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Algebraic geometry

Ulrich bundles on blowing ups

Fibrés de Ulrich sur les éclatements

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A R T I C L E   I N F O

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A B S T R A C T

We construct an Ulrich bundle on the blowup at a point where the original variety is embedded by a sufficiently positive linear system and carries an Ulrich bundle. In particular, we describe the relation between special Ulrich bundles on blown-up surfaces and the original surface.

R É S U M É

Nous construisons un fibré de Ulrich sur l'éclatée d'une variété dans un point, dans le cas où la variété d'origine est plongée dans un système linéaire suffisamment positif et admet un fibré de Ulrich. En particulier, nous décrivons la relation entre l'existence des fibrés spéciaux de Ulrich sur une surface éclatée et sur la surface d'origine.

Let $X \subset \mathbb{P}^N$ be a smooth projective variety of dimension n, embedded by a complete linear system $|\mathcal{O}_X(H)|$ for some very ample divisor H. An *Ulrich bundle* on X [10] is a vector bundle $\mathcal{F}$ on X whose twists satisfy a set of vanishing conditions on cohomology

$$H^i(X, \mathcal{F}(-iH)) = 0 \text{ for } i > 0 \text{ and } H^j(X, \mathcal{F}(-(j+1)H)) = 0 \text{ for } j < n.$$

In other words, $\mathcal{F}$ (resp., $\mathcal{F}^*(K_X)$) is 0-regular (resp., $(n+1)$-regular) in the sense of the Castelnuovo–Mumford regularity. Since being k-regular implies $(k+1)$-regular, we can easily see that the above vanishing conditions are equivalent to

$$H^i(X, \mathcal{F}(-jH)) = 0 \text{ for all } i \text{ and } 1 \le j \le n.$$

We refer to [10] for more various equivalent definitions and properties for Ulrich bundles.

Ulrich bundles appeared in commutative algebra in relation to maximally generated maximal Cohen–Macaulay modules [16]. In algebraic geometry, the notion of Ulrich bundles surprisingly appeared thanks to recent works by Beauville and

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Eisenbud–Schreyer. The importance is motivated by the relations between the Cayley–Chow forms [2,10] and with the cohomology tables [11].

Eisenbud and Schreyer made a conjecture that every projective variety admits an Ulrich bundle [10], which is wildly open even for smooth surfaces. The answer is known for a few cases, including curves [10], complete intersections [12], Grassmannians [9], del Pezzo surfaces [10], and more rational surfaces with an anticanonical pencil [14], general K3 surfaces [1], Abelian surfaces [3], Fano polarized Enriques surfaces [5], and surfaces with $q = p_g = 0$ embedded by a sufficiently large linear system [4].

In classical algebraic geometry, there are two fundamental operations, namely, the hyperplane cut and the linear projection. It is well known that the restriction of an Ulrich bundle to a general hyperplane section is also an Ulrich bundle (cf. [6]). It is also obvious that the vanishing conditions do not affect on taking a linear projection from a point $P \in \mathbb{P}^N \setminus X$. Hence, the only interesting case occurs from the "projection from a point inside of X", which can be realized as the blowup at a given point.

We briefly review the relation between inner projections and blowups. Let $P \in X$ be a point. The linear projection from P gives a rational map $\pi_P : X \dashrightarrow \mathbb{P}^{N-1}$ defined on $X \setminus \{P\}$. We can eliminate the point of indeterminacy by taking the blowup $\sigma : \tilde{X} \to X$ at P. The complete linear system $|\sigma^*\mathcal{O}_X(H) \otimes \mathcal{O}_{\tilde{X}}(-E)|$ induces a morphism from $\tilde{X}$ to $\mathbb{P}^{N-1}$ whose image is the closure of $\pi_P(X \setminus \{P\})$, where $E = \sigma^{-1}(P)$ is the exceptional divisor.

In this short note, we construct an Ulrich bundle on the blowup at a point from an Ulrich bundle on the original variety.

Theorem 0.1. *Assume that the divisor $\tilde{H} := \sigma^*H - E$ is very ample. Suppose we have an Ulrich bundle $\mathcal{F}$ on X with respect to the polarization $\mathcal{O}_X(H)$. Then the vector bundle*

$$\tilde{\mathcal{F}} := \sigma^*\mathcal{F} \otimes \mathcal{O}_{\tilde{X}}(-E)$$

is an Ulrich vector bundle on $\tilde{X}$ with respect to $\mathcal{O}_{\tilde{X}}(\tilde{H})$.

Proof. We need to show that $\tilde{\mathcal{F}}(-j\tilde{H}) = \sigma^*(\mathcal{F}(-jH)) \otimes \mathcal{O}_{\tilde{X}}((j-1)E)$ has no cohomology for every $1 \le j \le n$. We first claim that $R\sigma_*\mathcal{O}_{\tilde{X}}((j-1)E) \simeq \mathcal{O}_X$ for each $1 \le j \le n$, i.e., $\sigma_*\mathcal{O}_{\tilde{X}}((j-1)E) \simeq \mathcal{O}_X$ and the higher direct images R^i vanish for $i > 0$. Since $R\sigma_*\mathcal{O}_{\tilde{X}} = \mathcal{O}_X$ and $R\Gamma(E, \mathcal{O}_E((j-1)E)) = 0$, the claim becomes straightforward from the exact sequence

$$0 \to \mathcal{O}_{\tilde{X}}((j-1)E) \to \mathcal{O}_{\tilde{X}}(jE) \to \mathcal{O}_E(jE) \simeq \mathcal{O}_{\mathbb{P}^{n-1}}(-j) \to 0.$$

Applying the projection formula, we have

$$R^i\sigma_*(\tilde{\mathcal{F}}(-j\tilde{H})) = \mathcal{F}(-jH) \otimes R^i\sigma_*\mathcal{O}_{\tilde{X}}((j-1)E) = 0$$

for every $i > 0$ and $1 \le j \le n$. Hence, Leray's spectral sequence implies that the cohomology group

$$\begin{aligned} H^i(\tilde{X}, \tilde{\mathcal{F}}(-j\tilde{H})) &\simeq H^i(X, \sigma_*(\tilde{\mathcal{F}}(-j\tilde{H}))) \\ &\simeq H^i(X, \mathcal{F}(-jH) \otimes \sigma_*\mathcal{O}_{\tilde{X}}((j-1)E)) \\ &= H^i(X, \mathcal{F}(-jH)) \\ &= 0 \end{aligned}$$

vanishes for every i and $1 \le j \le n$, since $\mathcal{F}$ is Ulrich on (X, H). Therefore, we conclude that $\tilde{\mathcal{F}}$ is an Ulrich vector bundle on $(\tilde{X}, \tilde{H})$.

Example 1. It is well known that every Abelian surface $X \subset \mathbb{P}^N$ carries an Ulrich bundle of rank 2 [3]. By taking consecutive blowups, *Theorem 0.1* implies that blowups of an Abelian surface, embedded positively enough, at a few general points carry an Ulrich bundle of rank 2.

Example 2. Del Pezzo surfaces as blowups of $v_3(\mathbb{P}^2) \subset \mathbb{P}^9$ carry an Ulrich bundle of arbitrary rank $r \ge 2$ [10,15]. Note that the construction of higher-rank Ulrich bundles in [15] has a similarity with *Theorem 0.1*. Almost nothing is yet known for higher dimensional cases. Since every k-uple embedding of $\mathbb{P}^n$ carries an Ulrich bundle [10], we have the existence of Ulrich bundles on blowups of $\mathbb{P}^n$ at s general points where

$$s \le \binom{n+k}{k} - (n-1)(n+1) - 4,$$

with respect to the polarization $(\sigma^*\mathcal{O}_{\mathbb{P}^n}(k))(-E)$ which is very ample [7].

A particularly interesting case happens when X is a smooth surface. Except only for a few surfaces like del Pezzo surfaces, the study of Ulrich bundles has been focused only on minimal surfaces, since it is convenient to apply various vector bundle techniques. On the other hand, the above construction provides an Ulrich bundle on blown-up surfaces at a few points, by taking consecutive inner projections. In particular, we found that the procedure yields much more interesting phenomena on "special Ulrich bundles". Eisenbud and Schreyer introduced the notion of *special Ulrich bundles* on a surface X [10], which are Ulrich bundles $\mathcal{F}$ of rank 2 such that $\det \mathcal{F} = \mathcal{O}_X(K_X + 3H)$, where K_X denotes the canonical divisor of X. The existence of special Ulrich bundles yields a very nice presentation of the Cayley–Chow form of X. More precisely, a special Ulrich bundle provides a Pfaffian Bézout expression of the Cayley–Chow form of X in Plücker coordinates [10].

Note that special Ulrich bundles also carry a technical merit. Let $\mathcal{F}$ be a vector bundle of rank 2 on a smooth polarized surface (X, H) with $\det \mathcal{F} = \mathcal{O}_X(K_X + 3H)$. If $\mathcal{F}$ satisfies a partial vanishing condition $H^\bullet(X, \mathcal{F}(-H)) = 0$, then

$$H^i(X, \mathcal{F}(-2H)) \simeq H^{2-i}(X, \mathcal{F}^*(K_X + 2H))^* \simeq H^{2-i}(X, \mathcal{F}(K_X + 2H - (K_X + 3H))^* = 0$$

by Serre duality, which automatically implies that $\mathcal{F}$ is Ulrich.

As an immediate consequence of *Theorem 0.1*, a special Ulrich bundle gives rise to a special Ulrich bundle on a blown-up surface:

Corollary 0.2. *Let (X, H) be a smooth polarized surface satisfies the assumptions in Theorem 0.1. If $\mathcal{F}$ is a special Ulrich bundle on X, then $\tilde{\mathcal{F}}$ is also a special Ulrich bundle on $\tilde{X}$.*

Proof. It comes from a direct computation

$$\det \tilde{\mathcal{F}} = \sigma^*(\mathcal{O}_X(K_X + 3H)) \otimes \mathcal{O}_{\tilde{X}}(-2E) = \mathcal{O}_{\tilde{X}}(K_{\tilde{X}} + 3\tilde{H}).$$

It is quite natural to ask for the converse direction. Unfortunately, in general *Theorem 0.1* does not provide any information on Ulrich bundles on the original variety when we have an Ulrich bundle on a blowup. For instance, let $X \subset \mathbb{P}^5$ be the Veronese surface. If we blowup X at a point, we get a cubic scroll in $\mathbb{P}^4$ that carries an Ulrich line bundle since it is linear determinantal [10]. However, it is well known that X itself does not have any Ulrich line bundle. Therefore, it is very interesting that we are able to approach the converse direction when we focused on special Ulrich bundles. Not only showing the existence, but it also reveals the connection between special Ulrich bundles on upstairs and downstairs as follows:

Theorem 0.3. *Let (X, H) be a smooth polarized surface satisfies the assumptions in Theorem 0.1 as above. Let $\tilde{\mathcal{F}}$ be a special Ulrich bundle on $(\tilde{X}, \tilde{H})$. Then $\sigma_*(\tilde{\mathcal{F}}(E))$ is a special Ulrich bundle on (X, H).*

Proof. We first claim that $\sigma_*(\tilde{\mathcal{F}}(E))$ is a vector bundle on X. Since $c_1(\tilde{\mathcal{F}}(E)) = K_{\tilde{X}} + 3\tilde{H} + 2E = \sigma^*(K_X + 3H)$, we have $\deg \tilde{\mathcal{F}}(E)|_E = 0$. By Grothendieck's theorem, we have $\tilde{\mathcal{F}}(E)|_E \simeq \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(-a)$ for some $a \geq 0$. Since $\tilde{\mathcal{F}}$ is 0-regular with respect to $\tilde{H}$, it is globally generated, and furthermore, the restriction $\tilde{\mathcal{F}}|_E \simeq \mathcal{O}_{\mathbb{P}^1}(a+1) \oplus \mathcal{O}_{\mathbb{P}^1}(-a+1)$ is also globally generated. Hence either $a = 0$ or $a = 1$ holds. For the both cases, we obtain $h^0(E, \tilde{\mathcal{F}}(E)|_E) = 2$. Therefore,$h^0(\sigma^{-1}(Q), \tilde{\mathcal{F}}(E)|_{\sigma^{-1}(Q)}) = 2$ holds for every $Q \in X$, implying that $\sigma_*(\tilde{\mathcal{F}}(E))$ is locally free of rank 2 by Grauert's theorem. Note also that a similar computation implies that the higher direct image $R^1\sigma_*(\tilde{\mathcal{F}}(E)) = 0$, since $\tilde{\mathcal{F}}(E)|_E$ has no H^1.

To prove $\sigma_*(\tilde{\mathcal{F}}(E))$ is a special Ulrich bundle, it is enough to show that $\det \sigma_*(\tilde{\mathcal{F}}(E)) \simeq \mathcal{O}_X(K_X + 3H)$ and that the partial vanishing conditions $H^\bullet(X, \sigma_*(\tilde{\mathcal{F}}(E)) \otimes \mathcal{O}_X(-H)) = 0$ hold.

Since any coherent sheaf on X supported on a subset of codimension at least 2 has a trivial determinant (cf. [13]), it is enough to show that $\det \sigma_*(\tilde{\mathcal{F}}(E)) \simeq \mathcal{O}_X(K_X + 3H)$ on the subset $X \setminus \{P\}$. Now the claim becomes straightforward since σ induces an isomorphism $\tilde{X} \setminus E \simeq X \setminus \{P\}$ and $\det \tilde{\mathcal{F}}(E) \simeq \sigma^*\mathcal{O}_X(K_X + 3H)$.

By the projection formula and Leray spectral sequence, we have

$$H^i(X, \sigma_*(\tilde{\mathcal{F}}(E)) \otimes \mathcal{O}_X(-H)) \simeq H^i(\tilde{X}, \tilde{\mathcal{F}}(E) \otimes \sigma^*\mathcal{O}_X(-H)) = H^i(\tilde{X}, \tilde{\mathcal{F}}(-\tilde{H})) = 0$$

which completes the proof.

Remark 1. When X is a smooth surface of $p_g = q = 0$ embedded by a sufficiently positive line bundle, Beauville's result [4] overlaps with Theorem 0.3 as a special case. He constructed a rank-2 Ulrich bundle $\mathcal{F}$ from a Lazarsfeld–Mukai sequence. Indeed, it is special since $\det \mathcal{F} = \mathcal{O}_X(K_X + 3H)$.

Example 3. Let $X \subset \mathbb{P}^{s+1}$ be a K3 surface of genus $s \geq 6$ and $\mathrm{Pic}(X) = \mathbb{Z} \cdot [H]$. Let $P_1, \cdots, P_{s-4} \in X$ be general points, and let $\sigma : \tilde{X} \to X$ be the blowup at P_i's. Note that $\tilde{H} = \sigma^*\mathcal{O}_X(H) \otimes \mathcal{O}_{\tilde{X}}(-E)$ is very ample [8]. Since (X, H) carries a $(2s + 10)$-dimensional family of special Ulrich bundles [1], the same estimation holds for special Ulrich bundles on $(\tilde{X}, \tilde{H})$.

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