)^{q(x)}\mathrm{d}x$$
$$\ge\int_0^t \upsilon(x)^{q(x)}\left(\int_0^x\left(\int_0^t \omega(\tau)^{p'(\tau)}\mathrm{d}\tau\right)^{-\frac{1}{p(s)}}\omega(s)^{p'(s)}\mathrm{d}s\right)^{q(x)}\mathrm{d}x \tag{3.6}$$

(by Lemma 2.7),

$$\ge (2C)^{-q^+}\int_0^t \upsilon(x)^{q(x)}W(x)^{q(x)}W(t)^{-\frac{q(x)}{p(x)}}\mathrm{d}x$$

(by Lemma 2.8),

$$\ge\frac{2^{-q^+}}{C_1}W(t)^{-\frac{q(0)}{p(0)}}\int_0^t \upsilon(x)^{q(x)}W(x)^{q(x)}\mathrm{d}x$$

(by Lemma 2.9),

$$\geq \frac{1}{C_2}\left[W(t)^{-\frac{1}{p(0)}}\left(\int_0^t \upsilon(x)^{q(x)}W(x)^{q(0)}\mathrm{d}x\right)^{\frac{1}{q(0)}}\right]^{q(0)}$$

(by Theorem 2.4),

$$\geq C_3\left[V(t)^{\frac{1}{q(0)}}W(t)^{\frac{1}{p'(0)}}\right]^{q(0)}. \tag{3.7}$$

From this inequality and (3.5) it follows that

$$V(t)^{\frac{1}{q(0)}}W(t)^{\frac{1}{p'(0)}} \to 0 \text{ as } t\to 0. \tag{3.8}$$

Now, from (3.8) it follows that $B_a \to 0$ as $a\to 0$.

Notice, in the proof of inequality (3.7), we have applied Theorem 2.4 under setting

$$F(t) := V(t) = \int_t^\infty \upsilon(x)^{q(x)}\mathrm{d}x; \quad G(t) := W(t) = \int_0^t \omega(x)^{p'(x)}\mathrm{d}x$$

and

$$\alpha = \frac{1}{q(0)},\ \ \beta = \frac{1}{p'(0)},\ \ s = \frac{1}{p(0)},\ \ \phi(x) = \upsilon(x)^{q(x)},\ \ g(x) = \omega(x)^{p'(x)}.$$

Then

$$\begin{aligned} B_4(t;\alpha,\beta,s) &= \left(\int_0^t \phi(x)G(x)^{\frac{\beta+s}{\alpha}}\mathrm{d}x\right)^{\alpha} G(t)^{-s} \\ &= \left(\int_0^t \upsilon(x)^{q(x)}W(x)^{q(0)}\mathrm{d}x\right)^{\frac{1}{q(0)}} W(t)^{-\frac{1}{p(0)}} \end{aligned} \tag{3.9}$$

(by Theorem 2.4),

$$\geq CB_1(t,\alpha,\beta) = CF(t)^{\alpha}G(t)^{\beta} = CV(t)^{\frac{1}{q(0)}}W(t)^{\frac{1}{p'(0)}}.$$

To prove the necessity of the second condition (3.2), set the new family of test functions

$$f_t(x) = \left(\int_t^\infty \upsilon(\tau)^{q(\tau)}\mathrm{d}\tau\right)^{-\frac{1}{q'(x)}} \chi_{(t,\infty)}(x)\ \upsilon(x)^{q(x)-1},\ \ t>1.$$

Let the operator $H_{\upsilon,\omega}$ be compact from $L^{p(\cdot)}(0,\infty)$ to $L^{q(\cdot)}(0,\infty)$. It is not difficult to see that the conjugate operator is

$$H^*_{\upsilon,\omega}f(x) = \omega(x)\int_x^\infty f(t)\,\upsilon(t)\,\mathrm{d}t.$$

Then it follows from Theorem 2.3 that the conjugate operator $H^*_{\upsilon,\omega}$ is also compactly acting from $L^{q'(\cdot)}$ to $L^{p'(\cdot)}$.

Now,

$$\begin{aligned} \rho_{q'(\cdot)}(f_t) &= \\ &= \int_0^\infty\left(\left(\int_t^\infty \upsilon(\tau)^{q(\tau)}\mathrm{d}\tau\right)^{-\frac{1}{q'(x)}} \chi_{(t,\infty)}(x)\upsilon(x)^{q(x)-1}\right)^{q'(x)}\mathrm{d}x \\ &= \left(\int_t^\infty \upsilon(x)^{q(x)}\mathrm{d}x\right)\left(\int_t^\infty \upsilon(\tau)^{q(\tau)}\mathrm{d}\tau\right)^{-1} = 1. \end{aligned}$$

Hence,

$$\rho_{q'(\cdot)}(f_t) \leq 1.$$

Therefore,

$$\| f_t(x) \|_{L^{q'(\cdot)}(0,\infty)} \leq 1.$$

By Holder's inequality, we have

$$\left| \int_0^\infty f_t(x)\,\varphi(x)\,\mathrm{d}x \right| \leq k(p)\, \| f_t(\cdot)\|_{L^{q'(\cdot)}(0,\infty)} \, \|\chi_{(t,\infty)}(\cdot)\varphi(\cdot)\|_{L^{q(\cdot)}(0,\infty)} \to 0$$

as $t \to \infty$ for all $\varphi \in L^{q(\cdot)}(0,\infty)$. Since $L^{q(\cdot)}(0,\infty)$ is the conjugate space of $L^{q'(\cdot)}(0,\infty)$, from here we get that the sequence $\{f_t\}$ converges weakly to 0 in $L^{q'(\cdot)}(0,\infty)$ as $t \to \infty$. Then by assumption, the sequence $\{H^*_{\upsilon,\omega} f_t\}$ converges to 0 in the norm of $L^{p'(\cdot)}(0,\infty)$. Therefore,

$$\rho_{p'(\cdot)}(H^*_{\upsilon,\omega} f_t) \to 0 \text{ as } t \to \infty. \tag{3.10}$$

On the other hand,

$$\rho_{p'(\cdot)}(H^*_{\upsilon,\omega} f_t) =$$

$$= \int_t^\infty \left(\omega(x) \int_x^\infty \left(\int_t^\infty \upsilon(\tau)^{q(\tau)} \mathrm{d}\tau \right)^{-\frac{1}{q'(s)}} \chi_{(t,\infty)}(s)\upsilon(s)^{q(s)-1}\upsilon(s)\,\mathrm{d}s \right)^{p'(x)} \mathrm{d}x$$

$$\geq \int_t^\infty \omega(x)^{p'(x)} \left(\int_x^\infty \left(\int_t^\infty \upsilon(\tau)^{q(\tau)} \mathrm{d}\tau \right)^{-\frac{1}{q'(s)}} \upsilon(s)^{q(s)}\,\mathrm{d}s \right)^{p'(x)} \mathrm{d}x \tag{3.11}$$

(by Lemma 2.10),

$$\geq (2C)^{-q^+} \int_t^\infty \omega(x)^{p'(x)} V(x)^{p'(x)} V(t)^{-\frac{p'(x)}{q'(x)}} \mathrm{d}x$$

(by Lemma 2.11),

$$\geq \frac{2^{-q^+}}{C_1} V(t)^{-\frac{p'(\infty)}{q'(\infty)}} \int_t^\infty \omega(x)^{p'(x)} V(x)^{p'(x)} \mathrm{d}x$$

(by Lemma 2.12),

$$\geq \frac{1}{C_2} \left[V(t)^{-\frac{1}{q'(\infty)}} \left(\int_t^\infty \omega(x)^{p'(x)} V(x)^{p'(\infty)} \mathrm{d}x \right)^{\frac{1}{p'(\infty)}} \right]^{p'(\infty)}$$

(by Theorem 2.4),

$$\geq C_3 \left[V(t)^{\frac{1}{q(\infty)}} W(t)^{\frac{1}{p'(\infty)}} \right]^{p'(\infty)}. \tag{3.12}$$

From this inequality and (3.10), it follows that

$$V(t)^{\frac{1}{q(\infty)}} W(t)^{\frac{1}{p'(\infty)}} \to 0 \text{ as } t \to \infty. \tag{3.13}$$

Now, from (3.13) it follows that $C_b \to 0$ as $b \to \infty$.

The necessity of Theorem 3.1 has been proved.

Notice, in the proof (3.12), we have applied Theorem 2.4 under the settings:

$$F(t) := V(t) = \int_t^\infty \upsilon(x)^{q(x)} \mathrm{d}x; \quad G(t) := W(t) = \int_0^t \omega(x)^{p'(x)} \mathrm{d}x$$

and

$$\alpha = \frac{1}{q(\infty)}, \ \beta = \frac{1}{p'(\infty)}, \ s = \frac{1}{q'(\infty)}, \ \phi(x) = \upsilon(x)^{q(x)}, \ g(x) = \omega(x)^{p'(x)}.$$

Then

$$\begin{aligned} B_5(t;\alpha,\beta,s) &= \left(\int_t^b g(x) F(x)^{\frac{\alpha+s}{\beta}} \mathrm{d}x \right)^{\beta} F(t)^{-s} \\ &= \left(\int_t^b \omega(x)^{p'(x)} V(x)^{p'(\infty)} \mathrm{d}x \right)^{\frac{1}{p'(\infty)}} V(t)^{-\frac{1}{q'(\infty)}} \end{aligned} \tag{3.14}$$

(by Theorem 2.4),

$$\geq C B_1(t,\alpha,\beta) = C F(t)^{\alpha} G(t)^{\beta} = C V(t)^{\frac{1}{q(\infty)}} W(t)^{\frac{1}{p'(\infty)}}.$$

This completes the proof of Theorem 3.1 □

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