C. R. Acad. Sci. Paris, Ser. I 355 (2017) 45–64

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Ordinary differential equations

Topological properties of solution sets for partial functional evolution inclusions

Propriétés topologiques des ensembles de solutions d'inclusions fonctionnelles partielles d'évolution

Yong Zhou [a,b], Li Peng [a]

[a] *Faculty of Mathematics and Computational Science, Xiangtan University, Hunan 411105, PR China*
[b] *Nonlinear Analysis and Applied Mathematics (NAAM) Research Group, Faculty of Science, King Abdulaziz University, Jeddah 21589, Saudi Arabia*

ARTICLE INFO

Article history:
Received 6 May 2016
Accepted after revision 28 November 2016
Available online 13 December 2016

Presented by the Editorial Board

ABSTRACT

This paper deals with functional evolution inclusions of neutral type in Banach space when the semigroup is compact as well as noncompact. The topological properties of the solution set is investigated. It is shown that the solution set is nonempty, compact and an R_δ-set which means that the solution set may not be a singleton but, from the point of view of algebraic topology, it is equivalent to a point, in the sense that it has the same homology group as one-point space. As a sample of application, we consider a partial differential inclusion at end of the paper.

RÉSUMÉ

Cette Note traite des inclusions fonctionnelles d'évolution de type neutre dans les espaces de Banach, aussi bien lorsque le semi-groupe est compact que lorsqu'il est non compact. Nous étudions les propriétés topologiques de l'ensemble des solutions. Nous montrons que cet ensemble est non vide, compact, et qu'il est un R_δ-ensemble. Ceci signifie qu'il peut ne pas être réduit à un point, mais qu'il est équivalent, pour la topologie algébrique, à un espace réduit à un point. Plus précisément, l'ensemble des solutions a les mêmes groupes d'homologie qu'un ensemble réduit à un point. Comme exemple d'application, nous considérons enfin une inclusion différentielle partielle.

E-mail address: yzhou@xtu.edu.cn (Y. Zhou).

http://dx.doi.org/10.1016/j.crma.2016.11.011
1631-073X/

1. Introduction

In this paper, we study the following functional evolution inclusion of neutral type

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}[x(t)-h(t,x_t)] \in Ax(t)+F(t,x_t), & t\in[0,b], \\ x(t)=\phi(t), & t\in[-\tau,0], \end{cases} \tag{1.1}$$

where the state $x(\cdot)$ takes value in Banach space X with norm $|\cdot|$, F is a multimap defined on a subset of $[0,b]\times X$, A is the infinitesimal generator of an analytic semigroup $\{T(t)\}_{t\geq 0}$. For any continuous function x defined on $[-\tau,b]$ and any $t\in[0,b]$, we denote by x_t the element of $C([-\tau,0],X)$ defined by $x_t(\theta)=x(t+\theta), \theta\in[-\tau,0]$. Here, $x_t(\cdot)$ represents the history of the state from time $t-\tau$, up to the present time t. For any $c\in C([-\tau,0],X)$ the norm of c is defined by $\|c\|_*=\sup_{\theta\in[-\tau,0]}|c(\theta)|$.

The study of (1.1) is justified and motivated by a neutral partial differential inclusion of parabolic type

$$\begin{cases} \dfrac{\partial}{\partial t}\left(x(t,\xi)-\displaystyle\int_0^\pi U(\xi,\zeta)x_t(\theta,\zeta)\,\mathrm{d}\zeta\right) \in \dfrac{\partial^2}{\partial\xi^2}x(t,\xi)+F(t,\xi,x_t(\theta,\xi)), & t\in[0,1],\ \xi\in[0,\pi], \\ x(t,0)=x(t,\pi)=0, & t\in[0,1], \\ x(\theta,\xi)=\phi(\theta)(\xi), & \theta\in[-\tau,0],\ \xi\in[0,\pi], \end{cases}$$

where the functions U and ϕ satisfy appropriate conditions, $F:[0,1]\times[0,\pi]\to 2^{\mathbb{R}}$ is weakly upper semicontinuous with closed convex values.

Particularly, if $h=0$, inclusion (1.1) degenerates to

$$x'(t)\in Ax(t)+F(t,x_t).$$

A strong motivation for investigating this class of inclusions is that a lot of phenomena investigated in hybrid systems with dry friction, processes of controlled heat transfer, obstacle problems and others can be described with the help of various differential inclusions, both linear and nonlinear (cf. [9,16,21]). The theory of differential inclusions is highly developed and constitutes an important branch of nonlinear analysis (see, e.g., Bressan and Wang [7], Donchev et al. [11], Gabor and Quincampoix [14], and the references therein).

Since a differential inclusion usually has many solutions starting at a given point, new issues appear, such as investigation of topological and geometric properties of solution sets, selection of solutions with given properties, evaluation of the reachability sets, etc. An important aspect of topological structure is the R_δ-property, which means that an R_δ-set is acyclic (in particular, nonempty, compact and connected) and may not be a singleton but, from the point of view of algebraic topology, it is equivalent to a point, in the sense that it has the same homology groups as one point space. The topological structure of solution sets of differential inclusions on compact intervals has been investigated intensively by many authors—please see Aronszajn [3], Bothe [6], Deimling [9], Hu and Papageorgiou [15], Staicu [19], and references therein. Moreover, one can find results on the topological structure of solution sets for differential inclusions defined on non-compact intervals (including infinite intervals) from Andres and Pavlačková [2], Andres et al. [1], Bakowska and Gabor [4], Bressan and Wang [7], Chen et al. [8], Gabor and Grudzka [12,13], Gabor and Quincampoix [14], Staicu [20], Wang et al. [22], and references therein.

The researches on the theory for nonlinear evolution inclusion of neutral type are only on their initial stage of development, see [5,17,23]. However, to the best of our knowledge, nothing has been done with the topological properties of solution sets for nonlinear evolution inclusion of neutral type. Our purpose in this paper is to study the topological structure of solution sets for inclusion (1.1).

The paper is organized as follows. In Section 2, we recall some notations, definitions, and preliminary facts from multivalued analysis. Subsection 3.1 is devoted to proving that the solution set for inclusion (1.1) is nonempty compact in the case that the semigroup is compact, then proceed to study the R_δ-set. Subsection 3.2 provides that the solution set for inclusion (1.1) is nonempty compact in the case that the semigroup is noncompact, then proceed to study the R_δ-structure of the solution set of (1.1). Finally, an example is given to illustrate the obtained theory.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts from multivalued analysis which are used throughout this paper.

Let $(X,|\cdot|)$ be a Banach space. $\mathcal{L}(X)$ stands for the space of all linear bounded operators on X with norm $\|\cdot\|$, and $L^1([0,b],X)$ stands for the Banach space consisting of integrable functions from $[0,b]$ to X equipped with the norm

$$\|f\|_1=\int_0^b |f(t)|\,\mathrm{d}t.$$

We denote by $C([-\tau, b], X)$ the Banach space consisting of continuous functions from $[-\tau, b]$ to X equipped with the norm

$$\|x\|_{[-\tau,b]} = \max_{t\in[-\tau,b]} |x(t)|.$$

We assume that A is the infinitesimal generator of an analytic semigroup $\{T(t)\}_{t\geq 0}$ of uniformly bounded linear operators on X. Let $0 \in \rho(A)$, where $\rho(A)$ is the resolvent set of A. Under these conditions, it is possible to define the fractional power A^{β}, $0 < \beta \leq 1$, as a closed linear operator on its domain $D(A^{\beta})$. For the analytic semigroup $\{T(t)\}_{t\geq 0}$, the following properties will be used:

(i) there is a $M \geq 1$ such that $M := \sup_{t\geq 0} \|T(t)\| < \infty$;
(ii) for any $\beta \in (0, 1]$, there exists a positive constant C_{β} such that

$$\|A^{\beta}T(t)\| \leq \frac{C_{\beta}}{t^{\beta}},\ 0 < t \leq b.$$

It is clear that $A^{\beta}T(t)x = T(t)A^{\beta}x$ for $x \in D(A^{\beta})$. Then $AT(t)x = A^{1-\beta}T(t)A^{\beta}x$ for $x \in D(A^{\beta})$.

Lemma 2.1. [10] *Let X be reflexive and $K \subset L^1([0, b], X)$ be bounded. If K is uniformly integrable, then K is relatively weakly compact in $L^1([0, b], X)$.*

Let Y and Z be metric spaces. $P(Y)$ stands for the collection of all nonempty subsets of Y. As usual, we denote $P_{\mathrm{cl}}(Y) = \{D \in P(Y):\ \text{closed}\}$, $P_{\mathrm{cp}}(Y) = \{D \in P(Y):\ \text{compact}\}$, $P_{\mathrm{cl,cv}}(Y) = \{D \in P(Y):\ \text{closed and convex}\}$, $\mathrm{co}(D)$ (resp., $\overline{\mathrm{co}}(D)$) be the convex hull (resp., convex closed hull in D) of a subset D.

For the multimap $\varphi : Y \to P(Z)$, we let $Gra(\varphi)$ stand for the graph of φ. If D is a subset of Z, then we denote by $\varphi^{-1}(D) = \{y \in Y : \varphi(y) \cap D \neq \emptyset\}$ the complete preimage of D under φ. φ is called closed if $Gra(\varphi)$ is closed in $Y \times Z$, quasicompact if $\varphi(D)$ is relatively compact for each compact set $D \subset Y$, upper semi-continuous (shortly, u.s.c.) if $\varphi^{-1}(D)$ is closed for each closed set $D \subset Z$, and weakly upper semi-continuous (shortly, weakly u.s.c.) if $\varphi^{-1}(D)$ is closed for each weakly closed set $D \subset Z$.

The following facts will be used.

Lemma 2.2. [16] *Let Y and Z be metric spaces and $\varphi : Y \to P(Z)$ a closed quasicompact multimap with compact values. Then φ is u.s.c.*

Lemma 2.3. [6] *Let $\varphi : D \subset Y \to P(Z)$ be a multimap with weakly compact, convex values. Then φ is weakly u.s.c. iff $\{x_n\} \subset D$ with $x_n \to x_0 \in D$ and $y_n \in \varphi(x_n)$ implies $y_n \rightharpoonup y_0 \in \varphi(x_0)$, up to a subsequence.*

X is called an absolute retract (AR-space) if for any metric space Y and any closed subset $D \subset Y$, every continuous function $h : D \to X$ can be extended to a continuous function $\widetilde{h} : Y \to X$.

X is called an absolute neighborhood retract (ANR-space) if for any metric space Y, closed subset $D \subset Y$ and continuous function $h : D \to X$, there exists a neighborhood $U \supset D$ and a continuous extension $\widetilde{h} : U \to X$ of h.

Definition 2.1. A nonempty subset D of a metric space is said to be contractible if there exists a point $y_0 \in D$ and a continuous function $h : [0, 1] \times D \to D$ such that $h(0, y) = y_0$ and $h(1, y) = y$ for every $y \in D$.

Definition 2.2. A subset D of a metric space is called an R_{δ}-set if there exists a decreasing sequence $\{D_n\}$ of compact and contractible sets such that

$$D = \bigcap_{n=1}^{\infty} D_n.$$

A function $\gamma : P(X) \to \mathbb{R}^+$ defined by:

$$\gamma(D) = \inf\{r > 0 : D \text{ can be covered by finitely many balls of radius } r\}$$

is called the Hausdorff measure of noncompactness. A function $\mu : P(X) \to \mathbb{R}^+$ defined by:

$$\mu(D) = \inf\{r > 0 : D \subset \bigcup_{i=0}^{m} N_i \text{ and } \mathrm{diam}(N_i) \leq r\}$$

is called Kuratowski measure of noncompactness. Here $\mathrm{diam}(N_i)$ is the diameter of N_i. Moreover, we have $\gamma \leq \mu \leq 2\gamma$.

Lemma 2.4. [16] *Let X be a separable Banach space and $F:[0,b]\to P(X)$ be integrably bounded multifunction such that $\gamma(F(t))\le q(t)$ for a.e. $t\in[0,b]$, here, $q\in L^1([0,b],\mathbb{R}^+)$. Then*

$$\gamma\Big(\int_0^t F(s)\,\mathrm{d}s\Big)\le\int_0^t q(s)\,ds$$

for all $t\in[0,b]$.

Lemma 2.5. [16, Proposition 4.2.2] *Let X be a Banach space and $\mathcal{L}$ be an operator*

$$\mathcal{L}:L^1([0,b],X)\to C([0,b],X)$$

that satisfies the following conditions:

(S$_1$) *there exists a constant $c_0>0$, such that*

$$|\mathcal{L}(f)(t)-\mathcal{L}(g)(t)|\le c_0\int_0^t|f(s)-g(s)|\mathrm{d}s,\ t\in[0,b]$$

for every $f,g\in L^1([0,b],X)$;

(S$_2$) *for each compact set $K\subset X$ and sequence $\{f_n\}\subset L^1([0,b],X)$ such that $\{f_n(t)\}\subset K$ for a.e. $t\in[0,b]$, the weak convergence $f_n\rightharpoonup f_0$ implies the convergence $\mathcal{L}(f_n)\to\mathcal{L}(f_0)$.*

Then

(i) *if the sequence of functions $\{f_n\}\subset L^1([0,b],X)$ is integrably bounded for all $n=1,2,\ldots$ and $\gamma(f_n(t))\le q_1(t)$ for a.e. $t\in[0,b]$, where $q_1\in L^1([0,b],\mathbb{R}^+)$, then*

$$\gamma(\{\mathcal{L}(f_n)(t)\}_{n=1}^{\infty})\le 2c_0\int_0^t q_1(s)\,\mathrm{d}s;$$

(ii) *for every semicompact sequence $\{f_n\}\subset L^1([0,b],X)$, the sequence $\{\mathcal{L}(f_n)\}$ is relatively compact in $C([0,b],X)$ and, moreover, if $f_n\rightharpoonup f_0$, then $\mathcal{L}(f_n)\to\mathcal{L}(f_0)$.*

Theorem 2.1. [6] *Let X be a complete metric space, γ denote Hausdorff measure of noncompactness in X and let $\emptyset\ne D\subset X$. Then the following statements are equivalent:*

(i) *D is an R_δ-set;*
(ii) *D is an intersection of a decreasing sequence $\{D_n\}$ of closed contractible spaces with $\gamma(D_n)\to0$;*
(iii) *D is compact and absolutely neighborhood contractible, i.e., D is contractible in each neighborhood in $Y\in ANR$.*

Definition 2.3. A multimap $\varphi:X\to P_{cp}(X)$ is said to be condensing with respect to an MNC γ (γ-condensing) if for every bounded set $D\subset X$ which is not relatively compact, we have:

$$\gamma(\varphi(D))<\gamma(D).$$

In subsequent proofs, we shall also use the following fixed point results for multimaps.

Theorem 2.2. [16, Corollary 3.3.1] *Let D be a bounded convex closed subset of a Banach space X, and $\varphi:D\to P_{\mathrm{cp,cv}}(D)$ an u.s.c. γ-condensing multimap. Then the fixed point set $\mathrm{Fix}\varphi:=\{x:x\in\varphi(x)\}$ is a nonempty compact set.*

Theorem 2.3. *Let D be a bounded convex closed subset of a Banach space X. Let $\varphi_1:D\to X$ be a single-valued map and $\varphi_2:D\to P_{cp,cv}(X)$ be a multimap such that $\varphi_1(x)+\varphi_2(x)\in P(D)$ for $x\in D$. Suppose that*

(a) *φ_1 is a contraction with the contraction constant $k<\frac{1}{2}$, and*
(b) *φ_2 is u.s.c. and compact.*

Then the fixed point set $\mathrm{Fix}(\varphi_1+\varphi_2):=\{x:x\in\varphi_1(x)+\varphi_2(x)\}$ is a nonempty compact set.

Proof. Since φ_1 is a single-valued contraction, it is continuous on X. For $x \in D$, $\varphi_1(x) + \varphi_2(x) \in P(D)$. Therefore the multimap $\varphi : D \to P(D)$ defined by $\varphi(x) = \varphi_1(x) + \varphi_2(x)$ is u.s.c. Since φ_1 is a contraction with the contraction constant k, then we have that $\mu(\varphi_1(S)) \le k\mu(S)$ for any bounded subset S of X. Thus $\mu(\varphi_1(x)) \le k\mu(\{x\}) = 0$. Obviously, $\varphi_1 : D \to P_{\mathrm{cp,cv}}(D)$. As a result, we have $\varphi : D \to P_{\mathrm{cp,cv}}(D)$. Let S be a bounded subset of D. As φ_2 is compact, we have that $\gamma(\varphi_2(S)) = 0$. It follows:

$$\begin{aligned}\gamma(\varphi(S)) &\le \gamma(\varphi_1(S) + \varphi_2(S))\\ &\le \gamma(\varphi_1(S)) + \gamma(\varphi_2(S)) \le \mu(\varphi_1(S))\\ &\le k\mu(S) \le 2k\gamma(S)\\ &< \gamma(S),\end{aligned}$$

whenever $\gamma(S) > 0$. Hence, we have that $\gamma(\varphi(S)) < \gamma(S)$, $\gamma(S) > 0$ for all bounded sets S in D. So $\varphi : D \to P_{\mathrm{cp,cv}}(D)$ is a γ-condensing multimap. By Theorem 2.2, the fixed point set Fixφ is a nonempty compact set. This completes the proof. □

3. Topological structure of solution sets

In this section, let X be reflexive. We study the topological structure of solution sets in cases that $T(t)$ is compact and noncompact, respectively. Before stating and proving the main results, we introduce the following hypotheses:

(H_1) the multivalued nonlinearity $F : [0,b] \times C([-\tau,0],X) \to P_{\mathrm{cl,cv}}(X)$ satisfies
  (i) $F(t,\cdot)$ is weakly u.s.c. for a.e. $t \in [0,b]$, and the multimap $F(\cdot,c)$ has a strongly measurable selection for every $c \in C([-\tau,0],X)$;
  (ii) there exists a function $\alpha(t) \in L^1([0,b],\mathbb{R}^+)$ such that

$$|F(t,c)| \le \alpha(t)(1 + \|c\|_*) \text{ for a.e. } t \in [0,b] \text{ and } c \in C([-\tau,0],X).$$

(H_2) The function $h : [0,b] \times C([-\tau,0],X) \to X$ is continuous and there exists a constant $\beta \in (0,1)$ and $d, d_1 > 0$ with $d\Big(\|A^{-\beta}\| + \frac{C_{1-\beta}b^\beta}{\beta}\Big) < \frac{1}{2}$, such that $h \in D(A^\beta)$ and for any $c_1, c_2 \in C([-\tau,0],X)$, the function $A^\beta h(t,\cdot)$ is strongly measurable and $A^\beta h(t,\cdot)$ satisfies the Lipschitz condition

$$|A^\beta h(t,c_1) - A^\beta h(t,c_2)| \le d\|c_1 - c_2\|_*$$

and the inequality

$$|A^\beta h(t,c_1)| \le d_1(1 + \|c_1\|_*) \text{ for every } t \in [0,b].$$

Given $x \in C([-\tau,b],X)$, let us denote

$$Sel_F(x) = \{f \in L^1([0,b],X) : f(t) \in F(t,x_t) \text{ for a.e. } t \in [0,b]\}.$$

The set $Sel_F(x)$ is always nonempty, as Lemma 3.1 below shows.

Lemma 3.1. [8] (see also [6]) *Let condition (H_1) be satisfied. Then $Sel_F : C([-\tau,b],X) \to P(L^1([0,b],X))$ is weakly u.s.c. with nonempty, convex and weakly compact values.*

Definition 3.1. A continuous function $x : [-\tau,b] \to X$ is said to be a mild solution to inclusion (1.1) if $x(t) = \phi(t)$ for $t \in [-\tau,0]$ and if there exists $f(t) \in L^1([0,b],X)$ such that $f(t) \in F(t,x_t)$, and x satisfies the following integral equation:

$$x(t) = T(t)[\phi(0) - h(0,\phi)] + h(t,x_t) + \int_0^t AT(t-s)h(s,x_s)\,ds + \int_0^t T(t-s)f(s)\,ds, \quad \text{for } t \in [0,b].$$

Remark 3.1. For any $x \in C([-\tau,b],X)$, now define a solution multioperator $\mathcal{F} : C([-\tau,b],X) \to P(C([-\tau,b],X))$ as follows

$$\mathcal{F}(x) = \Gamma_1(x) + \Gamma_2(x),$$

where

$$\Gamma_1(x)(t) = \begin{cases} -T(t)h(0,\phi) + h(t,x_t) + \displaystyle\int_0^t AT(t-s)h(s,x_s)\,ds, & t \in [0,b],\\ 0, & t \in [-\tau,0],\end{cases}$$

and

$$\Gamma_2(x)(t) = \left\{ y(t) \in C([-\tau, b], X) : y(t) = \begin{cases} S(f)(t), \quad f \in Sel_F(x), & t \in [0, b], \\ \phi(t), & t \in [-\tau, 0], \end{cases} \right\},$$

here, the operator $S : L^1([0, b], X) \to C([0, b], X)$ is defined by

$$S(f) = T(t)\phi(0) + \int_0^t T(t-s) f(s)\,\mathrm{d}s.$$

It is easy to verify that the fixed points of the multioperator $\mathcal{F}$ are mild solutions to inclusion (1.1).

Lemma 3.2. [8, Lemma 3.3] *Let hypothesis* (H_1) *be satisfied. Then there exists a sequence* $\{F_n\}$ *with* $F_n : [0, b] \times C([-\tau, 0], X) \to P_{\mathrm{cl,cv}}(X)$ *such that*

(i) $F(t, c) \subset \cdots \subset F_{n+1}(t, c) \subset F_n(t, c) \subset \cdots \subset \overline{\mathrm{co}}(F(t, B_{3^{1-n}}(c)))$, $n \geq 1$, *for each* $t \in [0, b]$ *and* $c \in C([-\tau, 0], X)$;
(ii) $|F_n(t, c)| \leq \alpha(t)(2 + \|c\|_*)$, $n \geq 1$, *for a.e.* $t \in [0, b]$ *and each* $c \in C([-\tau, 0], X)$;
(iii) *there exists* $E \subset [0, b]$ *with* $mes(E) = 0$ *such that for each* $x^* \in X^*$, $\varepsilon > 0$ *and* $(t, c) \in [0, b] \setminus E \times C([-\tau, 0], X)$, *there exists* $N > 0$ *such that for all* $n \geq N$,

$$x^*(F_n(t, c)) \subset x^*(F(t, c)) + (-\varepsilon, \varepsilon);$$

(iv) $F_n(t, \cdot) : C([-\tau, 0], X) \to P_{\mathrm{cl,cv}}(X)$ *is continuous for a.e.* $t \in [0, b]$ *with respect to the Hausdorff metric for each* $n \geq 1$;
(v) *for each* $n \geq 1$, *there exists a selection* $g_n : [0, b] \times C([-\tau, 0], X) \to X$ *of* F_n *such that* $g_n(\cdot, c)$ *is measurable for each* $c \in C([-\tau, 0], X)$ *and for any compact subset* $\mathcal{D} \subset C([-\tau, 0], X)$, *there exist constants* $C_V > 0$ *and* $\delta > 0$ *for which the estimate*

$$|g_n(t, c_1) - g_n(t, c_2)| \leq C_V \alpha(t) \|c_1 - c_2\|_*$$

holds for a.e. $t \in [0, b]$ *and each* $c_1, c_2 \in C([-\tau, 0], X)$ *with* $V := \mathcal{D} + B_\delta(0)$;
(vi) F_n *verifies condition* (H_1)(i) *with* F_n *instead of* F *for each* $n \geq 1$, *provided that* X *is reflexive.*

3.1. Compact operator case

The following compactness characterizations of the solution set to inclusion (1.1) will be useful.

Lemma 3.3. *Suppose that* $\{T(t)\}_{t>0}$ *is compact and that there exists* $r \in L^1([0, b], \mathbb{R}^+)$ *such that*

$$|F(t, c)| \leq r(t) \text{ for a.e. } t \in [0, b] \text{ and } c \in C([-\tau, 0], X).$$

Then the multimap Γ_2 *is compact in* $C([-\tau, b], X)$.

Proof. Let D be a bounded set of $C([-\tau, b], X)$. We will prove that for each $t \in [-\tau, b]$, $V(t) = \{\Gamma_2(x)(t) : x \in D\}$ is relatively compact in X.

Obviously, for $t \in [-\tau, 0]$, $V(t) = \{\phi(t)\}$ is relatively compact in X. Let $t \in [0, b]$ be fixed, for $x \in D$ and $y \in V(t)$, there exists $f \in Sel_F(x)$ such that

$$y(t) = T(t)\phi(0) + \int_0^t T(t-s) f(s)\,\mathrm{d}s.$$

For an arbitrary $\varepsilon \in (0, t)$, define an operator $J_\varepsilon : V(t) \to X$ by

$$J_\varepsilon y(t) = T(t)\phi(0) + T(\varepsilon) \int_0^{t-\varepsilon} T(t-\varepsilon-s) f(s)\,\mathrm{d}s.$$

From the compactness of $T(t)$, $t > 0$, we get that the set $V_\varepsilon(t) = \{J_\varepsilon y(t) : y(t) \in V(t)\}$ is relatively compact in X for each $\varepsilon \in (0, t)$. Moreover, it follows

$$|y(t) - J_\varepsilon y(t)| \leq \left| \int_{t-\varepsilon}^t T(t-s) f(s)\,\mathrm{d}s \right| \leq M \int_{t-\varepsilon}^t r(s)\,\mathrm{d}s.$$

Therefore, there is a relatively compact set arbitrarily close to the set $V(t)$. Thus the set $V(t)$ is also relatively compact in X, which yields that $V(t) = \{\Gamma_2(x)(t) : x \in D\}$ is relatively compact in X for each $t \in [-\tau, b]$.

We proceed to verify that the set $\{\Gamma_2(x) : x \in D\}$ is equicontinuous on $[-\tau, b]$. Taking $0 < t_1 < t_2 \le b$ and $\delta > 0$ small enough, for any $y(t) \in \Gamma_2(x)$, we obtain:

$$\begin{aligned}
|y(t_2) - y(t_1)| \le \|T(t_2) - T(t_1)\||\phi(0)| &+ \left|\int_{t_1}^{t_2} T(t_2 - s) f(s)\,\mathrm{d}s\right| \\
&+ \left|\int_0^{t_1-\delta} [T(t_2 - s) - T(t_1 - s)] f(s)\,\mathrm{d}s\right| + \left|\int_{t_1-\delta}^{t_1} [T(t_2 - s) - T(t_1 - s)] f(s)\,\mathrm{d}s\right| \\
&\le \|T(t_2) - T(t_1)\|\|\phi\|_* + M\int_{t_1}^{t_2} r(s)\,\mathrm{d}s \\
&+ \sup_{s\in[0,t_1-\delta]} \|T(t_2 - s) - T(t_1 - s)\| \int_0^{t_1-\delta} r(s)\,\mathrm{d}s + 2M\int_{t_1-\delta}^{t_1} r(s)\,\mathrm{d}s.
\end{aligned}$$

The right-hand side tends to zero as $t_2 - t_1 \to 0$, since $T(t)$ is strongly continuous, and the compactness of $T(t)$ $(t > 0)$, implies the continuity in the uniform operator topology.

For $-\tau \le t_1 < 0 < t_2 \le b$, we have

$$|y(t_2) - y(t_1)| \le |T(t_2)\phi(0) - \phi(t_1)| + \left|\int_0^{t_2} T(t_2 - s) f(s)\,\mathrm{d}s\right| \le \|T(t_2) - I)\|\|\phi\|_* + |\phi(t_1) - \phi(0)| + M\int_0^{t_2} r(s)\,\mathrm{d}s.$$

The right-hand side tends to zero as $t_2 - t_1 \to 0$ $(t_2 \to 0^+,\ t_1 \to 0^-)$, since $\phi(t)$ is continuous. Note that for $t_1, t_2 \in [-\tau, 0]$, $|y(t_2) - y(t_1)| = |\phi(t_2) - \phi(t_1)| \to 0$ as $t_2 - t_1 \to 0$. Thus $\{\Gamma_2(x) : x \in D\}$ is equicontinuous as well. Thus, an application of the Arzela–Ascoli theorem justifies that $\{\Gamma_2(x) : x \in D\}$ is relatively compact in $C([-\tau, b], X)$. Hence Γ_2 is compact in $C([-\tau, b], X)$. This completes the proof. □

Let $a \in [0, b)$ and $\varphi \in C([-\tau, b], X)$. Consider the integral equation of the form

$$x(t) = \begin{cases} \varphi(t) + h(t, x_t) + \displaystyle\int_a^t AT(t - s)h(s, x_s)\,\mathrm{d}s + \int_a^t T(t - s)g(s, x_s)\,\mathrm{d}s, & t \in [a, b], \\ \varphi(t), & t \in [-\tau, a]. \end{cases} \tag{3.1}$$

Lemma 3.4. *Assume that for every $c \in C([a - \tau, a], X)$, $g(\cdot, c)$ is L^1-integrable, $\{T(t)\}_{t>0}$ is compact and (H_2) holds. Suppose in addition that*

(i) *for any compact subset $K \subset C([a - \tau, a], X)$, there exist $\delta > 0$ and $L_K \in L^1([a, b], \mathbb{R}^+)$ such that*

$$|g(t, c_1) - g(t, c_2)| \le L_K(t)\|c_1 - c_2\|_*, \text{ for a.e. } t \in [a, b] \text{ and each } c_1, c_2 \in B_\delta(K);$$

(ii) *there exists $r_1(t) \in L^1([a, b], \mathbb{R}^+)$ such that $|g(t, c)| \le r_1(t)(c' + \|c\|_*)$ for a.e. $t \in [a, b]$ and every $c \in C([a - \tau, a], X)$, where c' is arbitrary, but fixed.*

If $d_1\|A^{-\beta}\| < 1$, then integral equation (3.1) admits a unique solution for every $\varphi \in C([-\tau, b], X)$. Moreover, the solution to (3.1) depends continuously on φ.

Proof. *Step 1.* (Priori estimate). Assume that x is a solution to (3.1). We have

$$\begin{aligned}
|x(t)| &\le |A^{-\beta}A^\beta h(t, x_t)| + \left|\int_a^t A^{1-\beta}T(t - s)A^\beta h(s, x_s)\,\mathrm{d}s\right| + |\varphi(t)| + \left|\int_a^t T(t - s)g(s, x_s)\,\mathrm{d}s\right| \\
&\le d_1\|A^{-\beta}\|(1 + \|x_t\|_*) + d_1C_{1-\beta}\int_a^t (t - s)^{\beta-1}(1 + \|x_s\|_*)\,\mathrm{d}s + \max_{[a,b]} |\varphi(t)| + M\int_a^t r_1(s)(c' + \|x_s\|_*)\,\mathrm{d}s
\end{aligned}$$

$$\leq d_1\|A^{-\beta}\|\big(1+\|x\|_{[a-\tau,t]}\big)+d_1C_{1-\beta}\int_a^t(t-s)^{\beta-1}\big(1+\|x\|_{[a-\tau,s]}\big)\,\mathrm{d}s$$
$$+\max_{[-\tau,b]}|\varphi(t)|+M\int_a^t r_1(s)\big(c'+\|x\|_{[a-\tau,s]}\big)\,\mathrm{d}s,$$

for $t\in[a,b]$, and notice that $|x(t)|=|\varphi(t)|$ for $t\in[-\tau,a]$. Then

$$\|x\|_{[a-\tau,t]}\leq\frac{1}{1-d_1\|A^{-\beta}\|}\Bigg(d_1\|A^{-\beta}\|+\max_{[-\tau,b]}|\varphi(t)|+\frac{d_1C_{1-\beta}b^{\beta}}{\beta}+c'M\|r_1\|_1$$
$$+\int_a^t\Big[d_1C_{1-\beta}(t-s)^{\beta-1}+Mr_1(s)\Big]\|x\|_{[a-\tau,s]}\,\mathrm{d}s\Bigg).$$

By Gronwall's inequality, we get that there exists $M_1>0$ such that $\|x\|_{[-\tau,b]}\leq M_1$.

Step 2. Let $\varphi\in C([-\tau,b],X)$ be fixed. From $d_1\|A^{-\beta}\|<1$, we can find one ξ arbitrarily close to a such that

$$d_1\left(\|A^{-\beta}\|+\frac{C_{1-\beta}(\xi-a)^{\beta}}{\beta}\right)+M\|r_1\|_{L[a,\xi]}<1.$$

Then for one of such ξ, we choose one ρ satisfying

$$\rho\geq\frac{d_1\left(\|A^{-\beta}\|+\frac{C_{1-\beta}(\xi-a)^{\beta}}{\beta}\right)+\max\limits_{[-\tau,\xi]}|\varphi(t)|+Mc'\|r_1\|_{L[a,\xi]}}{1-d_1\left(\|A^{-\beta}\|+\frac{C_{1-\beta}(\xi-a)^{\beta}}{\beta}\right)-M\|r_1\|_{L[a,\xi]}},$$

that is,

$$d_1(1+\rho)\left(\|A^{-\beta}\|+\frac{C_{1-\beta}(\xi-a)^{\beta}}{\beta}\right)+\max_{[-\tau,\xi]}|\varphi(t)|+M(c'+\rho)\|r_1\|_{L[a,\xi]}\leq\rho.$$

Write

$$B_{\rho}(\xi)=\left\{x\in C([-\tau,\xi],X):\max_{t\in[-\tau,\xi]}|x(t)|\leq\rho\right\}.$$

Let us define the operator W:

$$Wx(t)=W_1x(t)+W_2x(t),$$

where

$$W_1x(t)=\begin{cases}h(t,x_t)+\displaystyle\int_a^t AT(t-s)h(s,x_s)\,\mathrm{d}s, & t\in[a,b],\\ 0, & t\in[-\tau,a],\end{cases}$$

and

$$W_2x(t)=\begin{cases}\varphi(t)+\displaystyle\int_a^t T(t-s)g(s,x_s)\,\mathrm{d}s, & t\in[a,b],\\ \varphi(t), & t\in[-\tau,a].\end{cases}$$

For $x\in B_{\rho}(\xi)$, we have

$$|W_1x(t)+W_2x(t)|\leq|A^{-\beta}A^{\beta}h(t,x_t)|+\left|\int_a^t A^{1-\beta}T(t-s)A^{\beta}h(s,x_s)\,\mathrm{d}s\right|+|\varphi(t)|+\left|\int_a^t T(t-s)g(s,x_s)\,\mathrm{d}s\right|$$
$$\leq\|A^{-\beta}\|d_1\big(1+\|x\|_{[a-\tau,t]}\big)+d_1C_{1-\beta}\int_a^t(t-s)^{\beta-1}\big(1+\|x\|_{[a-\tau,s]}\big)\,\mathrm{d}s$$

$$
\begin{aligned}
&+\max_{[-\tau,\xi]}|\varphi(t)|+M\int_a^t r_1(s)\big(c'+\|x\|_{[a-\tau,s]}\big)\,\mathrm{d}s\\
&\le d_1(1+\rho)\left(\|A^{-\beta}\|+\frac{C_{1-\beta}(\xi-a)^\beta}{\beta}\right)+\max_{[-\tau,\xi]}|\varphi(t)|+M(c'+\rho)\|r_1\|_{L[a,\xi]}\\
&\le\rho,
\end{aligned}
$$

for $t\in[a,\xi]$. Obviously, W maps $B_\rho(\xi)$ into itself.

For any $x, y\in B_\rho(\xi)$ and $t\in[a,b]$, we have

$$
\begin{aligned}
|W_1x(t)-W_1y(t)|&\le|h(t,x_t)-h(t,y_t)|+\left|\int_a^t AT(t-s)[h(s,x_s)-h(s,y_s)]\,\mathrm{d}s\right|\\
&=\left|A^{-\beta}A^\beta[h(t,x_t)-h(t,y_t)]\right|+\left|\int_a^t A^{1-\beta}T(t-s)A^\beta[h(s,x_s)-h(s,y_s)]\,\mathrm{d}s\right|\\
&\le d\,\|A^{-\beta}\|\,\|x_t-y_t\|_*+d\,C_{1-\beta}\int_a^t(t-s)^{\beta-1}\|x_s-y_s\|_*\,\mathrm{d}s\\
&\le d\,\|A^{-\beta}\|\,\|x-y\|_{[a-\tau,t]}+dC_{1-\beta}\int_a^t(t-s)^{\beta-1}\|x-y\|_{[a-\tau,s]}\,\mathrm{d}s\\
&\le d\left(\|A^{-\beta}\|+\frac{C_{1-\beta}(\xi-a)^\beta}{\beta}\right)\|x-y\|_{[a-\tau,\xi]}.
\end{aligned}
$$

Noting that $W_1x(t)=0$ for $t\in[-\tau,a]$, which implies that

$$
\|W_1x-W_1y\|_{[a-\tau,\xi]}\le d\left(\|A^{-\beta}\|+\frac{C_{1-\beta}\xi^\beta}{\beta}\right)\|x-y\|_{[a-\tau,\xi]},
$$

we get that W_1 is a contraction.

Next, we will prove that W_2 is continuous on $B_\rho(\xi)$. Let $x^n, x\in B_\rho(\xi)$ with $x^n\to x$ on $B_\rho(\xi)$. Then by (i) and the fact that $x^n_t\to x_t$ for $t\in[a,\xi]$, we have

$$
g(s,x^n_s)\to g(s,x_s),\ \text{for a.e. } s\in[a,\xi]\ \text{as } n\to\infty.
$$

Noting that $|g(s,x^n_s)-g(s,x_s)|\le 2r_1(t)(c'+\rho)$, by Lebesgue's dominated convergence theorem, we have

$$
|W_2x^n(t)-W_2x(t)|\le M\int_a^t\left|g(s,x^n_s)-g(s,x_s)\right|\mathrm{d}s\to 0,\ \text{as } n\to\infty.
$$

Moreover, from the proof of Lemma 3.3, we see that W_2 is a compact operator. Thus, W_2 is a completely continuous operator. Hence, Krasnoselskii's fixed point theorem shows that there is a fixed point of W, denoted by x, which is a local solution to equation (3.1).

Step 3. We prove that this solution is unique. In fact, let y be another local solution to equation (3.1). According to condition (i), we obtain

$$
\begin{aligned}
|x(t)-y(t)|&\le|h(t,x_t)-h(t,y_t)|+\left|\int_a^t AT(t-s)[h(s,x_s)-h(s,y_s)]\,\mathrm{d}s\right|+\left|\int_a^t T(t-s)[g(s,x_s)-g(s,y_s)]\,\mathrm{d}s\right|\\
&\le d\|A^{-\beta}\|\,\|x_t-y_t\|_*+dC_{1-\beta}\int_a^t(t-s)^{\beta-1}\|x_s-y_s\|_*\mathrm{d}s+M\int_a^t L_K(s)\|x_s-y_s\|_*\mathrm{d}s\\
&\le d\|A^{-\beta}\|\,\|x-y\|_{[a-\tau,t]}+\int_a^t[dC_{1-\beta}(t-s)^{\beta-1}+ML_K(s)]\|x-y\|_{[a-\tau,s]}\,\mathrm{d}s,
\end{aligned}
$$

for $t\in[a,\xi]$, and $|x(t)-y(t)|=0$ for $t\in[-\tau,a]$. It follows that

$$\|x-y\|_{[a-\tau,t]} \le \frac{1}{1-d\|A^{-\beta}\|}\int_a^t [dC_{1-\beta}(t-s)^{\beta-1}+ML_K(s)]\|x-y\|_{[a-\tau,s]}\,\mathrm{d}s.$$

Applying Gronwall's inequality, we get $\|x-y\|_{[a-\tau,t]}=0$, which implies $x(t)=y(t)$ for $t\in[-\tau,\xi]$.

Next, we continue the solution for $t\ge\xi$. For $t\in[\xi,\xi_1]$, where $\xi<\xi_1$, we say that a function $\hat{x}(t)$ is a continuation of $x(t)$ to the interval $[\xi,\xi_1]$, if

(a) $\hat{x}\in C([-\tau,\xi_1],X)$, and
(b) $\hat{x}(t)=\varphi(t)+h(t,\hat{x}_t)+\int_\xi^t AT(t-s)h(s,\hat{x}_s)\,\mathrm{d}s+\int_\xi^t T(t-s)g(s,\hat{x}_s)\,\mathrm{d}s$.

The terminology mild continuation applied to $\hat{x}(t)$ is justified by the observation that if we define a new function $v(t)$ on $[0,\xi_1]$ by setting

$$v(t)=\begin{cases} x(t), & \text{if } t\in[0,\xi],\\ \hat{x}(t), & \text{if } t\in[\xi,\xi_1],\end{cases}$$

and $v(t)=\varphi(t)$, $t\in[-\tau,a]$, then $v(t)$ is a mild solution to (3.1) on $[0,\xi_1]$. The existence and uniqueness of the mild continuation $\hat{x}(t)$ is demonstrated exactly as above with only some minor changes. The details are therefore omitted. Repeating this procedure and by the a priori estimate of the solution, one continues the solution till the time $\xi_m=\xi_{\max}$, where $[0,\xi_m]$ is the maximum interval of the existence and uniqueness of a solution, and $\tilde{x}$ denotes the solution on the interval $[0,\xi_{\max}]$. We prove $\xi_{\max}=b$. If this is not the case, then $\xi_{\max}<b$. Put

$$\hat{\varphi}(t)=\varphi(t)+\int_a^{\xi_{\max}} AT(t-s)h(s,\tilde{x}_s)\,\mathrm{d}s+\int_a^{\xi_{\max}} T(t-s)g(s,\tilde{x}_s)\,\mathrm{d}s,$$

with $\hat{\varphi}\in C([\xi_{\max},b],X)$. We consider the following integral equation:

$$x(t)=\hat{\varphi}(t)+h(t,x_t)+\int_{\xi_{\max}}^t AT(t-s)h(s,x_s)\,\mathrm{d}s+\int_{\xi_{\max}}^t T(t-s)g(s,x_s)\,\mathrm{d}s,$$

one can use the previous arguments to extend the solution beyond $\xi_{\max}$, which is a contradiction.

Step 4. Let $\varphi^n\to\varphi^0$ in $C([-\tau,b],X)$ as $n\to\infty$, and x^n be the solution to equation (3.1) with the perturbation φ^n, i.e.,

$$x^n(t)=\varphi^n(t)+h(t,x_t^n)+\int_a^t AT(t-s)h(s,x_s^n)\,\mathrm{d}s+\int_a^t T(t-s)g(s,x_s^n)\,\mathrm{d}s \tag{3.2}$$

for $t\in[a,b]$ and $x^n(t)=\varphi^n(t)$ for $t\in[-\tau,a]$. It is clear that $\lim_{n\to\infty}x^n$ exists in $C([-\tau,a],X)$. From condition (ii) and the compactness of $T(t)$ for $t>0$ it follows that the set

$$\left\{\int_a^t T(t-s)g(s,x_s^n)\,\mathrm{d}s : n\ge 1\right\}$$

is relatively compact in $C([a,b],X)$. This gives that the family

$$\left\{x^n(t)-h(t,x_t^n)-\int_a^t AT(t-s)h(s,x_s^n)\,\mathrm{d}s : n\ge 1\right\}$$

is relatively compact in $C([a,b],X)$. We only prove that $\lim_{n\to\infty}x^n$ exists in $C([a,b],X)$. On the contrary, if $\lim_{n\to\infty}x^n$ does not exist in $C([a,b],X)$, then for any $n\in\mathbb{N}$, we have n_1,n_2 with $n_1,n_2>n$ such that $\|x^{n_1}-x^{n_2}\|_{[a,b]}>\varepsilon_0$ ($\varepsilon_0>0$ is a constant), that is, there exists t^* such that

$$|x^{n_1}(t^*)-x^{n_2}(t^*)|=\|x^{n_1}-x^{n_2}\|_{[a,b]}>\varepsilon_0.$$

Let $u^n(t)=x^n(t)-h(t,x_t^n)-\int_a^t AT(t-s)h(s,x_s^n)\,\mathrm{d}s$. Using (H_2), we estimate

$$\begin{aligned}|u^{n_1}(t^*)-u^{n_2}(t^*)| &\ge |x^{n_1}(t^*)-x^{n_2}(t^*)|-|h(t^*,x_{t^*}^{n_1})-h(t^*,x_{t^*}^{n_2})|-\left|\int_a^{t^*} AT(t^*-s)[h(s,x_s^{n_1})-h(s,x_s^{n_2})]\,\mathrm{d}s\right|\\ &\ge |x^{n_1}(t^*)-x^{n_2}(t^*)|-d\|A^{-\beta}\|\|x_{t^*}^{n_1}-x_{t^*}^{n_2}\|_*-dC_{1-\beta}\int_a^{t^*}(t^*-s)^{\beta-1}\|x_s^{n_1}-x_s^{n_2}\|_*\mathrm{d}s\end{aligned}$$

$$\geq |x^{n_1}(t^*) - x^{n_2}(t^*)| - d\left(\|A^{-\beta}\| + \frac{C_{1-\beta}b^\beta}{\beta}\right)\|x^{n_1} - x^{n_2}\|_{[a,b]}$$
$$= \left[1 - d\left(\|A^{-\beta}\| + \frac{C_{1-\beta}b^\beta}{\beta}\right)\right]\varepsilon_0,$$

which contradicts the compactness of u^n in $C([a,b],X)$. Hence $\{x^n\}$ converges in $C([-\tau,b],X)$. We assume $x^n \to x$ in $C([a,b],X)$ as $n \to \infty$. Therefore, taking the limit in (3.2) as $n \to \infty$, one finds, again by (H_2) and Lebesgue's dominated convergence theorem, that x is the solution to equation (3.1) with the perturbation φ^0. This completes the proof. □

For convenience, define

$$\tilde{d} = d_1\left(\|A^{-\beta}\| + \frac{C_{1-\beta}b^\beta}{\beta}\right) + M\|\alpha\|_1.$$

Theorem 3.1. *Assume that (H_1) and (H_2) hold. In addition, suppose that $\{T(t)\}_{t>0}$ is compact in X. If $\tilde{d} < 1$, then the solution set of the inclusion (1.1) is a nonempty compact subset of $C([-\tau,b],X)$ for each $\phi \in C([-\tau,0],X)$.*

Proof. *Step 1.* Let $\phi \in C([-\tau,0],X)$ be fixed. Consider the set

$$B_R(b) = \{x \in C([-\tau,b],X) : \max_{t\in[-\tau,b]} |x(t)| \leq R\},$$

where

$$R > \frac{\|\phi\|_* + M[\|\phi\|_* + \|A^{-\beta}\|d_1(1+\|\phi\|_*)] + \tilde{d}}{1-\tilde{d}}.$$

It is clear that $B_R(b)$ is a bounded, closed and convex set of $C([-\tau,b],X)$. We first show that $\Gamma_1(B_R(b)) + \Gamma_2(B_R(b)) \subset B_R(b)$. Indeed, taking $x \in B_R(b)$ and $y(t) \in \Gamma_2(x)$, there exists $f \in Sel_F(x)$ such that

$$|\Gamma_1 x(t)| \leq |T(t)h(0,\phi)| + |A^{-\beta}A^\beta h(t,x_t)| + \left|\int_0^t A^{1-\beta}T(t-s)A^\beta h(s,x_s)\,\mathrm{d}s\right|$$
$$\leq M|A^{-\beta}A^\beta h(0,\phi)| + \|A^{-\beta}\|d_1(1+\|x_t\|_*) + d_1C_{1-\beta}\int_0^t (t-s)^{\beta-1}(1+\|x_s\|_*)\,\mathrm{d}s$$
$$\leq Md_1\|A^{-\beta}\|(1+\|\phi\|_*) + \|A^{-\beta}\|d_1\big(1+\|x\|_{[-\tau,t]}\big) + d_1C_{1-\beta}\int_0^t (t-s)^{\beta-1}\big(1+\|x\|_{[-\tau,s]}\big)\,\mathrm{d}s$$
$$\leq Md_1\|A^{-\beta}\|(1+\|\phi\|_*) + \|A^{-\beta}\|d_1(1+R) + d_1(1+R)\frac{C_{1-\beta}b^\beta}{\beta},$$

and

$$|y(t)| \leq |T(t)\phi(0)| + \left|\int_0^t T(t-s)f(s)\,\mathrm{d}s\right|$$
$$\leq M\|\phi\|_* + M\int_0^t \alpha(s)(1+\|x_s\|_*)\,\mathrm{d}s$$
$$\leq M\|\phi\|_* + M(1+R)\|\alpha\|_1,$$

it follows that

$$|\Gamma_1 x(t) + y(t)| \leq M[\|\phi\|_* + \|A^{-\beta}\|d_1(1+\|\phi\|_*)] + \tilde{d}(1+R)$$

for $t \in [0,b]$. From $\Gamma_1 x(t) + y(t) = \phi(t)$ for $t \in [-\tau,0]$, we know

$$|\Gamma_1 x(t) + \Gamma_2 x(t)| \leq \|\phi\|_* + M[\|\phi\|_* + \|A^{-\beta}\|d_1(1+\|\phi\|_*)] + \tilde{d}(1+R) \leq R$$

for $t \in [-\tau,b]$,

Step 2. We show that Γ_1 is a contraction on $C([-\tau,b],X)$. Let $x, y \in C([-\tau,b],X)$. Then

$$\begin{aligned}|\Gamma_1 x(t)-\Gamma_1 y(t)| &\le |h(t,x_t)-h(t,y_t)| + \left|\int_0^t AT(t-s)[h(s,x_s)-h(s,y_s)]\,\mathrm{d}s\right| \\ &\le d\|A^{-\beta}\|\|x_t-y_t\|_* + dC_{1-\beta}\int_0^t (t-s)^{\beta-1}\|x_s-y_s\|_*\mathrm{d}s \\ &\le d\|A^{-\beta}\|\|x-y\|_{[-\tau,t]} + dC_{1-\beta}\int_0^t (t-s)^{\beta-1}\|x-y\|_{[-\tau,s]}\,\mathrm{d}s \\ &\le d\Big(\|A^{-\beta}\| + \frac{C_{1-\beta}b^\beta}{\beta}\Big)\|x-y\|_{[-\tau,b]}.\end{aligned}$$

Noting that $\Gamma_1 x(t)=0$ for $t\in[-\tau,0]$, which implies that

$$\|\Gamma_1 x-\Gamma_1 y\|_{[-\tau,b]} \le d\Big(\|A^{-\beta}\| + \frac{C_{1-\beta}b^\beta}{\beta}\Big)\|x-y\|_{[-\tau,b]}.$$

This shows that Γ_1 is a contraction, since $d\Big(\|A^{-\beta}\| + \frac{C_{1-\beta}b^\beta}{\beta}\Big) < \frac{1}{2}$.

Step 3. An application of Lemma 3.3 enables us to find that Γ_2 is compact on $B_R(b)$. We only show that Γ_2 is u.s.c.

By Lemma 2.2, it suffices to show that Γ_2 has closed graph (and therefore has closed values). Let $x^n \subset B_R(b)$ with $x^n \to x$ and $y^n \in \Gamma_2(x^n)$ with $y^n \to y$. We shall prove that $y \in \Gamma_2(x)$. By the definition of Γ_2, there exist $f^n \in Sel_F(x^n)$ such that

$$y^n(t) = T(t)\phi(0) + \int_0^t T(t-s)f^n(s)\,\mathrm{d}s,\ \text{for } t\in[0,b],\ \text{and } y^n(t)=\phi(t),\ \text{for } t\in[-\tau,0].$$

We need to prove that there exists $f \in Sel_F(x)$ such that

$$y(t) = T(t)\phi(0) + \int_0^t T(t-s)f(s)\,\mathrm{d}s,\ \text{for } t\in[0,b],\ \text{and } y(t)=\phi(t),\ \text{for } t\in[-\tau,0]. \tag{3.3}$$

By (H_1)(ii), noticing that $Sel_F(x)$ is weakly u.s.c. with weakly compact and convex values due to Lemma 3.1, an application of Lemma 2.3 yields that there exists $f \in Sel_F(x)$ and a subsequence of f^n, still denoted by f^n, such that $f^n \rightharpoonup f$ in $L^1([0,b],X)$. From this and Lemma 3.3, we see

$$y_n(t) = T(t)\phi(0) + \int_0^t T(t-s)f^n(s)\,\mathrm{d}s \to T(t)\phi(0) + \int_0^t T(t-s)f(s)\,\mathrm{d}s,\ \text{as } n\to\infty.$$

By the uniqueness of the limit, (3.3) holds and $y\in\Gamma_2(x)$. It follows that Γ_2 is closed and therefore has compact values.

Therefore, the operators Γ_1 and Γ_2 satisfy all conditions of Theorem 2.3, thus the fixed points set of the operator $\Gamma_1+\Gamma_2$ is a nonempty compact subset of $C([-\tau,b],X)$. □

Now, let $\Theta(\phi)$ denote the set of all mild solutions for inclusion (1.1).

Theorem 3.2. *Under the conditions in Theorem 3.1, the solution set of (1.1) is an R_δ-set.*

Proof. To this aim, let us consider the following semilinear evolution inclusion

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}[x(t)-h(t,x_t)] \in Ax(t) + F_n(t,x_t), & t\in[0,b], \\ x(t)=\phi(t), & t\in[-\tau,0], \end{cases} \tag{3.4}$$

where multivalued functions $F_n : [0,b]\times C([-\tau,0],X) \to P_{\mathrm{cl,cv}}(X)$ are established in Lemma 3.2. Let $\Theta_n(\phi)$ denote the set of all mild solutions for inclusion (3.4).

From Lemma 3.2(ii) and (vi), it follows that $\{F_n\}$ verifies condition (H_1) for each $n\ge 1$. Then from Lemma 3.1, one finds that Sel_{F_n} is weakly u.s.c. with convex and weakly compact values. Moreover, one can see from Theorem 3.1 that each set $\Theta_n(\phi)$ is nonempty and compact in $C([-\tau,b],X)$ for each $n\ge 1$.

We show that the set $\Theta_n(\phi)$ is contractible for each $n \geq 1$. In fact, let $x \in \Theta_n(\phi)$. For any $\lambda \in [0,1]$, we consider the Cauchy problem of the form

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}[y(t) - h(t, y_t)] = Ay(t) + g_n(t, y_t), & t \in [\lambda b, b], \\ y(t) = x(t), & t \in [-\tau, \lambda b], \end{cases} \tag{3.5}$$

where g_n is the selection of F_n. Since the functions g_n satisfy the conditions in Lemma 3.4 due to Lemma 3.2(ii) and (v), by Lemma 3.4, we know that equation (3.5) has a unique solution for every $x(t) \in C([-\tau, \lambda b], X)$. Moreover, the solution to (3.5) depends continuously on (λ, x), denoted by $y(t, \lambda b, x)$.

Define the function $\tilde{h} : [0,1] \times \Theta_n(\phi) \to \Theta_n(\phi)$ by the formula

$$\tilde{h}(\lambda, x) = \begin{cases} x(t), & t \in [-\tau, \lambda b], \\ y(t, \lambda b, x), & t \in [\lambda b, b]. \end{cases}$$

Clearly $\tilde{h}(\lambda, x) \in \Theta_n(\phi)$. In fact, for each $x \in \Theta_n(\phi)$, there exists $\tilde{g} \in Sel_{F_n}(x)$ such that $x = \Gamma_1(x) + S(\tilde{g})$. Put

$$\hat{g}(t) = \tilde{g}(t)\chi_{[0,\lambda b]}(t) + g_n(t)\chi_{[\lambda b, b]}(t) \text{ for each } t \in [0, b].$$

It is clear that $\hat{g} \in Sel_{F_n}(\tilde{h})$. Also, it is readily checked that $\Gamma_1(\tilde{h}(\lambda, x)) + S(\hat{g})(t) = x(t)$ for all $t \in [-\tau, \lambda b]$ and $\Gamma_1(\tilde{h}(\lambda, x)) + S(\hat{g})(t) = y(t, \lambda b, x)$ for all $t \in [\lambda b, b]$, which gives $\Gamma_1(\tilde{h}(\lambda, x)) + S(\hat{g}) = \tilde{h}(\lambda, x)$ and hence $\tilde{h}(\lambda, x) \in \Theta_n(\phi)$.

To show that $\tilde{h}$ is a continuous homotopy, let $(\lambda^m, x^m) \in [0,1] \times \Theta_n(\phi)$ be such that $(\lambda^m, x^m) \to (\lambda, x)$ as $m \to \infty$. Then

$$\tilde{h}(\lambda^m, x^m) = \begin{cases} x^m, & t \in [-\tau, \lambda b], \\ y(t, \lambda^m b, x^m), & t \in [\lambda b, b]. \end{cases}$$

We shall prove that $\tilde{h}(\lambda^m, x^m) \to \tilde{h}(\lambda, x)$ as $m \to \infty$. Without loss of generality, we assume that $\lambda^m \leq \lambda$. If $t \in [-\tau, \lambda^m b]$, then

$$|\tilde{h}(\lambda^m, x^m)(t) - \tilde{h}(\lambda, x)(t)| = |x^m(t) - x(t)| \to 0, \text{ as } m \to \infty.$$

If $t \in [\lambda b, b]$, then

$$\|\tilde{h}(\lambda^m, x^m) - \tilde{h}(\lambda, x)\|_{[\lambda b, b]} = \sup_{t \in [\lambda b, b]} |y(t, \lambda^m b, x^m) - y(t, \lambda, x)|,$$

which tends to 0 as $m \to \infty$, since $y(t, \lambda b, x)$ depends continuously on (λ, x). If $t \in [\lambda^m b, \lambda b]$, then

$$\begin{aligned} |\tilde{h}(\lambda^m, x^m)(t) - \tilde{h}(\lambda, x)(t)| &= |y(t, \lambda^m b, x^m) - x(t)| \\ &\leq |y(t, \lambda^m b, x^m) - x^m(t)| + |x^m(t) - x(t)| \to 0, \text{ as } m \to \infty, \end{aligned}$$

due to $y(t, \lambda^m b, x^m) \to x^m(t)$ $(t \to \lambda_m b)$. But $\tilde{h}(0, \cdot) = y(t, 0, \phi)$ and $\tilde{h}(1, \cdot)$ is the identity, hence $\Theta_n(\phi)$ is contractible.

Finally, in view of Lemma 3.2(i), it is easy to verify that $\Theta(\phi) \subset \cdots \subset \Theta_n(\phi) \cdots \subset \Theta_2(\phi) \subset \Theta_1(\phi)$; this implies that $\Theta(\phi) \subset \bigcap_{n\geq 1} \Theta_n(\phi)$. To prove the reverse inclusion, we take $x \in \bigcap_{n \geq 1} \Theta_n(\phi)$. Therefore, there exists a sequence $\{g_n\} \subset L^1([0,b], \mathbb{R}^+)$ such that $g_n \in Sel_{F_n}(x)$, $x = \Gamma_1(x) + S(g_n)$ and for $n \geq 1$,

$$|g_n(t)| \leq \alpha(t)(2 + \|x_t\|_*), \text{ for a.e. } t \in [0, b],$$

in view of Lemma 3.2(ii). According to the reflexivity of the space X and Lemma 2.1, we have the existence of a subsequence, denoted as the sequence, such that $g_n \rightharpoonup g \in L^1([0,b], X)$. By Mazur's convexity theorem, we obtain a sequence $\tilde{g}_n \in \mathrm{co}\{g_k : k \geq n\}$ for $n \geq 1$ such that $\tilde{g}_n \to g$ in $L^1([0,b], X)$ and, up to subsequence, $\tilde{g}_n(t) \to g(t)$ for a.e. $t \in [0,b]$ and $g_n(t) \in F_n(t, x_t)$ for all $n \geq 1$.

Denote by $\mathcal{N}$ the set of all $t \in [0,b]$ such that $\tilde{g}_n(t) \to g(t)$ in X and $g_n(t) \in F_n(t, x_t)$ for all $n \geq 1$. According to Lemma 3.2(iii), we know that there exists $E \subset [0,b]$ with $mes(E) = 0$ such that for each $t \in ([0,b] \setminus E) \cap \mathcal{N}$ and $x^* \in X^*$, $\varepsilon > 0$

$$\langle x^*, \tilde{g}_n(t)\rangle \in \mathrm{co}\{\langle x^*, g_k(t)\rangle : k \geq n\} \subset \langle x^*, F_n(t, x_t)\rangle \subset \langle x^*, F(t, x_t)\rangle + (-\varepsilon, \varepsilon),$$

here, $\langle x^*, F(t, \cdot)\rangle$ denotes the duality product. Therefore, we obtain that $\langle x^*, g(t)\rangle \in \langle x^*, F(t, x_t)\rangle$ for each $x^* \in X$ and $t \in ([0,b] \setminus E) \cap \mathcal{N}$. Since F has convex and closed values, we conclude that $g(t) \in F(t, x_t)$ for each $t \in ([0,b] \setminus E) \cap \mathcal{N}$, which implies that $g \in Sel_F(x)$. Moreover, since

$$x(t) = T(t)[\phi(0) - h(0, \phi)] + h(t, x_t) + \int_0^t AT(t-s)h(s, x_s)\,\mathrm{d}s + \int_0^t T(t-s)g_n(s)\,\mathrm{d}s,$$

by Lemma 3.3, we know that $\int_0^t T(t-s)g_n(s)\,ds \to \int_0^t T(t-s)g(s)\,ds$, which implies that $x = \Gamma_1(x) + S(g)$. This proves that $x \in \bigcap_{n\ge 1}\Theta_n(\phi)$. We conclude that $\Theta(\phi) = \bigcap_{n\ge 1}\Theta_n(\phi)$. Consequently, we conclude that $\Theta(\phi)$ is an R_δ-set, completing this proof. □

3.2. Noncompact operator case

We study the semilinear differential inclusion (1.1) under the following assumptions:

$(H_2)'$ h satisfies (H_2) with

$$|A^\beta h(t,c_1) - A^\beta h(s,c_2)| \le d\|c_1 - c_2\|_*, \text{ for } t,s \in [0,b]$$

instead of

$$|A^\beta h(t,c_1) - A^\beta h(t,c_2)| \le d\|c_1 - c_2\|_*;$$

(H_3) for every $\varepsilon > 0$ and every bounded set $D \subset C([-\tau,0],X)$ there exists $\delta > 0$ and a function $k \in L^1([0,b],\mathbb{R}^+)$ such that

$$\gamma(F(t,B_\delta(D))) \le k(t) \sup_{-\tau\le\theta\le 0} \gamma(B_\varepsilon(D(\theta))) \text{ for a.e. } t \in [0,b],$$

where $B_\delta(D)$ denotes a δ-neighborhood of D defined as

$$B_\delta(D) := \{z \in C([-\tau,0],X) : \mathrm{dist}(z,D) < \delta\}.$$

The assumption (H_3) was introduced and used in [13] and it implies the compactness of values of F.

Theorem 3.3. *Let conditions (H_1), $(H_2)'$ and (H_3) be satisfied. If $\tilde{d} < 1$, then the solution set of inclusion (1.1) is a nonempty compact subset of $C([-\tau,b],X)$ for each $\phi \in C([-\tau,0],X)$.*

Proof. For the same $B_R(b)$, as the reason for Theorem 3.1, we see that $B_R(b)$ is a closed and convex subset of $C([-\tau,b],X)$.

Claim 1. The multimap $\mathcal{F}$ has closed graph with compact values. Let $x^n \subset B_R(b)$ with $x^n \to x$ and $y^n \in \mathcal{F}(x^n)$ with $y^n \to y$. We shall prove that $y \in \mathcal{F}(x)$. By the definition of $\mathcal{F}$, there exists $f_n \in Sel_F(x^n)$ such that

$$y^n(t) = \begin{cases} \Gamma_1(x^n)(t) + S(f_n)(t), & t \in [0,b], \\ \phi(t), & t \in [-\tau,0]. \end{cases}$$

The operator S satisfies the properties (S_1) and (S_2) of Lemma 2.5, since $T(t)$ is a strongly continuous operator. In view of (H_1)(ii), we have that $\{f_n\}$ is integrably bounded, and condition (H_3) implies

$$\gamma(\{f_n(t)\}) \le \gamma(F(t,x_t^n)) \le k(t) \sup_{-\tau\le\theta\le 0} \gamma(x_t^n(\theta)) \le k(t) \sup_{-\tau\le\theta\le t} \gamma(x^n(s)) = 0.$$

Then $\{f_n\}$ is a semicompact sequence. Consequently, $\{f_n\}$ is weakly compact in $L^1([0,b],X)$; we may assume, without loss of generality, that $f_n \rightharpoonup f$ in $L^1([0,b],X)$. By Lemma 2.5(ii), one obtains that $S(f_n) \to S(f)$ in $C([0,b],X)$. Since Sel_F is weakly u.s.c. with weakly compact and convex values (see Lemma 3.1), from Lemma 2.3, we have that $f \in Sel_F(x)$.

On the other hand, we have the inequalities:

$$\begin{aligned} |\Gamma_1(x^n)(t) - \Gamma_1(x)(t)| &\le |h(t,x_t^n) - h(t,x_t)| + \left|\int_0^t AT(t-s)[h(s,x_s^n) - h(s,x_s)]\,ds\right| \\ &\le d\|A^{-\beta}\|\|x^n - x\|_{[-\tau,t]} + dC_{1-\beta}\int_0^t (t-s)^{\beta-1}\|x^n - x\|_{[-\tau,s]}\,ds \\ &\le d\left(\|A^{-\beta}\| + \frac{C_{1-\beta}b^\beta}{\beta}\right)\|x^n - x\|_{[-\tau,b]}, \end{aligned}$$

for $t \in [0,b]$. For $t \in [-\tau,0]$, we have

$$|\Gamma_1(x^n)(t) - \Gamma_1(x)(t)| = 0.$$

Then

$$\|\Gamma_1(x^n) - \Gamma_1(x)\|_{[-\tau,b]} \le d\Big(\|A^{-\beta}\| + \frac{C_{1-\beta}b^\beta}{\beta}\Big)\|x^n - x\|_{[-\tau,b]} \to 0, \text{ as } n \to \infty.$$

It follows immediately that $y^n \to y$ with

$$y(t)=\begin{cases}\Gamma_1(x)(t)+S(f)(t), & t\in[0,b],\\ \phi(t), & t\in[-\tau,0],\end{cases}$$

where $f\in Sel_F(x)$ and $y\in\mathcal{F}(x)$. Hence, $\mathcal{F}$ is closed.

It remains to show that, for $x\in\mathcal{M}_0$ and $\{f_n\}$ chosen in $Sel_F(x)$, the sequence $\{S(f_n)\}$ is relatively compact in $C([-\tau,b],X)$. Hypotheses (H_1)(ii) and (H_3) imply that $\{f_n\}$ is semicompact. Using Lemma 2.5(ii), we obtain that $\{S(f_n)\}$ is relatively compact in $C([0,b],X)$. Thus $\mathcal{F}(x)$ is relatively compact in $C([-\tau,b],X)$, together with the closeness of $\mathcal{F}$, then $\mathcal{F}$ has compact values.

Claim 2. The multioperator $\mathcal{F}$ is u.s.c. In view of Lemma 2.2, it suffices to check that $\mathcal{F}$ is a quasicompact multimap. Let Q be a compact set. We prove that $\mathcal{F}(Q)$ is a relatively compact subset of $C([-\tau,b],X)$. Assume that $\{y^n\}\subset\mathcal{F}(Q)$. Then

$$y^n(t)=\begin{cases}\Gamma_1(x^n)(t)+S(f_n)(t), & t\in[0,b],\\ \phi(t), & t\in[-\tau,0],\end{cases}$$

where $\{f_n\}\in Sel_F(x^n)$, for a certain sequence $\{x^n\}\subset Q$. Hypotheses (H_1)(ii) and (H_3) yield the fact that $\{f_n\}$ is semicompact and then it is a weakly compact sequence in $L^1([0,b],X)$. Similar arguments as in the previous proof of closeness imply that $\{\Gamma_1(x^n)\}$ and $\{S(f_n)\}$ are relatively compact in $C([0,b],X)$. Thus, $\{y^n\}$ converges in $C([-\tau,b],X)$, so the multioperator $\mathcal{F}$ is u.s.c.

Claim 3. The multioperator $\mathcal{F}$ is a condensing multioperator. Now in the space $C([-\tau,b],X)$, we consider the measure of noncompactness ν defined as: for a bounded subset $\Omega\subset\mathcal{M}_0$, let $\mathrm{mod}_C(\Omega)$ be the modulus of equicontinuity of the set of functions Ω given by

$$\mathrm{mod}_C(\Omega)=\lim_{\delta\to0}\sup_{x\in\Omega}\max_{|t_2-t_1|<\delta}|x(t_2)-x(t_1)|.$$

Given Hausdorff MNC γ, let χ be the real MNC defined on bounded set $D\subset C([-\tau,b],X)$ by

$$\chi(D)=\sup_{t\in[0,b]}\mathrm{e}^{-Lt}\gamma(D(s)).$$

Here, the constant $L>0$ is chosen such that

$$l:=d\|A^{-\beta}\|+\sup_{t\in[0,b]}\left(c_0\int_0^t\mathrm{e}^{-L(t-s)}k(s)\,\mathrm{d}s+dC_{1-\beta}\int_0^t\mathrm{e}^{-L(t-s)}(t-s)^{\beta-1}\,\mathrm{d}s\right)<\frac12,$$

where $k(t)$ is the function from condition (H_3).

Consider the function $\nu(\Omega)=\max_{D\in\Delta(\Omega)}\Big(\gamma(D[-\tau,0]),\chi(D),\mathrm{mod}_C(D)\Big)$ in space $C([-\tau,b],X)$, where $\Delta(\Omega)$ is the collection of all countable subsets of Ω.

To show that $\mathcal{F}$ is ν-condensing, let $\Omega\subset\mathcal{M}_0$ be a bounded set in $\mathcal{M}_0$ such that

$$\nu(\Omega)\le\nu(\mathcal{F}(\Omega)). \tag{3.6}$$

We will show that Ω is relatively compact. Let $\nu(\mathcal{F}(\Omega))$ be achieved on a sequence $\{y_n\}\subset\mathcal{F}(\Omega)$, i.e.,

$$\nu(\{y^n\})=\Big(\gamma(\{y^n\}|_{[-\tau,0]}),\chi(\{y^n\}),\mathrm{mod}_C(\{y^n\})\Big).$$

Then

$$y^n(t)=\begin{cases}\Gamma_1(x^n)(t)+S(f_n)(t), & t\in[0,b],\\ \phi(t), & t\in[-\tau,0],\end{cases}$$

where $\{x^n\}\subset\Omega$ and $f_n\in Sel_F(x^n)$. From inequality (3.6), it follows that $\gamma(\{x^n\}|_{[-\tau,0]})=0$. Indeed, we have

$$\gamma(\{y^n\}|_{[-\tau,0]})=\gamma(\{\phi(t):t\in[-\tau,0]\})=0\ge\gamma(\{x^n\}|_{[-\tau,0]})\ge0.$$

Now we give an upper estimate of $\gamma(\{y^n(t)\})$ for any $t\in[0,b]$. Using (H_3), we have $\gamma(\{f_n(t)\})\le k(t)\sup_{-\tau\le\theta\le0}\gamma(\{x_t^n(\theta)\})$. Then

$$\begin{aligned}\gamma(\{f_n(t)\})&\le k(t)\left(\sup_{s\in[-\tau,0]}\gamma(\{x^n(s)\})+\sup_{s\in[0,t]}\gamma(\{x^n(s)\})\right)\\&\le\mathrm{e}^{Lt}k(t)\left(\sup_{s\in[0,t]}\mathrm{e}^{-Ls}\gamma(\{x^n(s)\})\right)\\&\le\mathrm{e}^{Lt}k(t)\chi(\{x_n\}).\end{aligned}$$

Then, from Lemma 2.5(i) with $c_0 = M$, we get

$$\begin{aligned} \mathrm{e}^{-Lt}\gamma(\{S(f_n)(t)\}) &\le 2M\mathrm{e}^{-Lt}\int_0^t \mathrm{e}^{Ls}k(s)\,\mathrm{d}s\cdot\chi(\{x_n\}) \\ &\le 2M\int_0^t \mathrm{e}^{-L(t-s)}k(s)\,\mathrm{d}s\cdot\chi(\{x_n\}). \end{aligned} \tag{3.7}$$

Since the measure γ is monotone, from $(H_2)'$, for $t\in[0,b]$ we get

$$\begin{aligned} \mathrm{e}^{-Lt}\gamma(\{h(t,x_t^n)\}) &\le \mathrm{e}^{-Lt}\gamma\left(\left\{A^{-\beta}A^{\beta}h(t,x_t^n)\right\}\right) \le \|A^{-\beta}\|\mathrm{e}^{-Lt}\gamma\left(\left\{A^{\beta}h(t,x_t^n)\right\}\right) \\ &\le \|A^{-\beta}\|\mathrm{e}^{-Lt}\mu\left(\left\{A^{\beta}h(t,x_t^n)\right\}\right) \le d\|A^{-\beta}\|\mathrm{e}^{-Lt}\left(\sup_{\theta\in[-\tau,0]}\mu(\{x_t^n\})\right) \\ &\le 2d\|A^{-\beta}\|\mathrm{e}^{-Lt}\left(\sup_{\theta\in[-\tau,0]}\gamma(\{x_t^n\})\right) \le 2d\|A^{-\beta}\|\mathrm{e}^{-Lt}\left(\sup_{s\in[0,t]}\gamma(\{x^n(s)\})\right) \\ &\le 2d\|A^{-\beta}\|\left(\sup_{s\in[0,t]}\mathrm{e}^{-Ls}\gamma(\{x^n(s)\})\right) \le 2d\|A^{-\beta}\|\chi(\{x^n\}). \end{aligned} \tag{3.8}$$

Let $t\in[0,b]$ and $s\in[0,t]$. Clearly, the function $G: s\mapsto AT(t-s)h(s,x_s^n)$ is integrable and integrably bounded. Since

$$\begin{aligned} \gamma(\left\{AT(t-s)h(s,x_s^n)\right\}) &= \gamma\left(\left\{A^{1-\beta}T(t-s)A^{\beta}h(s,x_s^n)\right\}\right) \\ &\le \|A^{1-\beta}T(t-s)\|\gamma\left(\left\{A^{\beta}h(s,x_s^n)\right\}\right) \\ &\le 2dC_{1-\beta}(t-s)^{\beta-1}\mathrm{e}^{Ls}\left(\sup_{s_1\in[0,s]}\mathrm{e}^{-Ls_1}\gamma(\{x^n(s_1)\})\right) \\ &\le 2dC_{1-\beta}(t-s)^{\beta-1}\mathrm{e}^{Ls}\chi(\{x^n\}), \end{aligned}$$

by Lemma 2.4, one obtains

$$\begin{aligned} \mathrm{e}^{-Lt}\int_0^t \gamma(\{AT(t-s)h(s,x_s^n)\})\,\mathrm{d}s &\le 2dC_{1-\beta}\chi(\{x^n\})\int_0^t \mathrm{e}^{-L(t-s)}(t-s)^{\beta-1}\,\mathrm{d}s \\ &\le 2dC_{1-\beta}\chi(\{x^n\})\sup_{t\in[0,b]}\int_0^t \mathrm{e}^{-L(t-s)}(t-s)^{\beta-1}\,\mathrm{d}s. \end{aligned} \tag{3.9}$$

From (3.7)–(3.9), and the fact that $d\|A^{-\beta}\| < \frac{1}{2}$, it follows

$$\begin{aligned} \chi(\{y^n\}) &= \sup_{t\in[0,b]}\mathrm{e}^{-Lt}\gamma(\{y^n(t)\}) \\ &\le \sup_{t\in[0,b]}\mathrm{e}^{-Lt}\gamma\left(\left\{S(f_n)(t)+h(t,x_t^n)+\int_0^t AT(t-s)h(s,x_s^n)\,\mathrm{d}s\right\}\right) \\ &\le 2\left[d\|A^{-\beta}\| + \sup_{t\in[0,b]}\left(M\int_0^t \mathrm{e}^{-L(t-s)}k(s)\,\mathrm{d}s + dC_{1-\beta}\int_0^t \mathrm{e}^{-L(t-s)}(t-s)^{\beta-1}\,\mathrm{d}s\right)\right]\chi(\{x^n\}) \\ &\le 2l\chi(\{x^n\}). \end{aligned}$$

But (3.6) implies

$$\chi(\{y^n\}) \ge \chi(\{x^n\}),$$

and consequently, $\chi(\{x^n\}) = 0$. This implies that $\gamma(\{x^n(t)\}) = 0$.

Using (H_1)(ii) and (H_3) again, one gets that $\{f_n\}$ is a semicompact sequence. Then Lemma 2.5(ii) ensures that $\{S(f_n)\}$ is relatively compact in $C([0,b],X)$. Hence, $\mathrm{mod}_C(\{S(f_n)\}) = 0$.

Now we will show that the set $\{\Gamma_1(x^n)(t)\}$ is equicontinuous on $C([-\tau,b],X)$. For $-\tau \le t_1 < t_2 \le 0$, we have

$$|\Gamma_1(x^n)(t_2) - \Gamma_1(x^n)(t_1)| = |\phi(t_2)-\phi(t_1)| \to 0, \text{ as } |t_1-t_2|\to 0.$$

For $0 < t_1 < t_2 \le b$, we obtain

$$
\begin{aligned}
&|\Gamma_1(x^n)(t_2) - \Gamma_1(x^n)(t_1)| \\
&\quad \le \|T(t_2) - T(t_1)\| |h(0,\phi)| + \left|h(t_2, x^n_{t_2}) - h(t_1, x^n_{t_1})\right| \\
&\qquad + \left| \int_{t_1}^{t_2} AT(t_2 - s)h(s, x^n_s)\,\mathrm{d}s \right| + \left| \int_0^{t_1} A[T(t_2 - s) - T(t_1 - s)]h(s, x^n_s)\,\mathrm{d}s \right| \\
&\quad \le \|T(t_2) - T(t_1)\| |h(0,\phi)| + d\|A^{-\beta}\| \|x^n_{t_2} - x^n_{t_1}\|_* + \frac{d_1 C_{1-\beta}(t_2 - t_1)^\beta}{\beta}\left(1 + \|x^n\|_{[t_1-\tau, t_2]}\right) \\
&\qquad + \left| [T(t_2 - t_1) - I] \int_0^{t_1} AT(t_1 - s)h(s, x^n_s)\,\mathrm{d}s \right|.
\end{aligned}
$$

Since $\gamma\left(\int_0^t AT(t-s)h(s,x^n_s)\,\mathrm{d}s\right) = 0$ for all $t \in [0,b]$, the last term on the right-hand side converges to zero when $t_2 - t_1$ tends uniformly to 0. From the inequality

$$
\mathrm{mod}_{\mathrm{C}}(\{y^n\}) \le \mathrm{mod}_{\mathrm{C}}(\{\Gamma_1(x^n)\}) + \mathrm{mod}_{\mathrm{C}}(\{S(f_n)\}),
$$

we get

$$
\mathrm{mod}_{\mathrm{C}}(\{y^n\}) \le d\|A^{-\beta}\| \|x^n_{t_2} - x^n_{t_1}\|_* \le d\|A^{-\beta}\| \mathrm{mod}_{\mathrm{C}}(\{x^n\}).
$$

In view of $d\|A^{-\beta}\| < 1$, from the last inequality and inequality (3.6) follows $\mathrm{mod}_{\mathrm{C}}(\{y^n\}) = 0$, which implies that $\mathrm{mod}_{\mathrm{C}}(\{x^n\}) = 0$. Hence, the subset $\{x^n\}$ is relatively compact, thus $\nu(\{x^n\}) = 0$, and so the map $\mathcal{F}$ is ν-condensing.

From Theorem 2.2, we deduce that the fixed point set $\mathrm{Fix}\mathcal{F}$ is a nonempty compact set. □

Before proving the main result of this subsection, we give an important lemma to prove the contractibility of the solution set.

Lemma 3.5. *Under the conditions in Lemma 3.4 except that $\{T(t)\}_{t>0}$ is compact, if*

(H_4) *there exists a function $k_1 \in L^1([a,b], \mathbb{R}^+)$ such that*

$$
\gamma(g(t,D)) \le k_1(t) \sup_{-\tau \le \theta \le 0} \gamma((D(\theta)) \quad \text{a.e. } t \in [a,b]
$$

for every bounded set $D \subset C([a-\tau, a], X)$,

then integral equation (3.1) admits a unique solution for every $\varphi(t) \in C([-\tau, b], X)$. Moreover, the solution to (3.1) depends continuously on φ.

Proof. Let $\varphi(t) \in C([-\tau, \xi], X)$ be fixed. In view of Lemma 3.4, we know that the operator $W : B_\rho \to B_\rho$ is continuous. Similar to the proof of Claim 3 in Theorem 3.3, it follows that W is a ν-condensing operator. So W has a fixed point, which implies that equation (3.1) has a local solution. According to Lemma 3.4, the uniqueness and continuation of the solution are obtained. Therefore, the first part of the lemma is proved.

We only prove that the solution to equation (3.1) depends continuously on φ. Let $\varphi^n \to \varphi^0$ in $C([-\tau, b], X)$ as $n \to \infty$, and x^n be the solution to equation (3.1) with the perturbation φ^n, i.e.,

$$
x^n(t) = \varphi^n(t) + h(t, x^n_t) + \int_a^t AT(t-s)h(s, x^n_s)\,\mathrm{d}s + \int_a^t T(t-s)g(s, x^n_s)\,\mathrm{d}s
$$

for $t \in [a,b]$ and $x^n(t) = \varphi^n(t)$ for $t \in [-\tau, a]$. By $(H_2)'$ and (H_4), together with similar argument as above, we have

$$
\begin{aligned}
\chi(\{x^n\}) &= \sup_{t\in[0,b]} \mathrm{e}^{-Lt}\gamma(\{x^n(t)\}) \\
&\le \sup_{t\in[0,b]} \mathrm{e}^{-Lt}\gamma\left(\left\{\varphi^n(t) + h(t, x^n_t) + \int_a^t AT(t-s)h(s, x^n_s)\,\mathrm{d}s + \int_a^t T(t-s)g(s, x^n_s)\,\mathrm{d}s\right\}\right) \\
&\le \sup_{t\in[0,b]} \mathrm{e}^{-Lt}\gamma(\{\varphi^n(t)\}) + \sup_{t\in[0,b]} \mathrm{e}^{-Lt}\gamma(\{h(t, x^n_t)\})
\end{aligned}
$$

$$
\begin{aligned}
&+\sup_{t\in[0,b]} \mathrm{e}^{-Lt}\gamma\left(\left\{\int_a^t AT(t-s)h(s,x_s^n)\,\mathrm{d}s+\int_a^t T(t-s)g(s,x_s^n)\,\mathrm{d}s\right\}\right)\\
&\le 2\left[d\|A^{-\beta}\|+\sup_{t\in[0,b]}\left(M\int_a^t \mathrm{e}^{-L(t-s)}k_1(s)\,\mathrm{d}s+dC_{1-\beta}\int_a^t \mathrm{e}^{-L(t-s)}(t-s)^{\beta-1}\,\mathrm{d}s\right)\right]\chi(\{x^n\})\\
&<\chi(\{x^n\}).
\end{aligned}
$$

Thus $\chi(\{x^n\})=0$, then $\gamma(\{x^n(t)\})=0$. On the other hand, as the reason for the proof of $\mathrm{mod}_C(\{x^n\})$ in Theorem 3.3, it follows that $\mathrm{mod}_C(\{x^n\})=0$. Hence, $\{x^n\}$ is relatively compact in $C([-\tau,b],X)$. Therefore, taking the limit in (3.2) as $n\to\infty$, one finds, again by (H_2) and Lebesgue's dominated convergence theorem, that x is the solution to equation (3.1) with the perturbation φ^0. The proof is completed. □

Theorem 3.4. *Under the conditions in Theorem 3.3, the solution set of (1.1) is an R_δ-set.*

Proof. We also consider inclusion (3.4), where the multivalued functions $F_n:[0,b]\times C([-\tau,0],X)\to P_{\mathrm{cl,cv}}(X)$ are established in view of Lemma 3.2, and F_n satisfy condition (H_1) for each $n\ge 1$.

Let $\Theta_n(\phi)$ denote the set of all mild solutions for inclusion (3.4).

We show that each sequence $\{x^n\}$ such that $x^n\in\Theta_n(\phi)$ for all $n\ge 1$ has a convergent subsequence $x^{n_k}\to x\in\Theta(\phi)$. At first we notice

$$
x^n(t)=T(t)[\phi(0)-h(0,\phi)]+h(t,x_t^n)+\int_0^t AT(t-s)h(s,x_s^n)\,\mathrm{d}s+\int_a^t T(t-s)g_n(s)\,\mathrm{d}s,\qquad g_n(s)\in F_n(s,x_s^n),
$$

for $t\in[0,b]$, and $x_n(t)=\phi(t)$ for $t\in[-\tau,0]$. It is easy to know that $\gamma(\{x^n(t)\})=0$ for $t\in[-\tau,0]$. By (H_3), for any $\varepsilon>0$ there exist some $N\in\mathbb{N}$ such that

$$
\begin{aligned}
\gamma(\{g_n(s)\}_{n\ge1})=\gamma(\{g_n(s)\}_{n\ge N})&\le\gamma(\{F_N(s,x_s^n)\}_{n\ge N})\\
&\le\gamma\left(\overline{\mathrm{co}}F(t,B_{3^{1-N}}(\{x_s^n\}_{n\ge N}))\right)\\
&\le k(s)\left(\sup_{-\tau\le\theta\le0}\gamma(\{x_s^n\}_{n\ge N})+\varepsilon\right)\\
&\le k(s)\left(\sup_{\theta\in[0,s]}\gamma(\{x^n(\theta)\}_{n\ge1})+\varepsilon\right).
\end{aligned}
\tag{3.10}
$$

Therefore,

$$
\gamma(\{g_n(s)\}_{n\ge1})\le k(s)\sup_{\theta\in[0,s]}\gamma(\{x^n(\theta)\}_{n\ge1}).
$$

This, together with (3.8) and (3.9) implies

$$
\begin{aligned}
\chi(\{x^n\})=&\sup_{t\in[0,b]}\mathrm{e}^{-Lt}\gamma(\{x^n(t)\})\\
&\le\sup_{t\in[0,b]}\mathrm{e}^{-Lt}\gamma(\{h(t,x_t^n)\})+\sup_{t\in[0,b]}\mathrm{e}^{-Lt}\gamma\left(\left\{\int_0^t AT(t-s)h(s,x_s^n)\,\mathrm{d}s\right\}\right)\\
&\quad+\sup_{t\in[0,b]}\mathrm{e}^{-Lt}\gamma\left(\left\{\int_0^t T(t-s)g_n(s)\,\mathrm{d}s\right\}\right)\\
&\le 2\left[d\|A^{-\beta}\|+\sup_{t\in[0,b]}\left(M\int_0^t \mathrm{e}^{-L(t-s)}k(s)\,\mathrm{d}s+dC_{1-\beta}\int_0^t \mathrm{e}^{-L(t-s)}(t-s)^{\beta-1}\,\mathrm{d}s\right)\right]\chi(\{x^n\})\\
&\le 2l\chi(\{x^n\})<\chi(\{x^n\}).
\end{aligned}
$$

Thus $\chi(\{x^n\})=0$, then $\gamma(\{x^n(t)\})=0$ for $t\in[0,b]$. From (3.10), we get that $\gamma(\{g_n(s)\})=0$. The fact that the equicontinuity of $\{x^n\}$ is proved in Theorem 3.3 implies the existence of a subsequence $\{x^{n_k}\}$ that is convergent on $[-\tau,b]$. Denote the limit by x.

Since $\gamma(\{g_n(s)\})=0$, we can assume, up to subsequence, that $g_n(s)\to g(s)$ in X for $s\in[0,t]$. From the above discussion, we have

$$x(t) = T(t)[\phi(0) - h(0,\phi)] + h(t,x_t) + \int_0^t AT(t-s)h(s,x_s)\,ds + \int_0^t T(t-s)g(s)\,ds,$$

for $t \in [0,b]$ and $x(t) = \phi(t)$ for $t \in [-\tau, 0]$. As the reason for Theorem 3.2, the fact that F_n satisfies condition (H_1) shows that $g(t) \in F(t,x_t)$ for a.e. $t \in [0,b]$.

It follows that $\sup\{d(x,\Theta(\phi)) : x \in \Theta_n(\phi)\} \to 0$ (an easy proof by contradiction). Therefore, $\sup\{d(x,\Theta(\phi)) : x \in \overline{\Theta_n(\phi)}\} \to 0$, as well. Since $\Theta(\phi)$ is compact and $\Theta_{n+1}(\phi) \subset \Theta_n(\phi)$, $\gamma(\Theta_n(\phi)) = \gamma(\overline{\Theta_n(\phi)}) \to 0$, as $n \to \infty$ and $\Theta(\phi) = \bigcap_{n=1}^{\infty} \overline{\Theta_n(\phi)}$.

By the same methods as in Theorem 3.1, together with Lemma 3.5, we know that $\Theta_n(\phi)$ is contractible for all $n \geq 1$. Consequently, we conclude that $\Theta(\phi)$ is an R_δ-set. The proof is completed. □

4. An example

Let $X = L^2([0,\pi],\mathbb{R})$, we consider the following partial differential inclusions of neutral type:

$$\begin{cases} \dfrac{\partial}{\partial t}\left(z(t,\xi) - \displaystyle\int_0^\pi U(\xi,y)z_t(\theta,y)\mathrm{d}y\right) \in \dfrac{\partial^2}{\partial \xi^2} z(t,\xi) + G(t, z_t(\theta,\xi)), & t \in [0,b],\ \xi \in [0,\pi], \\ z(t,0) = z(t,\pi) = 0, & t \in [0,b], \\ z(\theta,\xi) = \phi(\theta)(\xi), & \theta \in [-\tau,0],\ \xi \in [0,\pi], \end{cases} \tag{4.1}$$

where $\phi \in C([-\tau,0],X)$, that is, $\phi(\theta) \in X$ and $z_t(\theta,\xi) = z(t+\theta,\xi)$, $t \in [0,b]$, $\theta \in [-\tau,0]$.

We consider the operator $A : D(A) \subset X \to X$ defined as $Ay = -y''$ with the domain

$$D(A) = \{y(\cdot) \in X : y,\ y' \text{ absolutely continuous},\ y'' \in X \text{ and } y(0) = y(\pi) = 0\}.$$

Then A generates a strongly continuous semigroup $\{T(t)\}_{t\geq 0}$, which is compact, analytic and self-adjoint. Furthermore, A has a discrete spectrum, the eigenvalues are n^2 $(n \in \mathbb{N})$, with corresponding normalized eigenvectors $x_n(\xi) = \sqrt{\frac{2}{\pi}}\sin n\xi$. This implies that $\sup_{t\geq 0}\|T(t)\| < +\infty$ (see [18]). We also use the following properties:

(i) for each $y \in X$, $T(t)y = \sum_{n=1}^{\infty} e^{-n^2 t}\langle y, x_n\rangle x_n$;
(ii) for each $y \in X$, $A^{-\frac{1}{2}}y = \sum_{n=1}^{\infty} \frac{1}{n}\langle y, x_n\rangle x_n$;
(iii) the operator $A^{\frac{1}{2}}$ is given by

$$A^{\frac{1}{2}}y = \sum_{n=1}^{\infty} n\langle y, x_n\rangle x_n$$

on the space $D(A^{\frac{1}{2}}) = \{y(\cdot) \in X : \sum_{n=1}^{\infty} n\langle y, x_n\rangle x_n \in X\}$.

Then system (4.1) can be reformulated as

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}[x(t) - h(t,x_t)] \in Ax(t) + F(t,x_t), & t \in [0,b], \\ x(t) = \phi(t), & t \in [-\tau,0], \end{cases}$$

where $x(t)(\xi) = z(t,\xi)$, $x_t(\theta,\xi) = z_t(\theta,\xi)$, $F(t,x_t)(\xi) = G(t, z_t(\theta,\xi))$. The function $h(t,x_t) : [0,b] \times C([-\tau,0],X) \to X$ is defined by

$$h(t,x_t) = \int_0^\pi U(\xi,y)z_t(\theta,y)\,\mathrm{d}y.$$

Moreover, we assume that the following conditions hold:

(h_1) the function $U(\xi,y)$ is measurable and

$$\int_0^\pi\int_0^\pi U^2(\xi,y)\,\mathrm{d}y\,\mathrm{d}\xi < \infty;$$

(h_2) the function $\partial_\xi U(\xi, y)$ is measurable, $U(0, y) = U(\pi, y) = 0$, and let

$$\overline{H} = \left(\int_0^\pi \int_0^\pi (\partial_\xi U(\xi, y))^2 \mathrm{d}y \, \mathrm{d}\xi \right)^{\frac{1}{2}} < \infty.$$

Clearly, (H_2) is satisfied.

Let $F(t, z_t) = [f_1(t, z_t), f_2(t, z_t)]$. Now, we assume that:

$$f_i : [0, b] \times C([-\tau, 0], X) \to \mathbb{R}, \quad i = 1, 2$$

satisfy

(F_1) f_1 is l.s.c. and f_2 is u.s.c.;
(F_2) $f_1(t, \psi) \le f_2(t, \psi)$ for each $(t, \psi) \in [0, b] \times C([-\tau, 0], X)$;
(F_3) there exists $\alpha_1, \alpha_2 \in L^\infty([0, b], \mathbb{R}^+)$ such that

$$|f_i(t, \psi)| \le \alpha_i(t)(1 + \|\psi\|_*), \quad i = 1, 2,$$

for each $(t, \psi) \in [0, b] \times C([-\tau, 0], X)$.

From our assumptions on (F_1)–(F_3), it follows readily that the multivalued function $F(\cdot, \cdot) : [0, b] \times C([-\tau, 0], X) \to P(X)$ satisfies (H_1).

Thus, all the assumptions in Theorems 3.1 and 3.2 are satisfied, our results can be used to problem (4.1).

Acknowledgements

The authors acknowledges support from the National Natural Science Foundation of China (11671339) and the Hunan Provincial Innovation Foundation for Postgraduate (CX2015B204). The authors also want to express their thanks to the anonymous referees for their suggestions and comments that improved the quality of the paper.

References

[1] J. Andres, G. Gabor, L. Górniewicz, Topological structure of solution sets to multi-valued asymptotic problems, Z. Anal. Anwend. 19 (2000) 35–60.
[2] J. Andres, M. Pavlačková, Topological structure of solution sets to asymptotic boundary value problems, J. Differ. Equ. 248 (2010) 127–150.
[3] N. Aronszajn, Le correspondant topologique de l'unicité dans la théorie des équations différentielles, Ann. Math. (2) 43 (4) (1942) 730–738.
[4] A. Bakowska, G. Gabor, Topological structure of solution sets to differential problems in Fréchet spaces, Ann. Pol. Math. 95 (2009) 17–36.
[5] M. Benchohra, S. Abbas, Advanced Functional Evolution Equations and Inclusions, Springer, 2015.
[6] D. Bothe, Multi-valued perturbations of m-accretive differential inclusions, Isr. J. Math. 108 (1998) 109–138.
[7] A. Bressan, Z.P. Wang, Classical solutions to differential inclusions with totally disconnected right-hand side, J. Differ. Equ. 246 (2009) 629–640.
[8] D.H. Chen, R.N. Wang, Y. Zhou, Nonlinear evolution inclusions: topological characterizations of solution sets and applications, J. Funct. Anal. 265 (2013) 2039–2073.
[9] K. Deimling, Multivalued Differential Equations, de Gruyter, Berlin, 1992.
[10] J. Diestel, W.M. Ruess, W. Schachermayer, Weak compactness in $L^1(\mu; X)$, Proc. Amer. Math. Soc. 118 (1993) 447–453.
[11] T. Donchev, E. Farkhi, B.S. Mordukhovich, Discrete approximations, relaxation, and optimization of one-sided Lipschitzian differential inclusions in Hilbert spaces, J. Differ. Equ. 243 (2007) 301–328.
[12] G. Gabor, A. Grudzka, Structure of the solution set to impulsive functional differential inclusions on the half-line, Nonlinear Differ. Equ. Appl. 19 (2012) 609–627.
[13] G. Gabor, A. Grudzka, Erratum to: structure of the solution set to impulsive functional differential inclusions on the half-line, Nonlinear Differ. Equ. Appl. 22 (2015) 175–183.
[14] G. Gabor, M. Quincampoix, On existence of solutions to differential equations or inclusions remaining in a prescribed closed subset of a finite-dimensional space, J. Differ. Equ. 185 (2002) 483–512.
[15] S.C. Hu, N.S. Papageorgiou, On the topological regularity of the solution set of differential inclusions with constraints, J. Differ. Equ. 107 (1994) 280–289.
[16] M. Kamenskii, V. Obukhovskii, P. Zecca, Condensing Multi-valued Maps and Semilinear Differential Inclusions in Banach Spaces, Walter de Gruyter, Berlin, New York, 2001.
[17] S.K. Ntouyas, D. O'Regan, Existence results for semilinear neutral functional differential inclusions via analytic semigroups, Acta Appl. Math. 98 (2007) 223–253.
[18] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Applied Mathematical Sciences, vol. 44, Springer-Verlag, New York, 1983.
[19] V. Staicu, On the solution sets to nonconvex differential inclusions of evolution type, Discrete Contin. Dyn. Syst. 2 (1998) 244–252.
[20] V. Staicu, On the solution sets to differential inclusions on unbounded interval, Proc. Edinb. Math. Soc. 43 (2000) 475–484.
[21] I.I. Vrabie, Compactness Methods for Nonlinear Evolutions, second edition, Pitman Monographs and Surveys in Pure and Applied Mathematics, vol. 75, Longman and John Wiley & Sons, 1995.
[22] R.N. Wang, Q.H. Ma, Y. Zhou, Topological theory of non-autonomous parabolic evolution inclusions on a noncompact interval and applications, Math. Ann. 362 (2014) 173–203.
[23] J.H. Wu, Theory and Applications of Partial Functional Differential Equations, vol. 119, Springer Science & Business Media, 2012.