C. R. Acad. Sci. Paris, Ser. I 355 (2017) 80–83

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Partial differential equations

Boundedness in a full parabolic two-species chemotaxis system

Les solutions d'un système de chimiotaxie à deux espèces, complètement parabolique, sont bornées

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A R T I C L E   I N F O

Article history:
Received 6 September 2016
Accepted 26 October 2016
Available online 24 November 2016

Presented by the Editorial Board

A B S T R A C T

This paper is concerned with the two-species chemotaxis system

$$\begin{cases} u_t = \triangle u - \nabla\cdot(u\chi_1(w)\nabla w) + \mu_1 u(1-u-a_1 v), & x\in\Omega, t>0, \\ v_t = \triangle v - \nabla\cdot(v\chi_2(w)\nabla w) + \mu_2 v(1-a_2 u-v), & x\in\Omega, t>0, \\ w_t = d\Delta w - w + u + v, & x\in\Omega, t>0 \end{cases}$$

in a bounded smooth domain $\Omega\subset\mathbb{R}^n (n\geq 1)$, where $d>0$, $\mu_i\geq 0$ and $a_i\geq 0$ $(i=1,2)$ are parameters, χ_i are functions satisfying some conditions. The purpose of this paper is to show the global boundedness of solutions to the above system under weaker conditions than those assumed in the related literature.

R É S U M É

Cette Note étudie les systèmes de chimiotaxie à deux espèces du type

$$\begin{cases} u_t = \triangle u - \nabla\cdot(u\chi_1(w)\nabla w) + \mu_1 u(1-u-a_1 v), & x\in\Omega, t>0, \\ v_t = \triangle v - \nabla\cdot(v\chi_2(w)\nabla w) + \mu_2 v(1-a_2 u-v), & x\in\Omega, t>0, \\ w_t = d\Delta w - w + u + v, & x\in\Omega, t>0 \end{cases}$$

où Ω est un domaine borné de $\mathbf{R}^n$ avec $n\geq 1$, $d>0$, $\mu_i\geq 0$, $i=1,2$, sont des paramètres et χ_i, $i=1,2$, sont des fonctions satisfaisant certaines conditions. Notre propos est de montrer que, sous des conditions plus faibles que celles faites jusqu'à présent dans la littérature, les solutions d'un tel système sont globalement bornées.

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http://dx.doi.org/10.1016/j.crma.2016.10.024
1631-073X/

1. Introduction

This paper is concerned with the chemotaxis system for two-species that are attracted by the same signaling substance

$$\begin{cases} u_t = \triangle u - \nabla\cdot(u\chi_1(w)\nabla w) + \mu_1 u(1-u-a_1 v), & x\in\Omega, t>0,\\ v_t = \triangle v - \nabla\cdot(v\chi_2(w)\nabla w) + \mu_2 v(1-a_2 u - v), & x\in\Omega, t>0,\\ \tau w_t = d\triangle w - w + u + v, & x\in\Omega, t>0,\\ \nabla u\cdot\nu = \nabla v\cdot\nu = \nabla w\cdot\nu = 0, & x\in\partial\Omega, t>0,\\ u(x,0)=u_0(x), v(x,0)=v_0(x), \tau w(x,0)=\tau w_0(x), & x\in\Omega, \end{cases} \tag{1.1}$$

where $\Omega\subset\mathbb{R}^n$ $(n\ge 1)$ is a bounded domain with smooth boundary $\partial\Omega$, ν is the outward normal vector to $\partial\Omega$, the initial data u_0, v_0 and w_0 are nonnegative functions; the unknown functions $u(x,t)$ and $v(x,t)$ denote the population densities of two species, respectively, and $w(x,t)$ represents the concentration of the signaling substance. Here $d>0$, $\tau=1$, $\mu_i\ge 0$ and $a_i\ge 0$ $(i=1,2)$ are parameters, the nonnegative chemotactic sensitivity functions $\chi_i\in C^{1+\theta}([0,+\infty))$ with some $\theta\in(0,1)$.

The system (1.1) is a generalization of the celebrated Keller–Segel model, which describes the migration of the single-species in response to the chemical produced by themselves [7], to the case of two species [5]. Over the last decades, the Keller–Segel model has been extensively investigated; in particular, a large amount of work has been devoted to determining whether the solutions are global in time or blow up in finite time, see, for example, [4,6,15,17] and references therein.

Though multi-species chemotaxis systems have been proposed over the last 30 years [1,5,18], they have been intensively studied recently [3,8–10]. In [3,13,14], the asymptotic stability of homogeneous steady states of the parabolic–parabolic–elliptic version of (1.1) (i.e. $\tau=0$) was studied by some comparison techniques. For the full parabolic variant of (1.1) (i.e. $\tau=1$), the global existence and behavior of solutions are investigated in [2,10–12,19]. In particular, Zhang et al. [19] proved the global boundedness of solutions to (1.1) under the assumption that μ_i is small and $\chi_i(w)\le\frac{\chi_{0,i}}{(1+\alpha_i w)^{\sigma_i}}$ with $\sigma_i>1$ and $\chi_{0,i}>0$ being sufficiently small. When $\mu_i>0$, $a_i=0$, and $\chi_i(w)$ satisfies some conditions including

$$\exists p>n,\ 2d\,\chi_i'(w) + ((d-1)\,p + \sqrt{(d-1)^2\,p^2 + 4\,d\,p})\,\chi_i^2(w)\le 0,$$

Mizukami et al. [10] obtained the global boundedness of (1.1) with $u+v-w$ replaced by $h(u,v,w)$.

This paper extends the method in [16] to show the global boundedness of solutions to (1.1) under weaker conditions than those in [10,19].

Theorem 1.1. *Let $d>0$, $\mu_i\ge 0$ and $a_i\ge 0$ $(i=1,2)$. Assume that $\chi_i(w)\le\frac{\chi_{0,i}}{(1+\alpha_i w)^{\sigma_i}}$ for $\sigma_i>1,\alpha_i>0$ and $\chi_{0,i}>0$. Then for all nonnegative functions $(u_0,v_0)\in(C(\overline{\Omega}))^2$ and $w_0\in W^{1,q}(\Omega)$ for some $q>n$, (1.1) possesses a unique classical solution (u,v,w) which is globally bounded in $\Omega\times(0,\infty)$.*

2. Proof of Theorem 1.1

As a preliminary, let us state one result about the local existence and uniqueness of a classical solution to (1.1), which can be found in [9,10,19].

Lemma 2.1. *Assume that $(u_0,v_0)\in(C(\overline{\Omega}))^2$ and $w_0\in W^{1,q}(\Omega)$ for some $q>n$ are nonnegative functions. Then there exist $T_{\max}\in(0,\infty]$ and an exactly one triple $(u,v,w)\in C(\bar\Omega\times[0,T_{\max}))\cap C^{2,1}(\bar\Omega\times(0,T_{\max}))$, which solves (1.1) classically. In addition, if $T_{\max}<+\infty$, then $\|u(\cdot,t)\|_{L^\infty(\Omega)}+\|v(\cdot,t)\|_{L^\infty(\Omega)}+\|w(\cdot,t)\|_{L^\infty(\Omega)}\to\infty$ as $t\nearrow T_{\max}$. Moreover, $\|u(\cdot,t)\|_{L^1(\Omega)}\le M$ and $\|v(\cdot,t)\|_{L^1(\Omega)}\le M$ with $M=\max\{|\Omega|,\|u_0\|_{L^1(\Omega)},\|v_0\|_{L^1(\Omega)}\}$.*

Theorem 1.1 can be derived by a standard argument provided that the following lemma is proved, and we refer the reader to [16,19], for instance.

Lemma 2.2. *Under assumptions of Theorem 1.1, there exists a constant $C>0$ such that*

$$\|u(\cdot,t)\|_{L^{n+1}(\Omega)}\le C,\quad \|v(\cdot,t)\|_{L^{n+1}(\Omega)}\le C\ \text{ for all } t\in(0,T_{\max}). \tag{2.1}$$

Proof. Let $p=n+1$ and define $\varphi(s)=\mathrm{e}^{(1+\beta s)^{-r}}$ for $s\ge 0$, where r and β satisfy

$$0<r<\min\{\frac{p-1}{2\,p\,(d+1)^2}, 2\sigma_i-2, 1\} \tag{2.2}$$

and

$$\beta>\max\{\sqrt{\frac{2\,p\,(p-1)}{d\,r}}\chi_{0,i}, 2\alpha_i\}. \tag{2.3}$$

As in the proof of Lemma 3.1 of [19] and noting $1 \le \varphi(w) \le e$ for all $s \ge 0$, we can get

$$\begin{aligned}
&\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^p\varphi(w) + \frac{p-1}{2}\int_\Omega u^{p-2}\varphi(w)|\nabla u|^2 + \frac{d}{p}\int_\Omega u^p\varphi''(w)|\nabla w|^2 \\
&\le \frac{(d+1)^2}{p-1}\int_\Omega u^p\frac{\varphi'^2(w)}{\varphi(w)}|\nabla w|^2 + (p-1)\chi_{0,1}^2\int_\Omega u^p\varphi(w)(1+\alpha_1 w)^{-2\sigma_1}|\nabla w|^2 \\
&\quad + \frac{r}{p}\int_\Omega u^p\varphi(w) + \mu_1\int_\Omega u(1-u)\varphi(w) \\
&\le \frac{(d+1)^2}{p-1}\int_\Omega u^p\frac{\varphi'^2(w)}{\varphi(w)}|\nabla w|^2 + (p-1)\chi_{0,1}^2\int_\Omega u^p\varphi(w)(1+\alpha_1 w)^{-2\sigma_1}|\nabla w|^2 \\
&\quad + \frac{r}{p}\int_\Omega u^p\varphi(w) + \mu_1 e|\Omega|.
\end{aligned} \tag{2.4}$$

Let

$$\begin{aligned}
F_1 &:= \frac{d}{p}\varphi''(s) = \frac{d}{p}\beta^2 r(r+1)(1+\beta s)^{-r-2}\varphi(s) + \frac{d}{p}\beta^2 r^2(1+\beta s)^{-2r-2}\varphi(s), \\
F_2 &:= \frac{(d+1)^2}{p-1}\frac{\varphi'^2(s)}{\varphi(s)} = \frac{(d+1)^2}{p-1}\beta^2 r^2(1+\beta s)^{-2r-2}\varphi(s), \\
F_3 &:= (p-1)\chi_{0,1}^2(1+\alpha_1 s)^{-2\sigma_1}\varphi(s).
\end{aligned}$$

Hence $\frac{2F_2}{F_1} \le \frac{2\,p\,r\,(d+1)^2}{p-1} < 1$ due to (2.2). On the other hand,

$$\frac{2F_3}{F_1} \le \frac{(p-1)\chi_{0,1}^2(1+\alpha_1 s)^{-2\sigma_1}\varphi(s)}{\frac{1}{p}\beta^2 r\,(r+1)(1+\beta s)^{-r-2}\varphi(s)} = \frac{2\,p\,(p-1)\,\chi_{0,1}^2}{d\,\beta^2\,r\,(r+1)}(1+\alpha_1 s)^{-2\sigma_1}(1+\beta s)^{r+2}.$$

Putting $f(s) = (1+\alpha_1 s)^{-2\sigma_1}(1+\beta s)^{r+2}$ for all $s \ge 0$, we then get $f'(s) \le (1+\alpha_1 s)^{-2\sigma_1-1}(1+\beta s)^{r+}[r+2-2\sigma_1+((r+2)\alpha_1 - 2\sigma_1\beta)s] < 0$ by (2.2) and (2.3). So in view of (2.3), $\frac{2F_3}{F_1} \le \frac{2\,p\,(p-1)\chi_{0,1}^2}{d\ \beta^2\,r\,(r+1)} < 1$.

Therefore from (2.3), we have

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^p\varphi(w) + \frac{p-1}{2}\int_\Omega u^{p-2}\varphi(w)|\nabla u|^2 \le \frac{r}{p}\int_\Omega u^p\varphi(w) + \mu_1 e|\Omega|, \tag{2.5}$$

which together with the fact that $1 \le \varphi(s) \le e$, implies that

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^p\varphi(w) + \frac{2(p-1)}{p^2}\|\nabla u^{\frac{p}{2}}\|_{L^2(\Omega)}^2 \le \frac{r}{p}\int_\Omega u^p\varphi(w) + \mu_1 e|\Omega|. \tag{2.6}$$

According the Gagliardo–Nirenberg inequality and using Lemma 2.1, we have

$$\begin{aligned}
\int_\Omega u^p\varphi(w) \le e\|u^{\frac{p}{2}}\|_{L^2(\Omega)}^2 &\le e\,C_{GN}^2(\|\nabla u^{\frac{p}{2}}\|_{L^2(\Omega)}^a\|u^{\frac{p}{2}}\|_{L^{\frac{2}{p}}(\Omega)}^{1-a} + \|u^{\frac{p}{2}}\|_{L^{\frac{2}{p}}(\Omega)})^2 \\
&\le 2\,e\,C_{GN}^2(\|\nabla u^{\frac{p}{2}}\|_{L^2(\Omega)}^{2a}M^{p(1-a)} + M^p)
\end{aligned}$$

with $a = \frac{n^2}{n^2+2}$, which along with (2.6) yields that for some constant $c_i(p) > 0$ $(i = 1, 2)$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^p\varphi(w) + c_1(p)(\int_\Omega u^p\varphi(w))^{1/a} \le r\int_\Omega u^p\varphi(w) + c_2(p). \tag{2.7}$$

Therefore by a standard ODE comparison argument and noticing $a \in (0, 1)$, we can get $\|u(\cdot, t)\|_{L^p(\Omega)} \le C$. It is obvious that $\|v(\cdot, t)\|_{L^p(\Omega)} \le C$ can be proved similarly. □

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