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Differential geometry

An index formula for the intersection Euler characteristic of an infinite cone

Un théorème de l'indice pour la caractéristique d'Euler d'intersection d'un cône infini

Ursula Ludwig

Universität Duisburg–Essen, Fakultät für Mathematik, 45117 Essen, Germany

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A B S T R A C T

The aim of this note is to establish an index formula for the intersection Euler characteristic of a cone. The main actor of these notes is the model Witten Laplacian on the infinite cone. First, we study its spectral properties and establish a McKean–Singer-type formula. We also give an explicit formula for the zeta function of the model Witten Laplacian. In a second step, we apply local index techniques to the model Witten Laplacian. By combining these two steps, we express the absolute and relative intersection Euler characteristic of the cone as a sum of two terms, a term which is local, and a second term which is the Cheeger invariant.

R É S U M É

Le but de cette note est d'établir un théorème de l'indice pour la caractéristique d'Euler d'intersection d'un cône. Dans un premier temps, on étudie les propriétés spectrales du laplacien de Witten et on établit une formule de McKean–Singer. On donne aussi une formule explicite pour la fonction zêta associée au laplacien de Witten. Dans une deuxième partie, on applique des techniques d'indice local au laplacien de Witten. On obtient une formule qui exprime la caractéristique d'Euler d'intersection du cône comme la somme de deux termes, l'un local, l'autre étant l'invariant de Cheeger.

1. Introduction

The setting of this note is the following: let (L, g^{TL}) be a smooth connected compact Riemannian manifold (without boundary) of $\dim L \geq 1$, which in the following we will call the link manifold. We denote by

E-mail address: ursula.ludwig@uni-due.de.

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$$cL := \big([0,\infty)\times L\big)/_{(0,x)\sim(0,y)} \tag{1.1}$$

the cone over L, by 0 the cone tip and let $C := cL \setminus \{0\} \simeq \mathbb{R}_{>0} \times L$. We denote by $r \in [0,\infty)$ the radial coordinate on cL. We equip the cone C with the conic metric $g^{TC} := dr^2 + r^2 g^{TL}$. Let $\rho_L : \pi_1(L) \to U(q)$ be a representation of the fundamental group of the link L. We denote by $(F_L, \nabla^{F_L}, g^{F_L})$ the flat Hermitian vector bundle associated with the representation ρ_L. The flat Hermitian bundle $(F_L, \nabla^{F_L}, g^{F_L})$ over L can be extended in a trivial way to a flat Hermitian vector bundle over C, we denote by (F, ∇^F, g^F) this extension. We denote by F_L^* the flat bundle dual to F_L and by F^* its extension to C.

Set $n := \dim C = \dim L + 1$. In the whole note, we will assume that the Witt condition is satisfied, *i.e.* either n is even, or n is odd and $H^{\frac{n-1}{2}}(L, F_L) = H_{\frac{n-1}{2}}(L, F_L^*)^* = 0$. Let us denote by Δ_L the Hodge Laplacian on the link manifold L, acting on sections of $\Lambda(T^*L) \otimes F_L$. For $k = 0, \ldots, n-1$, we denote by $\Delta_L^{(k)}$ its restriction to sections of $\Lambda^k(T^*L) \otimes F_L$. We denote by $\operatorname{spec}\left(\Delta_{L,ccl}^{(k)}\right)$ the co-closed spectrum of $\Delta_L^{(k)}$. The following assumption will be in place for the whole note:

$$\begin{cases} \operatorname{spec}(\Delta_{L,ccl}^{(n/2-1)}) \cap (0,1) = \emptyset & \text{if } n \text{ is even,} \\ \left(\operatorname{spec}(\Delta_{L,ccl}^{(\lfloor n/2\rfloor-1)}) \cap \left(0, \tfrac{3}{4}\right)\right) \cup \left(\operatorname{spec}(\Delta_{L,ccl}^{(\lfloor n/2\rfloor)}) \cap \left[0, \tfrac{3}{4}\right)\right) = \emptyset & \text{if } n \text{ is odd.} \end{cases} \tag{1.2}$$

For a space satisfying the Witt condition, condition (1.2) can always be achieved by rescaling the metric g^{TL}. Using [3, Corollary 2.3 and Lemma 3.1], one can prove that, under the assumption (1.2), the Laplacian Δ associated with the Hilbert complex of L^2-forms on C with values in the flat bundle F equals the Friedrichs extension of the Hodge Laplacian acting on smooth compactly supported sections of $\Lambda(T^*C) \otimes F$.

We denote by N the number operator acting on sections of $\Lambda(T^*C) \otimes F$ by multiplication with the form degree. Let $T > 0$. The main actor in this note is the model Witten Laplacian:

$$\Delta_{T,\pm} := \Delta \pm T(2N - n) + T^2 r^2. \tag{1.3}$$

The model Witten Laplacian $\Delta_{T,\pm}$ is the Laplace operator associated with the Witten deformation of the complex of L^2-forms on C using the radial (resp. anti-radial) Morse function $f_\pm = \pm\frac{1}{2}r^2$. The operator $\Delta_{T,\pm}$ has discrete spectrum.

In the first part of this note, we study the spectral properties of the model Witten Laplacian, in particular we study its zeta function. Before stating the first main result, we introduce the notation needed for its precise statement: for $k = -1, \ldots, n-1$, we define

$$\alpha_k := \left(k + 1 - \frac{n}{2}\right). \tag{1.4}$$

For $\mu \in \operatorname{spec}\big(\Delta_{L,ccl}^{(k)}\big) \setminus \{0\}$, put

$$\beta_k(\mu) := \sqrt{\alpha_k^2 + \mu}. \tag{1.5}$$

Let $m(\mu)$ be the multiplicity of the eigenvalue μ. For $\Re(s) >> 0$, we define the functions

$$\zeta_{L,\pm}(s) := \pm\sum_{k=0}^{n-2}(-1)^k \cdot \left(\sum_{\mu\in\operatorname{spec}(\Delta_{L,ccl}^{(k)})\setminus\{0\}} \frac{m(\mu)}{(\beta_k(\mu) \pm \alpha_k)^s}\right). \tag{1.6}$$

We denote by ζ_R the Riemann zeta function. For $\alpha \in \mathbb{R}_{>0}$, we denote by $\zeta_H(s,\alpha)$ the Hurwitz zeta function, which is the meromorphic continuation of

$$\zeta_H(s,\alpha) := \sum_{j=0}^{\infty} \frac{1}{(j+\alpha)^s}, \ \Re(s) > 1. \tag{1.7}$$

We denote by $\Delta_{T,\pm}^{\perp}$ the restriction of $\Delta_{T,\pm}$ to $\big(\ker \Delta_{T,\pm}\big)^{\perp}$. For $k = 0, \ldots, n-1$, we denote by $b^k(L, F_L)$ the k-th Betti number for the cohomology of L with local coefficients in F_L. Let Tr_s denote the supertrace of an operator.

Theorem I. *For $\Re(s) >> 0$, the zeta function $\zeta_{T,\pm}(s) := -\operatorname{Tr}_s\left[N\left(\Delta_{T,\pm}^{\perp}\right)^{-s}\right]$ is a holomorphic function; we have*

$$\zeta_{T,+}(s) = (2T)^{-s}\zeta_{L,+}(s) + (4T)^{-s}\left(\sum_{0\le k\le \frac{n}{2}-1}(-1)^k b^k(L,F_L)\zeta_R(s) + \sum_{\frac{n}{2}\le k\le n-1}(-1)^k b^k(L,F_L)\zeta_H(s,\alpha_k)\right), \tag{1.8}$$

$$\zeta_{T,-}(s) = (2T)^{-s}\zeta_{L,-}(s) + (4T)^{-s}\left(\sum_{0\le k\le \frac{n}{2}-1}(-1)^k b^k(L,F_L)\zeta_H(s,-\alpha_{k-1}) + \sum_{\frac{n}{2}\le k\le n-1}(-1)^k b^k(L,F_L)\zeta_R(s)\right). \tag{1.9}$$

Moreover, $\zeta_{T,\pm}$ admits a meromorphic continuation to the complex plane, which is holomorphic at $s=0$.

We now explain the notation appearing in the second main result of this note: we denote by ∇^{TC} (resp. by ∇') the Levi-Civita connection on (TC, g^{TC}) (resp. on $(TC, g' := dr^2 + g^{TL})$). We denote by $e(TC, \nabla^{TC})$ (resp. by $e(TC,\nabla')$) the Chern–Weil form associated with (TC,∇^{TC}) (resp. (TC,∇')), which represents the Euler class of the vector bundle TC. We denote by R^{TC} the curvature tensor of ∇^{TC}. By [1, Proposition 1.2], we have that $R^{TC}\left(\frac{\partial}{\partial r}, -\right) = 0$ and hence $e(TC,\nabla^{TC}) = 0$. Since g' is a product metric, we also have $e(TC,\nabla') = 0$. We denote by $\widetilde{e}(TC,\nabla^{TC},\nabla')$ the Chern–Simons class of smooth forms of degree $n-1$ on C, which is defined modulo exact forms, such that

$$\mathrm{d}\widetilde{e}(TC,\nabla^{TC},\nabla') = e(TC,\nabla') - e(TC,\nabla^{TC}). \tag{1.10}$$

In case n is odd, one has $\widetilde{e}(TC,\nabla^{TC},\nabla') = 0$. Note that the vanishing of the Euler forms and (1.10) imply that $\widetilde{e}(TC,\nabla^{TC},\nabla')$ is a closed form, in case n is even. For $a>0$ let $i_a : L \simeq \{a\}\times L \to cL$ be the inclusion. Since $\widetilde{e}(TC,\nabla^{TC},\nabla')$ is closed, the integral $\int_L i_a^*\widetilde{e}(TC,\nabla^{TC},\nabla')$ does not depend on $a>0$.

We denote by $Q_t\big((r,y),(r',y')\big)$, $(r,y),(r',y') \in C \simeq \mathbb{R}_{>0}\times L$ the heat kernel for the Laplacian Δ (acting on forms on C with values in F). The following integral is well defined:

$$\gamma(F) := \frac{1}{2}\int_0^\infty \frac{\mathrm{d}u}{u}\int_L \mathrm{Tr_s}\left[Q_u((1,y),(1,y))\right]\mathrm{dvol}_L; \tag{1.11}$$

throughout this note, it will be called the Cheeger invariant. It has first been introduced in [5, Theorem 2.1] for the case of trivial coefficients. The Cheeger invariant is the contribution of the singularity to the Gauss–Bonnet theorem for even dimensional spaces with isolated cone-like singularities [5, Theorem 5.1]. The well-definedness of the integral in (1.11) is discussed in [1, Section 1(f)] for the even dimensional case. It can be seen as follows (in both even and odd dimension): using local index techniques, one gets the asymptotic expansion $\mathrm{Tr_s}[Q_t((r,y),(r,y))]\mathrm{dvol}_C = e(TC,\nabla^{TC}) + \mathcal{O}(\sqrt{t})$, uniformly on compact sets, as $t \searrow 0$. Since $e(TC,\nabla^{TC}) = 0$, this yields the well-definedness of the integral (1.11) at 0. Using the characterisation of $\mathrm{dom}(\Delta)$ one has, as $r\to 0$, $Q_1((r,y),(r,y)) \sim r^{2\epsilon-n}$ for some $\epsilon>0$ (see [7, Corollary 1.3.14 and Proposition 1.4.5]). This together with the scaling property $Q_1((r,y),(r,y)) = r^{-n}Q_{1/r^2}((1,y),(1,y))$ (see e.g. [1, Proposition 1.7]) gives the well-definedness of the integral (1.11) at ∞.

We denote by $I\chi(cL,F^*)$ (resp. by $I\chi(cL,L,F^*)$) the Euler characteristic for the absolute (resp. relative) intersection homology with middle perversity and coefficients in the local system associated with F^*.

Theorem II. *If* $\dim C$ *is even, we have*

$$I\chi(cL,F^*) = I\chi(cL,L,F^*) = -\mathrm{rk}(F)\cdot\int_L i_1^*\widetilde{e}(TC,\nabla^{TC},\nabla') + \gamma(F). \tag{1.12}$$

If $\dim C$ *is odd, we have*

$$I\chi(cL,F^*) = \frac{1}{2}\cdot\mathrm{rk}(F)\cdot\chi(L,\mathbb{C}) + \gamma(F); \quad I\chi(cL,L,F^*) = -\frac{1}{2}\cdot\mathrm{rk}(F)\cdot\chi(L,\mathbb{C}) + \gamma(F). \tag{1.13}$$

In the case of an odd-dimensional oriented cone equipped with the trivial bundle of rank 1, one has that $\gamma(\mathbb{C}) = 0$ and formula (1.13) reduces to the well-known identity $I\chi(cL,\mathbb{C}) = -I\chi(cL,L,\mathbb{C}) = \frac{1}{2}\cdot\chi(L,\mathbb{C})$ (see the local calculation for intersection homology [6, Section 2.4]).

2. Proof of Theorem I

The spectrum as well as the eigenfunctions of the model Witten Laplacian can be computed explicitly, using the separation of variables techniques. This explicit computation uses the description of the domain of the Friedrichs extension [4, Section 6]. The eigenvalues of $\Delta^{(k)}_{T,\pm}$ are:

$$\begin{aligned}
&(4j \mp (n-2k) + 2\beta_k(\mu) + 2)T, && j\in\mathbb{N}_0,\ \mu\in\mathrm{spec}(\Delta^{(k)}_{L,ccl}),\\
&(4j \mp (n-2(k-1)) + 2\beta_{k-1}(\mu) \pm 2)T, && j\in\mathbb{N}_0,\ \mu\in\mathrm{spec}(\Delta^{(k-1)}_{L,ccl}),\\
&(4j \mp (n-2(k-1)) + 2\beta_{k-1}(\mu) + 4 \pm 2)T, && j\in\mathbb{N}_0,\ \mu\in\mathrm{spec}(\Delta^{(k-1)}_{L,ccl}),\\
&(4j \mp (n-2(k-2)) + 2\beta_{k-2}(\mu) \pm (4\pm 2))T, && j\in\mathbb{N}_0,\ \mu\in\mathrm{spec}(\Delta^{(k-2)}_{L,ccl}),
\end{aligned} \tag{2.1}$$

the multiplicity being $m(\mu)$; and

$$\begin{aligned} &(4j \pm 2\alpha_k + 2|\alpha_k| + 2 \mp 2)T, \ j \in \mathbb{N}_0, && \text{with multiplicity } \dim\ker(\Delta_L^{(k)}), \\ &(4j \pm 2\alpha_{k-1} + 2|\alpha_{k-2}| + 2)T, \ j \in \mathbb{N}_0, && \text{with multiplicity } \dim\ker(\Delta_L^{(k-1)}). \end{aligned} \tag{2.2}$$

The trace class property of $N\left(\Delta_{T,\pm}^{\perp}\right)^{-s}$, $\Re(s) >> 0$, as well as (1.8), (1.9) follow from this explicit computation. For $\Re(s) >> 0$, the function $\zeta_{L,\pm}(s)$ is a well-defined holomorphic function. Moreover, it admits a meromorphic continuation, which is holomorphic at $s = 0$. The Riemann zeta function and the Hurwitz zeta function are holomorphic at $s = 0$.

3. Proof of Theorem II

We denote by $\dot{R}^{TC}$ the curvature of the Levi-Civita connection ∇^{TC} seen as a section of $\Lambda(T^*C) \widehat{\otimes} \Lambda(T^*C)$. Let $e_1, \dots, e_n$ be an orthonormal frame of TC, and $e^1, \dots, e^n$ the dual frame of T^*C. For $T \geq 0$, we denote $B_{T,\pm} := \frac{\dot{R}^{TC}}{2} \pm \sqrt{T} \sum_{i=1}^n e^i \wedge \widehat{e^i} + Tr^2$, which is a smooth section of $\Lambda(T^*C) \widehat{\otimes} \Lambda(T^*C)$ over C. The Berezin integral $\int^B$ maps smooth sections of $\Lambda(T^*C) \widehat{\otimes} \Lambda(T^*C)$ into smooth sections of $\Lambda(T^*C) \widehat{\otimes} o(TC)$ (see [2, Section 3]).

Theorem 3.1. *The integral*

$$\alpha_{\pm}(L) := \int_C \int^B \exp(-B_{T,\pm}) \tag{3.1}$$

is well defined and does not depend on $T > 0$. Moreover

$$\alpha_{\pm}(L) = \begin{cases} -\int_L i_1^* \widetilde{e}(TC, \nabla^{TC}, \nabla') & \text{if } n \text{ is even}, \\ \pm \frac{1}{2} \chi(L, \mathbb{R}) & \text{if } n \text{ is odd}. \end{cases} \tag{3.2}$$

Proof of Theorem 3.1. For $a \in \mathbb{R}_{>0}$, we denote by σ_a the dilation of the infinite cone $\sigma_a : (r, y) \mapsto (ar, y)$. The fact that the integral (3.1) does not depend on $T > 0$ follows from the scaling property $\sigma_a^* \left(\int^B \exp(-B_{T,\pm})\right) = \int^B \exp(-B_{a^2 T,\pm})$ and the homogeneity of the cone. Formula (3.2) can be proved by following the line of arguments in [2, p. 94 ff.] and by applying the Mathai–Quillen formalism (see [9], see also [2, Section 3]) to the infinite cone and the radial (resp. anti-radial) Morse function. □

Theorem 3.2. *For $t \in \mathbb{R}_{>0}$, $T \in \mathbb{R}_{>0}$ we have*

$$I\chi(cL, F^*) = \mathrm{Tr}_s \left[\exp(-t\Delta_{T,+})\right] = \mathrm{rk}(F) \cdot \alpha_+(L) + \gamma(F), \tag{3.3}$$

and

$$I\chi(cL, L, F^*) = \mathrm{Tr}_s \left[\exp(-t\Delta_{T,-})\right] = \mathrm{rk}(F) \cdot \alpha_-(L) + \gamma(F). \tag{3.4}$$

Proof of Theorem 3.2. From the spectral properties of the model Witten Laplacian we deduce that the heat operator $e^{-t\Delta_{T,\pm}}$ is trace class. By a McKean–Singer type argument applied to the deformed complex of L^2-forms, one has

$$\mathrm{Tr}_s \left[\exp(-t\Delta_{T,\pm})\right] = \dim\ker\left(\Delta_{T,\pm}^{\mathrm{ev}}\right) - \dim\ker\left(\Delta_{T,\pm}^{\mathrm{odd}}\right). \tag{3.5}$$

For $i = 0, \dots, n$, we denote by $\Delta_{T,\pm}^{(i)}$ the restriction of the Witten Laplacian acting on i-forms on C with values in F. A direct generalisation of [8, Theorem 4.2] to the case of coefficients in the flat bundle F yields

$$\ker(\Delta_{T,+}^{(i)}) \simeq IH_i(cL, F^*)^* \text{ resp. } \ker(\Delta_{T,-}^{(i)}) \simeq IH_i(cL, L, F^*)^*, \ i = 0, \dots, n. \tag{3.6}$$

The first identity in (3.3) resp. (3.4) follows from (3.5) and (3.6). The second identity follows by applying local index techniques to the model Witten Laplacian, as in [1, Theorem 1.9] and [2, Section XIII]. □

Proof of Theorem II. The claim of Theorem II follows by combining Theorems 3.1 and 3.2. □

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