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Partial differential equations/Numerical analysis

Space/time convergence analysis of a class of conservative schemes for linear wave equations

Convergence espace/temps d'une classe de schémas conservatifs pour les équations d'onde linéaires

Juliette Chabassier [a,b], Sébastien Imperiale [c]

[a] *Magique 3D team – Inria Bordeaux Sud-Ouest, 200, avenue de la Vieille-Tour, 33405 Talence cedex, France*
[b] *Université de Pau et des pays de l'Adour, avenue de l'Université, 64013 Pau cedex, France*
[c] *Inria and Paris-Saclay University, 1, rue Honoré-d'Estienne-d'Orves, 91120 Palaiseau, France*

A R T I C L E I N F O

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A B S T R A C T

This paper concerns the space/time convergence analysis of conservative two-step time discretizations for linear wave equations. Explicit and implicit, second- and fourth-order schemes are considered, while the space discretization is given and satisfies minimal hypotheses. Convergence analysis is done using energy techniques and holds if the time step is upper-bounded by a quantity depending on space discretization parameters. In addition to showing the convergence for recently introduced fourth-order schemes, the novelty of this work consists in the independency of the convergence estimates with respect to the difference between the time step and its greatest admissible value.

R É S U M É

Ce travail concerne l'analyse de convergence espace/temps de schémas en temps à deux pas pour les équations d'onde linéaires. Sont considérés des schémas explicites et implicites, d'ordre deux et quatre, tandis que la discrétisation spatiale est donnée et satisfait des hypothèses minimales. L'analyse de convergence est faite par techniques d'énergie et est valide si le pas de temps est borné par une quantité dépendant des paramètres de discrétisation spatiale. En plus de montrer la convergence pour des schémas d'ordre quatre récemments introduits, la nouveauté de ce travail réside dans le fait que les estimations ne dépendent pas de la différence entre le pas de temps et sa plus grande valeur admissible.

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E-mail addresses: juliette.chabassier@inria.fr (J. Chabassier), sebastien.imperiale@inria.fr (S. Imperiale).

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On considère le problème de propagation d'onde suivant : pour une source donnée $f \in C^0([0,T],H)$, trouver $u(t) \in V$ solution, pour tout $t \in [0,T]$, de

$$\partial_{tt} u + Au = f \quad \text{dans } V', \quad u(0) = 0 \quad \text{dans } V, \quad \partial_t u(0) = 0 \quad \text{dans } H,$$

où V et H sont deux espaces séparables de Hilbert (la norme sur H est notée $|\cdot|$) et V' le dual de V, l'opérateur A est autoadjoint de V dans V'. Considérons donnée une discrétisation en espace s'appuyant sur une famille d'espaces de dimension finie $\{V_h\}_{h>0}$ avec $V_h \subset V$. Le problème semi-discret s'écrit alors : pour un terme source $f_h \in C^2([0,T],V_h)$, trouver $u_h(t) \in V_h$, pour tout $t \in [0,T]$ satisfaisant

$$\partial_{tt} u_h + A_h u_h = f_h, \quad u_h(0) = 0, \quad \partial_t u_h(0) = 0, \quad \text{dans } V_h,$$

où A_h est un opérateur de V_h dans V_h autoadjoint borné (non uniformément par rapport à h) résultant de la discrétisation en espace choisie. Nous supposerons qu'il existe une fonction positive $\delta(h) = o(h)$ telle que, pour tout $h > 0$, on ait

$$\sup_{t\in[0,T]} |u(t) - u_h(t)| \le \delta(h).$$

Nous étudions les schémas suivants : le θ-schéma (noté *TS* et dont le schéma saute-mouton explicite est un cas particulier), le schéma « saute-mouton stabilisé » (noté *SLF*) et le (θ,φ)-schéma d'ordre quatre (noté *TPS*). Soit $\Delta t > 0$ le pas de temps ; définissons $t^n := n\Delta t$. Pour toute suite $\{v_h^k\}_{k\ge 0} \subset V_h$, nous notons

$$\left[v_h^n\right]_{\Delta t^2} := \frac{v_h^{n+1} - 2v_h^n + v_h^{n-1}}{\Delta t^2}, \qquad \{v_h^n\}_\theta := \theta\, v_h^{n+1} + (1-2\theta) v_h^n + \theta\, v_h^{n-1}, \qquad n \ge 1.$$

Nous cherchons une suite $\{u_h^k\}_{k\ge 0} \subset V_h$ telle que u_h^n approche $u_h(t^n)$. Les deux premiers termes sont supposés donnés et, pour tout $n \ge 1$,

- *TS* : $\left[u_h^n\right]_{\Delta t^2} + A_h\{u_h^n\}_\theta = f_h(t^n)$. • *SLF* : $\left[u_h^n\right]_{\Delta t^2} + A_h u_h^n + \dfrac{\Delta t^2}{16} A_h^2 u_h^n = f_h(t^n)$.
- *TPS* : $\left[u_h^n\right]_{\Delta t^2} + A_h\{u_h^n\}_\theta + \Delta t^2 \left(\theta - \dfrac{1}{12}\right) A_h^2 \{u_h^n\}_\varphi = f_h(t^n) + \Delta t^2 \left[\dfrac{\partial_{tt}}{12} + \left(\theta - \dfrac{1}{12}\right) A_h\right] f_h(t^n)$.

Pour chaque schéma, on peut définir une énergie discrète associée ; une condition de type CFL assure sa positivité en tant que fonctionnelle :

$$\Delta t^2 \le \frac{\alpha}{\rho(A_h)}$$

où α est positif, le plus grand possible, et dépend du schéma. Nous montrons, sous des hypothèses portant sur la discrétisation spatiale du problème ainsi que sous condition CFL, le résultat de convergence espace/temps suivant : il existe $C > 0$ indépendant de Δt et h tel que

- θ-schéma (*TS*) et schéma saute-mouton stabilisé (*SLF*)
$$\sup_{t^n\in[0,T]} |u(t^n) - u_h^n| \le C\left[\Delta t^2\, T^2 + \delta(h)\right].$$
- (θ,φ)-schéma (*TPS*)
$$\sup_{t^n\in[0,T]} |u(t^n) - u_h^n| \le C\left[\Delta t^4\, T^2 + \delta(h)\right].$$

Notons qu'en particulier ces résultats de convergence sont valables lorsque $\Delta t^2 = \alpha/\rho(A_h)$. Pour obtenir ce résultat, nous nous ramenons à l'étude de polynômes dont les coefficients sont fonction des paramètres des schémas.

1. Abstract and semi-discrete wave propagation problem

In this work, we are interested in the simulation of linear wave propagation problems. The most simple example one could think of is given by the following problem: find $u(t) \in H_0^1(\Omega)$, for all $t \in [0,T]$ such that

$$\partial_{tt} u - \Delta u = f, \quad u(0) = 0, \quad \partial_t u(0) = 0, \quad \text{in } \Omega, \quad u = 0 \text{ on } \partial\Omega. \tag{1}$$

To study wave propagation in a more abstract framework, following [3], chapter XVIII, we assume given separable Hilbert spaces H and V. The space H is equipped with the scalar product $(\cdot,\cdot)_H$. The corresponding norms are denoted $|\cdot|$ and $\|\cdot\|$, respectively. Moreover, we assume that V is dense in H, H is identified with its dual H' and V is continuously embedded in H. Note that to solve (1) it is standard to choose $H = L^2(\Omega)$ and $V = H_0^1(\Omega)$. We assume given a continuous Hermitian bilinear form $a : V \times V \to \mathbb{R}$ that satisfies

$$a(v,v) \geq 0, \quad a(v,v) + C_o|v|^2 \geq C_a\|v\|^2, \quad \forall v \in V, \tag{2}$$

where (C_a, C_o) are real positive scalars. From this bilinear form, we construct a self-adjoint positive unbounded operator $A : V \mapsto V'$. It is defined by

$$A : u \to Au \quad \text{such that} \quad \langle Au, v\rangle = a(u,v), \quad \forall v \in V.$$

where V' denotes the dual of V and $\langle\cdot,\cdot\rangle$ denotes the duality product. We consider the following abstract wave propagation problems: for a given $f \in C^0([0,T],H)$ find $u(t) \in V$ solution, for all $t \in [0,T]$, of

$$\partial_{tt}u + Au = f \quad \text{in } V', \quad u(0) = 0 \quad \text{in } V, \quad \partial_t u(0) = 0 \quad \text{in } H. \tag{3}$$

Existence and uniqueness results for this problem are well known (see again [3], chapter XVIII): there exists a unique function u solution to (3), it satisfies

$$u \in C^0([0,T],V), \quad \partial_t u \in C^0([0,T],H).$$

We introduce the family of finite dimensional spaces $\{V_h\}_{h>0}$ with $V_h \subset V$. As usual, the parameter h is devoted to tend to 0 and represents an approximation parameter of V_h to V. For each h we define the operator A_h as $A_h : V_h \mapsto V_h$ and

$$A_h : u_h \to A_h u_h \qquad \text{such that} \quad (A_h u_h, v_h)_h = a_h(u_h, v_h), \quad \forall v_h \in V_h,$$

where the scalar product on V_h denoted by $(\cdot,\cdot)_h$ as well as the continuous bilinear forms $a_h(\cdot,\cdot)$ represent some approximation of $(\cdot,\cdot)_H$ and $a(\cdot,\cdot)$, respectively, that account for instance for the use of quadrature formulae in the computation of integrals. Similarly, we denote by $|\cdot|_h$ the norm induced by the scalar product $(\cdot,\cdot)_h$. We assume that there exists a positive constant C_H such that

$$|v_h| \leq C_H\,|v_h|_h, \quad \forall v_h \in V_h$$

and we assume that a_h is an Hermitian bilinear form that satisfies the same property as in equation (2)

$$a_h(v_h,v_h) \geq 0, \quad a_h(v_h,v_h) + C_o|v_h|_h^2 \geq C_a\|v_h\|^2, \quad \forall v_h \in V_h.$$

Note that this assumption implies that the operator A_h is self-adjoint and non-negative. Its spectrum, denoted $Sp(A_h)$, is a set of finite number of real non-negative eigenvalues (eigenvalues of multiplicity greater than one are counted accordingly). With each eigenvalue λ we associate a real eigenvector w_h^λ defined by

$$(A_h w_h^\lambda, v_h)_h = \lambda\,(w_h^\lambda, v_h)_h, \quad \forall v_h \in V_h, \quad |w_h^\lambda|_h = 1,$$

such that the family $\{w_h^\lambda\}_{\lambda\in Sp(A_h)}$ is an orthonormal basis of V_h. The semi-discrete equation we consider reads: for any given source term $f_h \in C^2([0,T],V_h)$ find $u_h(t) \in V_h$, for all $t \in [0,T]$ such that

$$\partial_{tt}u_h + A_h u_h = f_h, \quad u_h(0) = 0, \quad \partial_t u_h(0) = 0, \quad \text{in } V_h. \tag{4}$$

Existence and uniqueness results for this problem are direct consequences of the theory developed in infinite dimensional space (choose $H = V = V_h$ in the paragraph above): there exists a unique solution u_h of (4), it satisfies $u_h \in C^1([0,T],V_h)$. We introduce the discrepancy error $e_h(t) = u(t) - u_h(t)$; it depends on the approximation of the scalar product on H, on the bilinear form a_h, on the approximation of the space V itself and on the approximation of the source term f. It is expected that e_h vanishes when $h \to 0$. Therefore we make the following assumption.

Hypothesis 1.1 *(Convergence of the semi-discrete problem). There exists a positive function $\delta(h) = o(h)$ such that for all $h > 0$ we have*

$$\sup_{t\in[0,T]} |e_h(t)| \leq \delta(h).$$

Such a result is proved for continuous finite element approximation in [7] in the case $a(\cdot,\cdot)_h \equiv a(\cdot,\cdot)$. In [4], the case of spectral elements is given (see [8] for an introduction to spectral elements). For these convergence results to hold, the continuous solution must be regular enough in time and space. We do not detail the assumptions here. In what follows, we specify the extra regularity in time and space we require for the semi-discrete solution u_h in order to state the space/time convergence results; such regularity is a consequence of the space/time regularity of the source term.

Hypothesis 1.2 *(Stability of the semi-discrete problem). For a given couple (p,q) with p even, we assume that $\partial_t^q u_h(t) \in V_h$ for all $t \in [0,T]$ and there exists a constant $C_{p,q}$ such that for all $h > 0$ we have*

$$\sup_{t\in[0,T]} |\,A_h^{p/2}\partial_t^q u_h(t)\,|_h \leq C_{p,q}.$$

2. Analysis of a family of time discretizations

In the sequel, we study the following two-step conservative time discretizations: θ-Scheme (*TS*), which is the family of conservative centered Newmark schemes, see [3], such as the classical leap-frog scheme, the Stabilized Leap-Frog scheme (*SLF*), as introduced in [5], and the higher-order (θ,φ)-Scheme (*TPS*) developed in [2]. Let $\Delta t > 0$ be the time step, a small parameter devoted to tend to zero and let define $t^n := n\Delta t$. To shorten the writing, we introduce the following notations, for a series $\{v_h^k\}_{k\geq 0} \subset V_h$,

$$\big[v_h^n\big]_{\Delta t^2} := \frac{v_h^{n+1} - 2v_h^n + v_h^{n-1}}{\Delta t^2}, \qquad \{v_h^n\}_\theta := \theta\, v_h^{n+1} + (1-2\theta)v_h^n + \theta\, v_h^{n-1}, \qquad n \geq 1.$$

We seek a series $\{u_h^k\}_{k\geq 0} \subset V_h$ such that u_h^n is an approximation of $u_h(t^n)$ solution to equation (4). The two first terms of this series are considered given.[1] The following terms of the series are computed using one of the schemes below, for $n \geq 1$,

- *TS*: $\big[u_h^n\big]_{\Delta t^2} + A_h\{u_h^n\}_\theta = f_h(t^n)$. • *SLF*: $\big[u_h^n\big]_{\Delta t^2} + A_h u_h^n + \dfrac{\Delta t^2}{16} A_h^2 u_h^n = f_h(t^n)$.
- *TPS*: $\big[u_h^n\big]_{\Delta t^2} + A_h\{u_h^n\}_\theta + \Delta t^2\left(\theta - \dfrac{1}{12}\right) A_h^2 \{u_h^n\}_\varphi = f_h(t^n) + \Delta t^2\left[\dfrac{\partial_{tt}}{12} + \left(\theta - \dfrac{1}{12}\right) A_h\right] f_h(t^n)$.

By construction, (SLF) is an explicit scheme, whereas (TS) is explicit when $\theta = 0$ and (TPS) is explicit when $(\theta,\varphi) = (0,0)$. In this latter case, the scheme corresponds to the modified equation approach presented in [9]. Notice that for the (TPS) the source term has to be modified in order to get adequate accuracy.

All these schemes can be written

$$P_{\mathcal{K}}(\Delta t^2 A_h)\big[u_h^n\big]_{\Delta t^2} + P_{\mathcal{P}}(\Delta t^2 A_h)\, A_h\, \{u_h^n\}_{1/4} = f_h^n, \tag{5}$$

with f_h^n the source term and with the following definitions of the polynomial functions $P_{\mathcal{K}}$ and $P_{\mathcal{P}}$

- *TS*: $P_{\mathcal{K}}(x) = 1 + \left(\theta - \dfrac{1}{4}\right)x, \quad P_{\mathcal{P}}(x) = 1$. • *SLF*: $P_{\mathcal{K}}(x) = 1 - \dfrac{x}{4} + \dfrac{x^2}{64}, \quad P_{\mathcal{P}}(x) = 1 - \dfrac{x}{16}$.
- *TPS*: $P_{\mathcal{K}}(x) = 1 + \left(\theta - \dfrac{1}{4}\right)x + \left(\varphi - \dfrac{1}{4}\right)\left(\theta - \dfrac{1}{12}\right)x^2, \quad P_{\mathcal{P}}(x) = 1 + \left(\theta - \dfrac{1}{12}\right)x$

In the sequel, we aim at establishing energy and error estimates for general schemes of the form (5) under some assumptions on $P_{\mathcal{K}}$ and $P_{\mathcal{P}}$. Let the time discretization error be defined as $e_h^n := u_h(t^n) - u_h^n$, it satisfies

$$P_{\mathcal{K}}(\Delta t^2 A_h)\big[e_h^n\big]_{\Delta t^2} + P_{\mathcal{P}}(\Delta t^2 A_h)\, A_h\, \{e_h^n\}_{1/4} = r_h^n. \tag{6}$$

Assuming sufficient regularity of the semi-discrete solution, there exist $t^{n-1} \leq t^{n,\diamond}, t^{n,\heartsuit}, t^{n,\clubsuit} \leq t^{n+1}$ such that for each of the considered schemes, the remainder writes

- *TS*: $r_h^n = \Delta t^2\left(\dfrac{1}{12}\partial_t^4 u_h(t^{n,\diamond}) + \theta A_h\, \partial_{tt} u_h(t^{n,\heartsuit})\right)$. • *SLF*: $r_h^n = \Delta t^2\left(\dfrac{1}{12}\partial_t^4 u_h(t^{n,\diamond}) + \dfrac{1}{16} A_h^2 u_h(t^n)\right)$.
- *TPS*: $r_h^n = \Delta t^4\left(\dfrac{1}{360}\partial_t^6 u_h(t^{n,\diamond}) + \dfrac{\theta}{12} A_h\, \partial_t^4 u_h(t^{n,\heartsuit}) + \varphi\left(\theta - \dfrac{1}{12}\right) A_h^2\, \partial_{tt} u_h(t^{n,\clubsuit})\right)$.

This suggests that (TS) and (SLF) are second-order accurate and (TPS) is fourth-order accurate in time.

For all these schemes, a discrete energy is preserved (in the absence of sources). This energy is computed at the intermediate time step $t^{n+1/2}$ and is denoted $\mathcal{E}_h^{n+1/2}$. It is the sum of kinetic and potential energies, more precisely $\mathcal{E}_h^{n+1/2} = \mathcal{E}_{\mathcal{K}}^{n+1/2} + \mathcal{E}_{\mathcal{P}}^{n+1/2}$. When considering equation (6) satisfied by the error, we have

$$\mathcal{E}_{\mathcal{K}}^{n+1/2} = \frac{1}{2}\left(P_{\mathcal{K}}\left(\Delta t^2 A_h\right)\frac{e_h^{n+1} - e_h^n}{\Delta t}, \frac{e_h^{n+1} - e_h^n}{\Delta t}\right)_h \quad \text{and} \tag{7}$$

$$\mathcal{E}_{\mathcal{P}}^{n+1/2} = \frac{1}{2}\left(P_{\mathcal{P}}(\Delta t^2 A_h)\, A_h\, \frac{e_h^{n+1} + e_h^n}{2}, \frac{e_h^{n+1} + e_h^n}{2}\right)_h. \tag{8}$$

It can be shown that the total energy satisfies

[1] In practice, u_h^0 and u_h^1 are computed using the initial data and a sufficiently accurate one-step method (such as high-order Runge–Kutta methods).

$$\frac{\mathcal{E}_h^{n+1/2}-\mathcal{E}_h^{n-1/2}}{\Delta t}=\left(r_h^n,\,\frac{e_h^{n+1}-e_h^{n-1}}{2\Delta t}\right)_h. \tag{9}$$

A necessary condition for the schemes' stability is the non-negativity of the kinetic energy (7) and the potential energy (8). Such a condition is achieved if the following hypothesis holds.

Hypothesis 2.1. *There exists $\alpha>0$ such that $P_{\mathcal{K}}(x)$ and $P_{\mathcal{P}}(x)$ are non-negative for all $x\in[0,\,\alpha]$.*

For the schemes we consider we have

- *TS* $(0\le\theta<1/4):\ \alpha=4/(1-4\theta)$. • *TS* $(\theta\ge 1/4):\ \alpha=+\infty$. • *SLF* $:\ \alpha=16$.
- *TPS* $(\theta=0,\varphi=0):\ \alpha=12$. • *TPS* $(\theta\ge 1/4,\varphi\ge 1/4):\ \alpha=+\infty$.
- *TPS* $\forall\,(\theta,\varphi):$ see theorem 5.1 of [2].

Lemma 2.2. *Assume Hypothesis 2.1 holds. The energies $\mathcal{E}_{\mathcal{K}}^{n+1/2}$ and $\mathcal{E}_{\mathcal{P}}^{n+1/2}$ are non-negative for all $n\ge 0$ if Δt satisfies*

$$\Delta t^2\le\frac{\alpha}{\rho(A_h)}\qquad\textit{(CFL condition)} \tag{10}$$

where $\rho(A_h)$ is the largest eigenvalue of A_h.

Note that when α is finite (resp. infinite) the scheme is conditionally (resp. unconditionally) stable. The non-negativity of the energy $\mathcal{E}_h^{n+1/2}$ is however not sufficient to ensure the stability of the schemes in H since it is only a semi-norm. In the sequel, the polynomial functions $P_{\mathcal{P}}$ and $P_{\mathcal{K}}$ will be assumed to satisfy the following hypothesis.

Hypothesis 2.3. *We assume that there exists a partitioning of $[0,\alpha]$ into two disjoint subsets $\mathcal{J}_{\mathcal{K}}$ and $\mathcal{J}_{\mathcal{P}}$ such that there exist two strictly positive constants $C_{\mathcal{K}}$, $C_{\mathcal{P}}$, such that*

$$C_{\mathcal{K}}\le P_{\mathcal{K}}(x),\quad\forall x\in\mathcal{J}_{\mathcal{K}},\qquad\qquad C_{\mathcal{P}}\le x\,P_{\mathcal{P}}(x),\quad\forall x\in\mathcal{J}_{\mathcal{P}}.$$

Definition 2.4. We define for any $v_h\in V_h$, the projection operators:

$$\Pi_{\mathcal{K}}(v_h)=\sum_{\substack{\Delta t^2\lambda\in\mathcal{J}_{\mathcal{K}}\\ \lambda\in Sp(A_h)}}(v_h,w_h^\lambda)_h\,w_h^\lambda\quad\text{and}\quad\Pi_{\mathcal{P}}(v_h)=\sum_{\substack{\Delta t^2\lambda\in\mathcal{J}_{\mathcal{P}}\\ \lambda\in Sp(A_h)}}(v_h,w_h^\lambda)_h\,w_h^\lambda.$$

The operator $\Pi_{\mathcal{K}}$ (resp. $\Pi_{\mathcal{P}}$) corresponds to modal projection operators on subsets of V_h for which the kinetic (resp. potential) part of the energy is a uniform upper bound of the H-norm with respect to h and Δt. These ideas are specified in the following Lemma.

Lemma 2.5. *Assume Hypothesis 2.3 holds. Then, for any $h>0$ and $\Delta t>0$ satisfying the CFL condition (10), and for any $v_h\in V_h$, $v_h=\Pi_{\mathcal{K}}(v_h)+\Pi_{\mathcal{P}}(v_h)$ and*

$$|\,\Pi_{\mathcal{K}}(v_h)\,|_h^2\le C_{\mathcal{K}}^{-1}\left(P_{\mathcal{K}}(\Delta t^2A_h)v_h,\,v_h\right)_h,\qquad|\,\Pi_{\mathcal{P}}(v_h)\,|_h^2\le C_{\mathcal{P}}^{-1}\left(\Delta t^2A_h\,P_{\mathcal{P}}(\Delta t^2A_h)v_h,\,v_h\right)_h. \tag{11}$$

Then the following estimates can be written.

Lemma 2.6. *Let the series $\{e_h^n\}$ satisfy (6) and Hypothesis 2.3 holds. Then for all $h>0$ and $\Delta t>0$ satisfying the CFL condition (10), and for all $n\ge 1$,*

$$\sqrt{\mathcal{E}^{n+1/2}}\le\sqrt{\mathcal{E}^{1/2}}+\sqrt{2}\,\gamma\,\Delta t\sum_{\ell=1}^{n}\left|r_h^\ell\right|_h, \tag{12}$$

$$\left|e_h^{n+1}\right|_h\le\sqrt{2}\left|e_h^1\right|_h+2\,\gamma\,t^n\sqrt{2\,\mathcal{E}^{1/2}}+4\,\gamma^2\,\Delta t^2\sum_{\ell=1}^{n}\sum_{k=1}^{\ell}\left|r_h^k\right|_h, \tag{13}$$

where $\gamma=C_{\mathcal{P}}^{-1/2}+C_{\mathcal{K}}^{-1/2}/2$.

It is important to notify here that γ is independent of h and Δt; this is the key point to obtain uniform estimates in Δt. Indeed, as pointed out in Remark 10 of [6], the classical stability proof for the (TS) blows up when the time step approaches its greatest admissible value in inequality (10). The constant $C_{\mathcal{K}}$ and $C_{\mathcal{P}}$ (and therefore γ) are computed for the considered schemes in the appendix; we find $\gamma \leq \sqrt{2}$ for all schemes.

Proof of Lemma 2.6. From the Cauchy–Schwartz inequality applied to (9), we get

$$\frac{\mathcal{E}_h^{n+1/2}-\mathcal{E}_h^{n-1/2}}{\Delta t} \leq |r_h^n|_h \left|\frac{e_h^{n+1}-e_h^{n-1}}{2\Delta t}\right|_h \leq |r_h^n|_h \underbrace{\left|\Pi_{\mathcal{K}}\left(\frac{e_h^{n+1}-e_h^{n-1}}{2\Delta t}\right)\right|_h}_{\Xi} + |r_h^n|_h \underbrace{\left|\Pi_{\mathcal{P}}\left(\frac{e_h^{n+1}-e_h^{n-1}}{2\Delta t}\right)\right|_h}_{\Phi},$$

where $\Pi_{\mathcal{K}}$ and $\Pi_{\mathcal{P}}$ are the projection operators given by Definition 2.4. Now we use the two inequalities of (11) on respectively Ξ and Φ and then recognize the square root of the kinetic and potential parts of the energy. To shorten the notation we introduce the time finite difference of the error: $d_h^{n+1/2} := (e_h^{n+1}-e_h^n)/\Delta t$. We have

$$\begin{aligned}
2\,\Xi &\leq \left|\Pi_{\mathcal{K}}(d_h^{n+1/2})\right|_h + \left|\Pi_{\mathcal{K}}(d_h^{n-1/2})\right|_h \\
&\leq C_{\mathcal{K}}^{-1/2}\sqrt{\Big(P_{\mathcal{K}}(\Delta t^2 A_h)\,d_h^{n+1/2}, d_h^{n+1/2}\Big)_h} + C_{\mathcal{K}}^{-1/2}\sqrt{\Big(P_{\mathcal{K}}(\Delta t^2 A_h)\,d_h^{n-1/2}, d_h^{n-1/2}\Big)_h} \\
&\leq C_{\mathcal{K}}^{-1/2}\left[\sqrt{2\,\mathcal{E}_h^{n+1/2}} + \sqrt{2\,\mathcal{E}_h^{n-1/2}}\right],
\end{aligned}$$

note that the second inequality is obtained by taking $v_h = d_h^{n+1/2}$ and $v_h = d_h^{n-1/2}$ in the first inequality of (11). A similar procedure leads to

$$\Phi \leq C_{\mathcal{P}}^{-1/2}\left[\sqrt{2\,\mathcal{E}_h^{n+1/2}} + \sqrt{2\,\mathcal{E}_h^{n-1/2}}\right].$$

By denoting $\gamma = C_{\mathcal{P}}^{-1/2} + C_{\mathcal{K}}^{-1/2}/2$, we get from

$$\frac{\mathcal{E}_h^{n+1/2}-\mathcal{E}_h^{n-1/2}}{\Delta t} \leq \gamma\, |r_h^n|_h \left[\sqrt{2\,\mathcal{E}_h^{n+1/2}} + \sqrt{2\,\mathcal{E}_h^{n-1/2}}\right],$$

which classically leads to inequality (12). The other part of the proof concerns the upper bound on $\left|e_h^{n+1}\right|_h$. We write:

$$\left|e_h^{n+1}\right|_h \leq \underbrace{\left|\Pi_{\mathcal{K}}(e_h^{n+1})\right|_h}_{\Omega} + \underbrace{\left|\Pi_{\mathcal{P}}(e_h^{n+1})\right|_h}_{\Upsilon}.$$

Introducing artificially the term e_h^n we can write

$$\begin{aligned}
\Omega &\leq \left|\Pi_{\mathcal{K}}(e_h^n)\right|_h + \Delta t\left|\Pi_{\mathcal{K}}(d_h^{n+1/2})\right|_h \leq \left|\Pi_{\mathcal{K}}(e_h^n)\right|_h + \Delta t\, C_{\mathcal{K}}^{-1/2}\sqrt{\Big(P_{\mathcal{K}}(\Delta t^2 A_h) d_h^{n+1/2}, d_h^{n+1/2}\Big)_h} \\
&\leq \left|\Pi_{\mathcal{K}}(e_h^n)\right|_h + \Delta t\, C_{\mathcal{K}}^{-1/2}\sqrt{2\,\mathcal{E}_h^{n+1/2}}.
\end{aligned}$$

The same procedure leads to

$$\Upsilon \leq \left|\Pi_{\mathcal{P}}(e_h^n)\right|_h + 2\,\Delta t\, C_{\mathcal{P}}^{-1/2}\sqrt{2\,\mathcal{E}_h^{n+1/2}}.$$

We use recursively the former inequalities down to $n=1$ to get,

$$\left|e_h^{n+1}\right|_h \leq \underbrace{\left|\Pi_{\mathcal{K}}(e_h^1)\right|_h + \left|\Pi_{\mathcal{P}}(e_h^1)\right|_h}_{\leq \sqrt{2}\,|e_h^1|_h} + 2\sqrt{2}\,\Delta t\,\gamma \sum_{\ell=1}^{n}\sqrt{\mathcal{E}_h^{\ell+1/2}}.$$

We can reuse identity (12) to get (13). □

3. Space–time convergence result

Let ε_h^n be the error between the continuous solution and the fully discrete solution. Note that we have $u(t^n) - u_h^n = e_h(t^n) + e_h^n$, thanks to this decomposition, Hypothesis 1.1 and Lemma 2.6 we can state our final result.

Theorem 3.1. *Assume that $e_h^0 = e_h^1 = 0$ and assume that Hypothesis 2.3 holds and that Hypothesis 1.2 holds for every sufficiently large (p,q), then, for all $h > 0$ and all Δt satisfying the CFL condition (2.2) the following uniform convergence results hold*

- *θ-scheme (TS)*

$$\sup_{t^n\in[0,T]} |u(t^n) - u_h^n| \leq C_H \left[\Delta t^2 \gamma^2 T^2 \left(\frac{C_{0,4}}{6} + 2\theta\, C_{2,2} \right) + \delta(h) \right].$$

- *Stabilized leap-frog scheme (SLF)*

$$\sup_{t^n\in[0,T]} |u(t^n) - u_h^n| \leq C_H \left[\Delta t^2 \gamma^2 T^2 \left(\frac{C_{0,4}}{6} + \frac{C_{4,0}}{8} \right) + \delta(h) \right].$$

- *(θ,φ)-scheme (TPS)*

$$\sup_{t^n\in[0,T]} |u(t^n) - u_h^n| \leq C_H \left[\Delta t^4 \gamma^2 T^2 \left(\frac{C_{0,6}}{180} + \theta \frac{C_{2,4}}{6} + 2\varphi\, (\theta - \frac{1}{12})\, C_{4,2} \right) + \delta(h) \right].$$

Note that, in particular, these convergence results hold if $\Delta t^2 = \alpha/\rho(A_h)$.

Appendix. Expression of γ for the considered numerical schemes

• *TS* ($0 \leq \theta < 1/4$). We are looking for a partitioning $\mathcal{J}_\mathcal{K} \cup \mathcal{J}_\mathcal{P}$ of the interval $[0, \frac{4}{(1-4\theta)}]$ and two positive constant $C_\mathcal{K}$ and $C_\mathcal{P}$ such that

$$C_\mathcal{K} \leq P_\mathcal{K}(x) = 1 + \left(\theta - \frac{1}{4}\right) x, \quad \forall x \in \mathcal{J}_\mathcal{K}, \qquad C_\mathcal{P} \leq x\, P_\mathcal{P}(x) = x, \quad \forall x \in \mathcal{J}_\mathcal{P}.$$

We see that the first inequality cannot be fulfilled for $x = \frac{4}{(1-4\theta)} \in \mathcal{J}_\mathcal{K}$ and the second one for $x = 0 \in \mathcal{J}_\mathcal{P}$. The proposed partitioning is the following

$$\mathcal{J}_\mathcal{K} = \left[0, a^2\right[, \; C_\mathcal{K} = 1 + \left(\theta - \frac{1}{4}\right) a^2, \qquad \mathcal{J}_\mathcal{P} = \left[a^2, \frac{1}{(1/4-\theta)}\right], \; C_\mathcal{P} = a^2$$

where a can be chosen according to θ in order to minimize $\gamma = C_\mathcal{P}^{-1/2} + C_\mathcal{K}^{-1/2}/2$. The peculiar optimal value is given in Lemma 2.3 of [1], it reads

$$a(\theta) = \sqrt{\frac{4}{(1-4\theta)^{2/3} + (1-4\theta)}}; \qquad \gamma(\theta) = \frac{1}{a(\theta)} + \frac{1}{\sqrt{4-(1-4\theta)a(\theta)^2}}.$$

Especially if $\theta = 0$ we get $\gamma = \sqrt{2}$.

• *TS* ($\theta \geq 1/4$). In this case the polynomial functions $P_\mathcal{K}(x)$ and $P_\mathcal{P}(x)$ are positive on the whole interval $[0, +\infty]$ and moreover $P_\mathcal{K}(x) \geq 1\ \forall x \geq 0$. Therefore, one can choose $\mathcal{J}_\mathcal{K} = [0, +\infty]$ and $\mathcal{J}_\mathcal{P} = \{\emptyset\}$. As a consequence, one can set $C_\mathcal{K} = 1$ and since $C_\mathcal{P}$ has no influence and can be chosen formally equal to $+\infty$. We get $\gamma = 1/2$.

• *SLF*. We are looking for a partitioning $\mathcal{J}_\mathcal{K} \cup \mathcal{J}_\mathcal{P}$ of the interval $[0, 16]$ and two constants $C_\mathcal{K}$ and $C_\mathcal{P}$ such that

$$C_\mathcal{K} \leq \left(1 - \frac{x}{8}\right)^2 = 1 - \frac{x}{4} + \frac{x^2}{64}, \quad \forall x \in \mathcal{J}_\mathcal{K}, \qquad C_\mathcal{P} \leq x\left(1 - \frac{x}{16}\right), \quad \forall x \in \mathcal{J}_\mathcal{P}.$$

Out of symmetry with respect to $x = 8$, we look for $\mathcal{J}_\mathcal{K} = [0,\ 8(1-a)[\cup]8(1+a),\ 16]$ and $\mathcal{J}_\mathcal{P} = [8(1-a),\ 8(1+a)]$. Then $C_\mathcal{K} = a^2$ and $C_\mathcal{P} = 4(1-a^2)$. Therefore,

$$\gamma = C_\mathcal{P}^{-1/2} + \frac{C_\mathcal{K}^{-1/2}}{2} = \frac{1}{2\sqrt{1-a^2}} + \frac{1}{2a}.$$

The choice of a that minimizes the value of γ is $a = \frac{1}{\sqrt{2}}$ and leads to $\gamma = \sqrt{2}$.

• *TPS* ($\theta = 0, \varphi = 0$). The approach is similar to the one presented above. We are looking for a partitioning $\mathcal{J}_\mathcal{K} \cup \mathcal{J}_\mathcal{P}$ of the interval $[0, 12]$ and two constant $C_\mathcal{K}$ and $C_\mathcal{P}$ such that

$$C_{\mathcal{K}} \leq 1 - \frac{x}{4} + \frac{x^2}{48}, \quad \forall x \in \mathcal{J}_{\mathcal{K}}, \qquad C_{\mathcal{P}} \leq x\left(1 - \frac{x}{12}\right), \quad \forall x \in \mathcal{J}_{\mathcal{P}}.$$

We look for $\mathcal{J}_{\mathcal{K}} = [0,\ 6(1-a)[\cup]6(1+a),\ 12]$ and $\mathcal{J}_{\mathcal{P}} = [6(1-a),\ 6(1+a)]$, this choice gives some values for $C_{\mathcal{K}}$ and $C_{\mathcal{P}}$ that can be optimized to minimize γ. The optimal choice is $a = \frac{1}{\sqrt{3}}$ and $\gamma = \sqrt{2}$.

• *TPS* $(\theta \geq 1/4, \varphi \geq 1/4)$. In the case of the (θ, φ)-scheme with $\theta \geq 1/4$ and $\varphi \geq 1/4$, the polynomial functions $P_{\mathcal{K}}(x)$ and $P_{\mathcal{P}}(x)$ read

$$P_{\mathcal{K}}(x) = 1 + \left(\theta - \frac{1}{4}\right)x + \left(\varphi - \frac{1}{4}\right)\left(\theta - \frac{1}{12}\right)x^2, \quad P_{\mathcal{P}}(x) = 1 + \left(\theta - \frac{1}{12}\right)x.$$

The polynomial $P_{\mathcal{K}}(x)$ and $P_{\mathcal{P}}(x)$ are positive on the whole interval $[0, +\infty]$ and $P_{\mathcal{K}}(x) \geq 1\ \forall x \geq 0$. Therefore, one can choose $\mathcal{J}_{\mathcal{K}} = [0, +\infty]$ and $\mathcal{J}_{\mathcal{P}} = \{\emptyset\}$. We get $\gamma = 1/2$.

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