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Partial differential equations

Lipschitz dependence of the coefficients on the resolvent and greedy approximation for scalar elliptic problems ☆

Dépendance Lipschitz des coefficients comme fonction de la résolvante et approximation greedy pour les problèmes elliptiques scalaires

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A R T I C L E   I N F O

A B S T R A C T

We analyze the inverse problem of identifying the diffusivity coefficient of a scalar elliptic equation as a function of the resolvent operator. We prove that, within the class of measurable coefficients, bounded above and below by positive constants, the resolvent determines the diffusivity in an unique manner. Furthermore, we prove that the inverse mapping from resolvent to the coefficient is Lipschitz in suitable topologies. This result plays a key role when applying greedy algorithms to the approximation of parameter-dependent elliptic problems in an uniform and robust manner, independent of the given source terms. In one space dimension, the results can be improved using the explicit expression of solutions, which allows us to link distances between one resolvent and a linear combination of finitely many others and the corresponding distances on coefficients. These results are also extended to multi-dimensional elliptic equations with variable density coefficients. We also point out some possible extensions and open problems.

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R É S U M É

Nous examinons le problème inverse de l'identification du coefficient de diffusion comme fonction de la résolvante pour des équations elliptiques scalaires. Nous établissons, pour des topologies appropriées, un résultat de stabilité Lipschitz pour une classe de coefficients de diffusion mesurables, minorés et majorés par des constantes positives fixées a priori. Ce résultat intervient de manière essentielle dans le développement d'algorithmes greedy pour l'approximation d'une famille paramétrée de problèmes elliptiques de manière robuste et uniforme par rapport au terme source. Nous traitons séparément le cas de la dimension un, pour lequel nous disposons de formules explicites de représentation des solutions permettant de comparer la distance entre une résolvante et une combinaison linéaire d'un

nombre fini d'autres et des coefficients correspondants, et un développement complet de l'approche *greedy*. Nous étendons ces résultats au problème de l'identification de la densité à partir de l'opérateur résolvant correspondant. Nous signalons aussi quelques problèmes ouverts, en particulier dans le cas multi-dimensionnel.

1. Introduction

Let Ω be a bounded domain of $\mathbb{R}^n$, $n \geq 1$. Fix $0 < \sigma_0 < \sigma_1$ and consider the class of scalar diffusivity coefficients

$$\Sigma = \{\sigma \in L^\infty(\Omega); \ \sigma_0 \leq \sigma \leq \sigma_1 \text{ a.e. in } \Omega\}.$$

In the sequel $H_0^1(\Omega)$ is endowed with the norm

$$\|w\|_{H_0^1(\Omega)} = \|\nabla w\|_{L^2(\Omega)^n}.$$

For $\sigma \in \Sigma$, let $A_\sigma : H_0^1(\Omega) \to H^{-1}(\Omega)$ be the bounded operator given by

$$A_\sigma u = -\text{div}(\sigma \nabla u).$$

The inverse or resolvent operator R_σ maps continuously $H^{-1}(\Omega)$ into $H_0^1(\Omega)$.

To be more precise, denote by $\langle \cdot, \cdot \rangle_{-1,1}$ the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$. For $f \in H^{-1}(\Omega)$, consider the variational problem of finding $w \in H_0^1(\Omega)$ so that

$$\int_\Omega \sigma \nabla w \cdot \nabla v = \langle f, v \rangle_{-1,1} \text{ for any } v \in H_0^1(\Omega). \tag{1.1}$$

According to Lax–Milgram's lemma, (1.1) has a unique solution $u_\sigma \in H_0^1(\Omega)$. Moreover, the energy estimate yields

$$\|u_\sigma\|_{H_0^1(\Omega)} \leq \sigma_0^{-1} \|f\|_{H^{-1}(\Omega)}. \tag{1.2}$$

Indeed, using the solution itself u_σ as test function we have

$$\sigma_0 \int_\Omega |\nabla u_\sigma|^2 \, \mathrm{d}x \leq \int_\Omega \sigma(x)|\nabla u_\sigma|^2 \, \mathrm{d}x = \langle f, u \rangle_{-1,1} \leq \|f\|_{H^{-1}(\Omega)} \|\nabla u_\sigma\|_{L^2(\Omega)^n}$$

and consequently

$$\sigma_0 \|\nabla u_\sigma\|_{L^2(\Omega)^n} \leq \|f\|_{H^{-1}(\Omega)}.$$

As we have seen, the coefficient σ of the elliptic equation determines uniquely the resolvent operator R_σ. We address the inverse problem consisting in identifying the coefficient σ in terms of the resolvent R_σ.

The main result of this paper ensures the Lipschitz character of the inverse map in suitable topologies:

Theorem 1.1. *For any* $\sigma, \widetilde{\sigma} \in \Sigma$,

$$\sigma_0^2 \|R_\sigma - R_{\widetilde{\sigma}}\|_{-1,1} \leq \|\sigma - \widetilde{\sigma}\|_{L^\infty(\Omega)} \leq \sigma_1^2 \|R_\sigma - R_{\widetilde{\sigma}}\|_{-1,1}. \tag{1.3}$$

Here and henceforth $\|\cdot\|_{-1,1}$ *denotes the norm in* $\mathscr{B}(H^{-1}(\Omega), H_0^1(\Omega))$.

As observed by Albert Cohen [4], this result can be easily extended to the case of continuous matrix valued coefficients. The proof is the same as the one we shall develop, but using test functions that are scaled in a more pronounced manner in a distinguished direction. The extension of this result to the general case of measurable matrix valued coefficients seems however more delicate and requires further work. See Remark 2.1 below.

As we shall see below, this question and result arise in the context of parameter-dependent elliptic equations and it is of potential use (but not sufficient) to develop greedy algorithms to build fast and efficient approximation methods.

By inspecting the proof, one can see that Theorem 1.1 holds for any domain Ω for which Poincaré's inequality holds.

Let d_∞ be the distance (between diffusivity coefficients) induced by the L^∞-norm. Inequality (1.3) in Theorem 1.1 can be rephrased as

$$\sigma_0^2 d_R \leq d_\infty \leq \sigma_1^2 d_R \text{ on } \Sigma,$$

where d_R is the metric on Σ defined as follows

$$d_R(\sigma, \widetilde{\sigma}) = \|R_\sigma - R_{\widetilde{\sigma}}\|_{-1,1}, \ \ \sigma, \widetilde{\sigma} \in \Sigma.$$

The rest of the paper is organized as follows. We prove Theorem 1.1 in Section 2. In Section 3 we adapt the proof of Theorem 1.1 in order to establish a Lipschitz stability estimate in the case of the Neumann or Robin boundary conditions. The case of a BVP with non-homogeneous boundary values is treated in Section 4. In that case, the resolvent is obtained by varying the boundary data. Due to the elliptic smoothing effect, which prevents the information to propagate completely from the boundary to the interior, in the present case we are only able to prove Hölder's stability. We devote Section 5 to the one-dimensional case. Taking advantage of the explicit representation formula for the solution to the BVP, we establish a Lipschitz stability property and also estimate the distance from a resolvent to the linear subspace generated by a finite number of them. In Section 6, we present the motivation of this paper in the context of greedy algorithms for parameter-dependent elliptic equations and we fully develop it in the one-dimensional case, using the results of Section 5. We also added a remark in Section 6 in order to explain how the same program can be fully developed, in any dimension, for elliptic equations with variable density coefficients. We close with Section 7 devoted to some open problems, and in particular to the extension of the greedy approach to the multi-dimensional diffusivity model.

2. Proof of Theorem 1.1

The first inequality in (1.3) is contained in the following elementary lemma.

Lemma 2.1. *For any $\sigma, \widetilde{\sigma} \in L^\infty(\Omega)$ satisfying $\sigma_0 \le \sigma, \widetilde{\sigma}$,*

$$\|R_\sigma - R_{\widetilde{\sigma}}\|_{-1,1} \le \sigma_0^{-2} \|\sigma - \widetilde{\sigma}\|_{L^\infty(\Omega)}.$$

Proof. From (1.1) we have

$$\int_\Omega \sigma \nabla u_\sigma \cdot \nabla v \, dx = \int_\Omega \widetilde{\sigma} \nabla u_{\widetilde{\sigma}} \cdot \nabla v \, dx \text{ for any } v \in H_0^1(\Omega).$$

Hence

$$\int_\Omega \sigma \nabla (u_\sigma - u_{\widetilde{\sigma}}) \cdot \nabla v \, dx = \int_\Omega (\widetilde{\sigma} - \sigma) \nabla u_{\widetilde{\sigma}} \cdot \nabla v \, dx \text{ for any } v \in H_0^1(\Omega).$$

The particular choice of $v = u_\sigma - u_{\widetilde{\sigma}}$ in the identity above, thanks to (1.2), yields

$$\sigma_0 \int_\Omega |\nabla (u_\sigma - u_{\widetilde{\sigma}})|^2 \, dx \le \int_\Omega \sigma |\nabla (u_\sigma - u_{\widetilde{\sigma}})|^2 \, dx \le \|\widetilde{\sigma} - \sigma\|_{L^\infty(\Omega)} \|\nabla u_{\widetilde{\sigma}}\|_{L^2(\Omega)^n} \|\nabla (u_\sigma - u_{\widetilde{\sigma}})\|_{L^2(\Omega)^n} \tag{2.1}$$

$$\le \sigma_0^{-1} \|\widetilde{\sigma} - \sigma\|_{L^\infty(\Omega)} \|f\|_{H^{-1}(\Omega)} \|\nabla (u_\sigma - u_{\widetilde{\sigma}})\|_{L^2(\Omega)^n}. \quad \square$$

From (2.1) we deduce immediately the expected inequality.

Next, we establish the key lemma that we will use to prove the second inequality in (1.3).

Lemma 2.2. *Let $\gamma \in L^\infty(\Omega)$. For a.e. $x_0 \in \Omega$, there exists a sequence $(u_{x_0,\epsilon})$ in $H_0^1(\Omega)$ so that $\|u_{x_0,\epsilon}\|_{H_0^1(\Omega)} = 1$, for each ϵ, and*

$$\lim_\epsilon \int_\Omega \gamma(x) |\nabla u_{x_0,\epsilon}|^2 \, dx = \gamma(x_0).$$

Proof. Let $\epsilon > 0$ and set

$$\varphi_\epsilon(r) = 0, \ r \le 0, \ \ \varphi_\epsilon(r) = r, \ 0 < r < \epsilon, \ \ \varphi_\epsilon(r) = \epsilon, \ r \ge \epsilon,$$

the continuous function such that $\varphi_\epsilon' = \chi_{(0,\epsilon)}$, where $\chi_{(0,\epsilon)}$ is the characteristic function of the interval $(0, \epsilon)$.

Let $x_0 \in \Omega$ and ϵ_0 be sufficiently small is such a away that $B(x_0, \epsilon_0) \subset \Omega$. Define $u_{x_0,\epsilon}$, $0 < \epsilon \le \epsilon_0$ by

$$u_{x_0,\epsilon}(x) = \frac{1}{\sqrt{|B(x_0, \epsilon)|}} \varphi_\epsilon(|x - x_0|).$$

It is easy to see that

$$\nabla u_{x_0,\epsilon}(x) = \chi_{B(x_0,\epsilon)}(x) \frac{x-x_0}{|x-x_0|\sqrt{|B(x_0,\epsilon)|}},$$

and that $u_{x_0,\epsilon}$ belongs to $H_0^1(\Omega)$. Whence

$$|\nabla u_{x_0,\epsilon}|^2 = \frac{1}{|B(x_0,\epsilon)|} \chi_{B(x_0,\epsilon)}.$$

Therefore $\|u_{x_0,\epsilon}\|_{H_0^1(\Omega)} = 1$ and by Lebesgue's differentiation theorem

$$\int_{\Omega} \gamma(x)|\nabla u_{x_0,\epsilon}|^2\,dx = \frac{1}{|B(x_0,\epsilon)|}\int_{B(x_0,\epsilon)} \gamma(x)\,dx \underset{\epsilon\to 0}{\longrightarrow} \gamma(x_0) \text{ a.e. } x_0 \in \Omega. \quad \square$$

Corollary 2.1. *Let $\gamma \in L^\infty(\Omega)$ be so that*

$$\int_{\Omega} \pm\gamma|\nabla u|^2\,dx \le C, \text{ for any } u \in H_0^1(\Omega),\ \|u\|_{H_0^1(\Omega)} = 1, \tag{2.2}$$

for some constant $C > 0$. Then

$$\|\gamma\|_{L^\infty(\Omega)} \le C. \tag{2.3}$$

Proof. In light of Lemma 2.2, for a.e. $x_0 \in \Omega$, there exists a sequence $(u_n^\pm)$ in $H_0^1(\Omega)$ so that $\|u_n^\pm\|_{H_0^1(\Omega)} = 1$, for each n, and

$$\lim_n \int_{\Omega} \pm\gamma(x)|\nabla u_n^\pm|^2\,dx = \pm\gamma(x_0).$$

Therefore, in view of (2.2), $|\gamma(x_0)| \le C$ a.e. $x_0 \in \Omega$, implying (2.3). $\quad \square$

We are now ready to complete the proof of Theorem 1.1. Fix $\sigma, \widetilde{\sigma} \in \Sigma_0$. Starting from the identity

$$A_\sigma - A_{\widetilde{\sigma}} = A_\sigma(R_{\widetilde{\sigma}} - R_\sigma)A_{\widetilde{\sigma}},$$

we get

$$\|A_\sigma - A_{\widetilde{\sigma}}\|_{1,-1} \le \sigma_1^2\|R_\sigma - R_{\widetilde{\sigma}}\|_{-1,1}. \tag{2.4}$$

On the other hand,

$$\langle (A_\sigma - A_{\widetilde{\sigma}})u, v\rangle_{-1,1} = \int_{\Omega} (\sigma - \widetilde{\sigma})\nabla u \cdot \nabla v\,dx, \ \ u, v \in H_0^1(\Omega),$$

implying

$$\int_{\Omega} (\sigma - \widetilde{\sigma})\nabla u \cdot \nabla v\,dx \le \|A_\sigma - A_{\widetilde{\sigma}}\|_{1,-1}\|u\|_{H_0^1(\Omega)}\|v\|_{H_0^1(\Omega)}, \ \ u, v \in H_0^1(\Omega).$$

Combined with (2.4), this estimate yields

$$\int_{\Omega} \gamma\nabla u \cdot \nabla v\,dx \le \sigma_1^2\|R_\sigma - R_{\widetilde{\sigma}}\|_{-1,1}\|u\|_{H_0^1(\Omega)}\|v\|_{H_0^1(\Omega)}, \ \ u, v \in H_0^1(\Omega), \tag{2.5}$$

where we set $\gamma = \sigma - \widetilde{\sigma}$. Hence

$$\int_{\Omega} \gamma|\nabla u|^2\,dx \le \sigma_1^2\|R_\sigma - R_{\widetilde{\sigma}}\|_{-1,1}, \ \ u \in H_0^1(\Omega),\ \|u\|_{H_0^1(\Omega)} = 1. \tag{2.6}$$

But, by symmetry, (2.6) holds when γ is substituted by $-\gamma = \widetilde{\sigma} - \sigma$. That is we have

$$\int_{\Omega} \pm\gamma|\nabla u|^2\,dx \le \sigma_1^2\|R_\sigma - R_{\widetilde{\sigma}}\|_{-1,1}, \ \ u \in H_0^1(\Omega),\ \|u\|_{H_0^1(\Omega)} = 1$$

which, by Corollary 2.1, yields the second inequality of (1.3).

Remark 2.1. Following interesting discussions with A. Cohen [4], here we present possible extensions to the anisotropic case that can be obtained following the method of proof of Theorem 1.1.

(i) Consider, to begin with, in dimension two, the case of an anisotropic diagonal conductivity

$$\sigma = \operatorname{diag}(\sigma_1, \sigma_2).$$

In the sequel, for the sake of simplicity, we identify σ with (σ_1, σ_2).

Fix $0 < a_0 < a_1$ and let Σ' be the set of $\sigma = (\sigma_1, \sigma_2) \in C(\overline{\Omega}) \oplus C(\overline{\Omega})$ satisfying

$$a_0 |\xi|^2 \leq \sigma_1(x)\xi_1^2 + \sigma_2(x)\xi_2^2 \quad \text{and} \quad \sigma_1(x), \sigma_2(x) \leq a_1 \quad \text{for any } x \in \Omega, \ \xi \in \mathbb{R}^2.$$

Let $\sigma, \widetilde{\sigma} \in \Sigma'$. With similar notations (γ denotes the difference of two diffusivity pairs), instead of (2.6) we have in the present case

$$\int_{\Omega} \left[\gamma_1 (\partial_1 u)^2 + \gamma_2 (\partial_2 u)^2 \right] \mathrm{d}x \leq a_1^2 \|R_\sigma - R_{\widetilde{\sigma}}\|_{-1,1}, \quad u \in H_0^1(\Omega), \ \|u\|_{H_0^1(\Omega)} = 1. \tag{2.7}$$

Without loss of generality, we can assume that

$$\|\gamma_1\|_{C(\overline{\Omega})} = \max \left(\|\gamma_1\|_{C(\overline{\Omega})}, \|\gamma_2\|_{C(\overline{\Omega})} \right)$$

Fix $0 < \delta < 1$ and let $x_0 \in \Omega$ so that

$$|\gamma_1(x_0)| = (1 - \delta) \|\gamma_1\|_{C(\overline{\Omega})}.$$

Substituting $\sigma - \widetilde{\sigma}$ by $\widetilde{\sigma} - \sigma$, we can always assume that $\gamma_1(x_0) > 0$.

Let φ_ϵ be as in Lemma 2.2, $D_K(x_0, \epsilon) = \{(x_1, x_2); \ |(K(x_1 - x_{0,1}), x_2 - x_{0,2})| \leq \epsilon\}$ and consider the test function

$$\varphi_{x_0, \epsilon}(x_1, x_2) = \frac{1}{\sqrt{D_K(x_0, \epsilon)}} \varphi_\epsilon(|(K(x_1 - x_{0,1}), x_2 - x_{0,2})|),$$

where the scaling parameter K is chosen in such a away that $K\gamma_1(x_0) - |\gamma_2(x_0)| \geq \gamma_1(x_0)$, and ϵ is sufficiently small so that $\operatorname{supp}(\varphi_{x_0, \epsilon}) \Subset \Omega$.

Define $\psi_{x_0, \epsilon}$ by $\psi_{x_0, \epsilon}(x_1, x_2) = \varphi_{x_0, \epsilon}(x_2, x_1)$ and observe that we still have $\operatorname{supp}(\psi_{x_0, \epsilon}) \Subset \Omega$ provided that ϵ is sufficiently small. Then (2.7) with $u = \varphi_{x_0, \epsilon}$ and $u = \psi_{x_0, \epsilon}$ successively yields in a straightforward manner

$$\frac{1}{|D_K(x_0, \epsilon)|} \int_{D_K(x_0, \epsilon)} (K\gamma_1 + \gamma_2) \, \mathrm{d}x \leq 2a_1^2 \|R_\sigma - R_{\widetilde{\sigma}}\|_{-1,1}.$$

Whence

$$\frac{1}{|D_K(x_0, \epsilon)|} \int_{D_K(x_0, \epsilon)} \gamma_1 \, \mathrm{d}x \leq 2a_1^2 \|R_\sigma - R_{\widetilde{\sigma}}\|_{-1,1}.$$

A standard continuity argument leads

$$(1 - \delta) \max \left(\|\gamma_1\|_{C(\overline{\Omega})}, \|\gamma_2\|_{C(\overline{\Omega})} \right) = \gamma_1(x_0) \leq 2a_1^2 \|R_\sigma - R_{\widetilde{\sigma}}\|_{-1,1}.$$

Letting δ tend to zero, we get

$$\max \left(\|\gamma_1\|_{C(\overline{\Omega})}, \|\gamma_2\|_{C(\overline{\Omega})} \right) \leq 2a_1^2 \|R_\sigma - R_{\widetilde{\sigma}}\|_{-1,1}.$$

On the other hand, we can proceed as in the proof of Lemma 2.1 in order to get

$$\frac{1}{2} a_0^2 \|R_\sigma - R_{\widetilde{\sigma}}\|_{-1,1} \leq \max \left(\|\gamma_1\|_{C(\overline{\Omega})}, \|\gamma_2\|_{C(\overline{\Omega})} \right).$$

In other words, we established the following two-sided estimate

$$\frac{1}{2} a_0^2 \|R_\sigma - R_{\widetilde{\sigma}}\|_{-1,1} \leq \max \left(\|\gamma_1\|_{C(\overline{\Omega})}, \|\gamma_2\|_{C(\overline{\Omega})} \right) \leq 2a_1^2 \|R_\sigma - R_{\widetilde{\sigma}}\|_{-1,1}.$$

(ii) The same arguments apply in the any space dimension for continuous anisotropic diagonal conductivities of the form

$$\sigma = \operatorname{diag}(\sigma_1, \ldots \sigma_n).$$

(iii) The case of general symmetric continuous conductivities σ can be handled by an extra diagonalization argument that can be performed at each point x_0 in Ω.

(iv) Handling the more general case of measurable matrix valued diffusivities requires significant extra work [4].

3. Neumann and Robin boundary conditions

We explain briefly how Theorem 1.1 can be adapted to both Neumann and Robin BVP's.

3.1. The Neumann case

For $\sigma \in \Sigma$, define $A_\sigma^N : H^1(\Omega) \to (H^1(\Omega))'$ by

$$\langle A_\sigma^N u, v \rangle := \int_\Omega \sigma \nabla u \cdot \nabla v \, dx + \int_\Omega u v \, dx, \ \ u, v \in H^1(\Omega),$$

where $\langle \cdot, \cdot \rangle$ is the duality pairing between $(H^1(\Omega))'$ and $H^1(\Omega)$.

Clearly A_σ^N is bounded and, with $\underline{\sigma}_1 = \max(\sigma_1, 1)$,

$$\|A_\sigma^N u\|_{(H^1(\Omega))'} \le \underline{\sigma}_1 \|u\|_{H^1(\Omega)}, \ \ u \in H^1(\Omega).$$

We claim that A_σ^N is invertible. Indeed, if $f \in (H^1(\Omega))'$, we get by applying Lax–Milgram's lemma that the variational problem

$$\int_\Omega \sigma \nabla u_\sigma \cdot \nabla v \, dx + \int_\Omega u_\sigma v \, dx = \langle f, v \rangle, \ \ v \in H^1(\Omega) \tag{3.1}$$

has a unique solution $u_\sigma \in H^1(\Omega)$.

Taking $v = u_\sigma$ in (3.1), we get in a straightforward manner that

$$\|u_\sigma\|_{H^1(\Omega)} \le \underline{\sigma}_0^{-1} \|f\|_{(H^1(\Omega))'}, \ \ \text{with } \underline{\sigma}_0 = \min(\sigma_0, 1). \tag{3.2}$$

As a consequence of (3.1), $A_\sigma^N u_\sigma = f$. Thus A_σ^N has a bounded inverse

$$R_\sigma^N := (A_\sigma^N)^{-1} : (H^1(\Omega))' \to H^1(\Omega).$$

This operator is nothing but the resolvent of the operator $-\text{div}(\sigma \nabla \cdot) + 1$ under the Neumann boundary condition. When Ω and u are sufficiently smooth, this boundary condition can be written as $\partial_\nu u = 0$ on $\partial\Omega$, where $\partial_\nu = \nu \cdot \nabla$ with ν the exterior normal unit normal vector field on Γ.

As

$$\langle (A_\sigma^N - A_{\tilde{\sigma}}^N) u, v \rangle = \int_\Omega (\sigma - \tilde{\sigma}) \nabla u \cdot \nabla v \, dx, \ \ u, v \in H^1(\Omega),$$

we can mimic the proof of Theorem 1.1 in order to get

$$\|\sigma - \tilde{\sigma}\|_{L^\infty(\Omega)} \le \underline{\sigma}_1^2 \|R_\sigma^N - R_{\tilde{\sigma}}^N\|_{-1,1}.$$

On the other hand, we have, similarly to Lemma 2.1,

$$\underline{\sigma}_0^2 \|R_\sigma^N - R_{\tilde{\sigma}}^N\|_{-1,1} \le \|\sigma - \tilde{\sigma}\|_{L^\infty(\Omega)}.$$

In other words, we proved

$$\underline{\sigma}_0^2 \|R_\sigma^N - R_{\tilde{\sigma}}^N\|_{-1,1} \le \|\sigma - \tilde{\sigma}\|_{L^\infty(\Omega)} \le \underline{\sigma}_1^2 \|\sigma - \tilde{\sigma}\|_{L^\infty(\Omega)}.$$

3.2. The Robin case

In the present subsection, we assume that Ω has Lipschitz boundary Γ.

We examine the case of a BVP with a Robin boundary condition. To this end, pick $\beta \in L^\infty(\Gamma)$ so that $\beta \ge 0$ and $\beta \ge \beta_0$ on a measurable subset Γ_0 of Γ of positive measure, where $\beta_0 > 0$ is some constant. Consider the Robin BVP

$$-\text{div}(\sigma \nabla u) = f \text{ in } \Omega \text{ and } \sigma \partial_\nu u + \beta u = 0 \text{ on } \Gamma. \tag{3.3}$$

If $\sigma \in \Sigma$, define $A_\sigma^R : H^1(\Omega) \to (H^1(\Omega))'$ by

$$\langle A_\sigma^R u, v \rangle := \int_\Omega \sigma \nabla u \cdot \nabla v \, dx + \int_\Gamma \beta u v \, dS(x), \ \ u, v \in H^1(\Omega).$$

In the sequel, we equip $H^1(\Omega)$ with the norm

$$\|u\|_{H^1(\Omega)} = \left(\|\nabla u\|_{L^2(\Omega)^n}^2 + \|u\|_{L^2(\Gamma_0)}^2 \right)^{1/2}. \tag{3.4}$$

It is not hard to check that A_σ^R is bounded and

$$\|A_\sigma^R u\|_{(H^1(\Omega))'} \le \underline{\sigma}_1 \|u\|_{H^1(\Omega)}, \ \ u \in H^1(\Omega), \ \ \text{with } \underline{\sigma}_1 = \max(\sigma_1, \kappa \|\beta\|_{L^\infty(\Gamma)}),$$

where κ is the norm of the trace operator $u \in H^1(\Omega) \to u_{|\Gamma} \in L^2(\Gamma)$ when $H^1(\Omega)$ is endowed with the norm (3.4).

Similarly to the Neumann case, we show that A_σ^R is invertible and we calculate its inverse. To do that we consider the bilinear form

$$a(u,v) = \int_\Omega \sigma \nabla u \cdot \nabla v \, dx + \int_\Gamma \beta \, u \, v \, dS(x), \ \ u, v \in H^1(\Omega).$$

One can check that $u \to a(u,u)$ defines a norm on $H^1(\Omega)$ equivalent to the usual one on $H^1(\Omega)$. Let $f \in (H^1(\Omega))'$. Then according to Riesz's representation theorem, there exists a unique $u_\sigma \in H^1(\Omega)$ satisfying

$$a(u_\sigma, \psi) = \int_\Omega \sigma \nabla u_\sigma \cdot \nabla \psi \, dx + \int_\Gamma \beta \, u_\sigma \, \psi \, dS(x) = \langle f, \psi \rangle, \ \ \psi \in H^1(\Omega). \tag{3.5}$$

Note that u_σ is nothing but the variational solution to the BVP (3.3).

From (3.5), we easily get

$$\|u_\sigma\|_{H^1(\Omega)} \le \underline{\sigma}_0 \|f\|_{(H^1(\Omega))'}, \ \ \text{with } \underline{\sigma}_0 = \min(\sigma_0, \beta_0).$$

Consequently, A_σ^R possesses a bounded inverse $R_\sigma^R = (A_\sigma^R)^{-1} : (H^1(\Omega))' \to H^1(\Omega)$ defined by $R_\sigma f := u_\sigma$ for $f \in (H^1(\Omega))'$.

Concerning the inverse problem for Robin boundary conditions, starting from

$$\langle (A_\sigma^R - A_{\widetilde\sigma}^R)u, v \rangle_{-1,1} = \int_\Omega (\sigma - \widetilde\sigma) \nabla u \cdot \nabla v \, dx, \ \ u, v \in H^1(\Omega),$$

we get, similarly to the Neumann case,

$$\underline{\sigma}_0^2 \|R_\sigma - R_{\widetilde\sigma}\|_{-1,1} \le \|\sigma - \widetilde\sigma\|_{L^\infty(\Omega)} \le \underline{\sigma}_1^2 \|R_\sigma - R_{\widetilde\sigma}\|_{-1,1}.$$

4. Non-homogeneous BVPs

In this section, Ω is a C^2-smooth bounded domain of $\mathbb{R}^n$, $n \ge 2$, diffeomorphic to the unit ball of $\mathbb{R}^n$. Its boundary is denoted again by Γ.

Let $\sigma \in \Sigma$. For $g \in H^{\frac{1}{2}}(\Gamma)$, we denote by $u_\sigma \in H^1(\Omega)$ the unique weak solution to the BVP:

$$\text{div}(\sigma \nabla u) = 0 \text{ in } \Omega \text{ and } u = g \text{ on } \Gamma.$$

Let $G \in H^1(\Omega)$ so that $G = g$ on Γ and $\|G\|_{H^1(\Omega)} = \|g\|_{H^{\frac{1}{2}}(\Gamma)}$, where we identified $H^{\frac{1}{2}}(\Gamma)$ to the quotient space $H^1(\Omega)/H_0^1(\Omega)$. Then $f = -\text{div}(\sigma \nabla G) \in H^{-1}(\Omega)$ and

$$\|f\|_{H^{-1}(\Omega)} \le \sigma_1 \|G\|_{H^1(\Omega)} = \sigma_1 \|g\|_{H^{\frac{1}{2}}(\Gamma)}.$$

On the other hand, it is straightforward to check that $u_\sigma = G + R_\sigma f$, R_σ being the Dirichlet resolvent defined above. Therefore

$$\|u_\sigma\|_{H^1(\Omega)} \le \|G\|_{H^1(\Omega)} + \sigma_0^{-1} \|f\|_{H^{-1}(\Omega)} \le (1 + \sigma_0^{-1}\sigma_1)\|g\|_{H^{\frac{1}{2}}(\Gamma)}.$$

Then Λ_σ given by $\Lambda_\sigma g := u_\sigma$ defines a bounded operator from $H^{\frac{1}{2}}(\Gamma)$ into $H^1(\Omega)$ and

$$\|\Lambda_\sigma\|_{\frac{1}{2},1} \le 1 + \sigma_0^{-1}\sigma_1.$$

Here and in the sequel, $\|\cdot\|_{\frac{1}{2},1}$ denotes the norm in $\mathscr{B}(H^{\frac{1}{2}}(\Gamma), H^1(\Omega))$.

Fix $\overline{g} \in C^2(\Gamma)$ so that

$$\Gamma_- = \{x \in \Gamma; \ \overline{g}(x) = \min \overline{g}\}, \quad \Gamma_+ = \{x \in \Gamma; \ \overline{g}(x) = \max \overline{g}\}$$

are nonempty and connected, and the following condition is fulfilled: there exists a continuous strictly increasing function $\psi : [0, \infty) \to [0, \infty)$ with $\psi(0) = 0$ and $\rho_0 > 0$ so that, for any $0 < \rho \leq \rho_0$,

$$|\nabla_\tau \overline{g}| \geq \psi(\rho), \text{ on } \{x \in \Gamma; \text{ dist}(x, \Gamma_- \cup \Gamma_+) \geq \rho\},$$

where ∇_τ denotes the tangential gradient.

Such a function is called quantitatively unimodal in [1].

We point out that the existence of a quantitatively unimodal function is guaranteed by the assumption that Ω is diffeomorphic to the unit ball.

For $\sigma_1 > \sigma_0$, define

$$\mathcal{E} = \{\sigma \in W^{1,\infty}(\Omega); \; \sigma_0 \leq \sigma \text{ and } \|\sigma\|_{W^{1,\infty}(\Omega)} \leq \sigma_1\}.$$

Theorem 4.1. ([1, Theorem 3.5]) *There exist two constants $C > 0$ and $\gamma > 0$, that can depend on Ω, $\mathcal{E}$ and $\overline{g}$, so that*

$$\|\sigma - \widetilde{\sigma}\|_{L^\infty(\Omega)} \leq C \|\Lambda_\sigma \overline{g} - \Lambda_{\widetilde{\sigma}} \overline{g}\|_{L^2(\Omega)}^\gamma, \;\; \sigma, \widetilde{\sigma} \in \mathcal{E}_0,$$

where $\mathcal{E}_0 = \{\sigma \in \mathcal{E}; \; \sigma = \overline{\sigma} \text{ on } \Gamma\}$, for some fixed $\overline{\sigma} \in \mathcal{E}$.

This result is essential in the stability issue of the problem of determining the conductivity coefficient from two attenuated energy densities obtained from well chosen two illuminations. This problem is related to the so-called qualitative photo-acoustic tomography. We refer to [1] and the references therein for more details on this topic.

As a consequence of Theorem 4.1, we readily obtain:

Corollary 4.1. *There exist two constants $C > 0$ and $\gamma > 0$, that can depend on Ω and $\mathcal{E}$, so that*

$$\|\sigma - \widetilde{\sigma}\|_{L^\infty(\Omega)} \leq C \|\Lambda_\sigma - \Lambda_{\widetilde{\sigma}}\|_{\frac{1}{2},1}^\gamma, \;\; \sigma, \widetilde{\sigma} \in \mathcal{E}_0,$$

where $\mathcal{E}_0$ is as in the preceding theorem.

This result can be interpreted as a Hölder stability estimate on the determination of σ from Λ_σ.

Remark 4.1. Denote the lifting operator $g \to G$, defined above, by E and, for $\sigma \in \mathcal{E}$, consider the operator L_σ given by

$$L_\sigma : F \in H^1(\Omega) \mapsto L_\sigma F = \text{div}(\sigma \nabla F) \in H^{-1}(\Omega).$$

Then one can check in a straightforward manner that $\Lambda_\sigma = E + R_\sigma L_\sigma E$. Therefore, the mapping

$$\sigma \in \mathcal{E} \mapsto \Lambda_\sigma \in \mathscr{B}(H^{\frac{1}{2}}(\Gamma), H^1(\Omega))$$

is Lipschitz continuous. Whence, with reference to Corollary 4.1, we get, for some constants $c > 0$ and $C > 0$,

$$\|\sigma - \widetilde{\sigma}\|_{L^\infty(\Omega)} \leq C c^\gamma \|\sigma - \widetilde{\sigma}\|_{L^\infty(\Omega)}^\gamma, \;\; \sigma, \widetilde{\sigma} \in \mathcal{E}_0,$$

or equivalently

$$c \|\sigma - \widetilde{\sigma}\|_{L^\infty(\Omega)} \leq C c \left(c \|\sigma - \widetilde{\sigma}\|_{L^\infty(\Omega)}\right)^\gamma, \;\; \sigma, \widetilde{\sigma} \in \mathcal{E}_0.$$

As $\|\sigma - \widetilde{\sigma}\|_{L^\infty(\Omega)}$ can be chosen arbitrarily small, we conclude that $\gamma \leq 1$.

We do not know whether, actually, $\gamma < 1$ or not. Explicit computations can be carried out in the one-dimensional case, very much as in the next section. But Theorem 4.1, which genuinely of multi-dimensional nature, fails in this case since two different diffusivities, one multiple of the other, cannot be distinguished from boundary values.

5. The one-dimensional case

5.1. An explicit representation formula

For the sake of simplicity, we limit our analysis to a BVP with mixed boundary conditions. Specifically, we consider the BVP

$$-(\sigma(x)u_x)_x = f \text{ in } (0, 1), \; u_x(0) = 0 \text{ and } u(1) = 0. \tag{5.1}$$

Let $\sigma_0 < \sigma_1$ be two positive constants and

$$\Sigma^0 = \{\sigma \in L^\infty(0,1);\ 0 < \sigma_0 \le \sigma \le \sigma_1 \text{ a.e. in } (0,1)\}$$

and

$$H = \{u \in H^1(0,1);\ u(1) = 0\}.$$

It is a classical result that $u \in H \to \|u_x\|_{L^2(0,1)}$ defines a norm on H, which is equivalent to the norm induced by the usual norm on $H^1(0,1)$. In the sequel, H is equipped with this norm.

According to Lax–Milgram's lemma or Riesz's representation theorem, for each $f \in H'$, there exists a unique $u = u_\sigma \in H$ so that

$$\int_0^1 \sigma(x) u_x(x) v_x(x)\, dx = \langle f, v \rangle, \quad \text{for any } v \in H,$$

where $\langle \cdot, \cdot \rangle$ is the duality pairing between H and its dual H'. Note that u_σ is nothing but the variational solution to the BVP (5.1).

Therefore $R_\sigma : f \in H' \to u_\sigma \in H$ defines a bounded operator with

$$\|R_\sigma f\|_H \le \sigma_0^{-1} \|f\|_{H'}.$$

Pick $f \in L^2(0,1)$ and set

$$v(x) = \int_x^1 \frac{1}{\sigma(t)} \int_0^t f(s)\, ds\, dt, \quad x \in [0,1].$$

Clearly v is absolutely continuous, $v(1) = 0$ and

$$v_x(x) = -\frac{1}{\sigma(x)} \int_0^x f(t)\, dt \ \text{ a.e. } (0,1). \tag{5.2}$$

On the other hand, if $w \in H$, we get by applying Green's formula

$$\int_0^1 \sigma(x) v_x(x) w_x(x) = -\int_0^1 w_x(x) \left(\int_0^x f(t)\, dt \right) dx = \int_0^1 w(x) f(x)\, dx.$$

In other words, $v = R_\sigma f$.

5.2. Lipschitz stability

In view of the explicit representation formula above it is convenient to introduce the space $W^{-1,1}(0,1)$, the closure of $C_0^\infty(0,1)$ for the norm

$$\|f\|_{W^{-1,1}(0,1)} = \left\| \int_0^x f(t)\, dt \right\|_{L^1(0,1)}.$$

The resolvent operators, according to the explicit representation formula above, can be represented as

$$T_m f(x) = m(x) \int_0^x f(t)\, dt, \quad \text{a.e. } x \in (0,1), \tag{5.3}$$

where

$$m = \frac{1}{\sigma}.$$

Given $m \in L^\infty(0,1)$, these operators can be naturally understood in the functional setting of linear bounded operators, $T_m : W^{-1,1}(0,1) \to L^1(0,1)$. We denote by $\mathscr{B}(W^{-1,1}(0,1), L^1(0,1))$ this Banach space and by $\|\cdot\|_{-1,1}$ its norm.

The following holds:

Lemma 5.1. *Let* $m \in L^\infty(0,1)$ *and* $T_m : W^{-1,1}(0,1) \to L^1(0,1)$ *as in* (5.3). *Then*

$$\|T_m\|_{-1,1} = \|m\|_{L^\infty(0,1)}. \tag{5.4}$$

Proof. Firstly, it is straightforward to check that

$$\|T_m\|_{-1,1} \leq \|m\|_{L^\infty((0,1))}.$$

The reverse inequality can be easily derived as in Lemma 2.2, by taking a sequence f_ε so that the corresponding primitives

$$F_\varepsilon = \int_0^x f_\varepsilon(t)\, dt,$$

constitute an approximation of the identity around each $x_0 \in (0,1)$. □

Given two diffusivity coefficients $\sigma, \widetilde{\sigma} \in \Sigma^0$, formula (5.2) yields

$$(R_\sigma f - R_{\widetilde{\sigma}} f)_x = \left(\frac{1}{\widetilde{\sigma}(x)} - \frac{1}{\sigma(x)} \right) \int_0^x f(t)\, dt \quad \text{a.e. } (0,1).$$

In view of Lemma 5.1, it is natural to analyze the norms of these resolvent operators and their distances in the norm

$$\|R_\sigma\|_* = \|T_{1/\sigma}\|_{-1,1}.$$

As a consequence of Lemma 5.1,

$$\|R_\sigma - R_{\widetilde{\sigma}}\|_* = \left\| \frac{1}{\widetilde{\sigma}(x)} - \frac{1}{\sigma(x)} \right\|_{L^\infty(0,1)}.$$

Obviously, using the uniform upper and lower bounds on the coefficients, this also allows us to get estimates in terms of the $L^\infty(0,1)$-distances between coefficients:

$$\sigma_1^{-2} \|\widetilde{\sigma} - \sigma\|_{L^\infty(0,1)} \leq \|R_\sigma - R_{\widetilde{\sigma}}\|_* \leq \sigma_0^{-2} \|\widetilde{\sigma} - \sigma\|_{L^\infty(0,1)}. \tag{5.5}$$

This is so because

$$\frac{\sigma - \widetilde{\sigma}}{\sigma \widetilde{\sigma}} = \frac{1}{\widetilde{\sigma}} - \frac{1}{\sigma}.$$

5.3. Distance to a subspace

As we shall see in the following section, in the application of greedy algorithms, we need to further develop the computations above to achieve precise Lipschitz stability estimates on the distance from one given resolvent to the subspace generated by a finite number of others.

To do this, we consider a distinguished coefficient that we denote by $\tau(x)$ and $N \geq 2$ others, $\sigma_1(x), \cdots, \sigma_N(x)$, and denote the corresponding resolvents by R_τ and $R_1, \cdots, R_N$, respectively.

As a straightforward application of identity (5.2), we have

$$\left(R_\tau f - \sum_{i=1}^N a_i R_i f \right)_x = \left(\sum_{i=1}^N \frac{a_i}{\sigma_i(x)} - \frac{1}{\tau(x)} \right) \int_0^x f(t)\, dt \quad \text{a.e. } (0,1) \tag{5.6}$$

which yields the representation of the difference of a resolvent with respect to the linear combination of a finite number of others.

Arguing as above, we can conclude that

$$\left\| R_\tau - \sum_{i=1}^N a_i R_i \right\|_* = \left\| \sum_{i=1}^N \frac{a_i}{\sigma_i(x)} - \frac{1}{\tau(x)} \right\|_{L^\infty((0,1))}. \tag{5.7}$$

In other words, we have shown that the L^∞-distance between inverses of coefficients yields an adequate surrogate for the distance between the resolvents:

$$\text{dist}_*(R_\tau, \text{span}[R_i, 1 \le i \le N]) = \text{dist}_{L^\infty(0,1)}\left(\frac{1}{\tau(x)}, \text{span}\left[\frac{1}{\sigma_i(x)}, 1 \le i \le N\right]\right). \tag{5.8}$$

Here the distance dist_* stands for the one given in terms of the $\|\cdot\|_*$-norm.

Note that the methods of the previous sections do not allow us to achieve similar results in the multi-dimensional case. In particular, the analysis of the one-dimensional case shows that when dealing with the distance between a resolvent to the span of several others, one has to analyze nonlinear expressions involving the diffusivity coefficients.

6. Application to greedy algorithms for parameter depending elliptic equations

In recent years, there has been a significant body of literature developed on greedy methods to approximate parameter-dependent elliptic problems of the form

$$-\text{div}(\sigma(x, \mu)\nabla u) = f \text{ in } \Omega, \quad u = 0 \text{ on } \Gamma. \tag{6.1}$$

We refer for instance to [2,3,5,6,8,9].

Roughly, the problem has been formulated and addressed as follows.

Assume that $\sigma(x, \mu) \in \Sigma$ (with Σ as in previous sections) depends on a multi-parameter μ living on a compact set $\mathcal{K}$ of $\mathbb{R}^d$, with $d \ge 1$ finite. We denote by $\mathscr{S}$ the parametrized set of coefficients $\sigma(x, \mu) \in \Sigma$ for all values of μ.

Given a fixed $f \in H^{-1}(\Omega)$ and solving (6.1) we get the set $\mathscr{U}$ of the corresponding solutions $u(x, \mu) \in H_0^1(\Omega)$, $\mu \in \mathcal{K}$. This set inherits the regularity of the coefficients $\sigma(x, \mu) \in \Sigma$ in its dependence with respect to μ.

The question that has been considered so far consists in identifying the most distinguished values of the parameter μ to better approximate the set of solutions $\mathscr{U}$, for that specific given right-hand-side term $f \in H^{-1}(\Omega)$. This has been done applying (weak) greedy algorithms obtaining optimal approximation rates for $\mathscr{U}$. But, proceeding that way, the sequence of most relevant snapshots μ_n that the algorithm gives depends on the right-hand-side term $f \in H^{-1}(\Omega)$ and different right-hand-side terms f lead to different choices of the snapshots of μ.

Theorem 1.1 was developed in an attempt to apply the same methods independently of the specific value of the right-hand-side term $f \in H^{-1}(\Omega)$. However, this program can only be achieved so far in the one-dimensional case where our analysis was much more complete.

For this to be done, one needs to deal with the set $\mathscr{R}$ of resolvent operators $R(\mu)$ in $\mathscr{B}(H^{-1}(\Omega), H_0^1(\Omega))$ for all $\mu \in \mathcal{K}$ that inherit the continuity and regularity properties of the coefficients $\sigma(x, \mu) \in \Sigma$ in its dependence with respect to μ. For instance, if the map $\mu \in \mathcal{K} \to \sigma(x, \mu) \in L^\infty(\Omega)$ is continuous, the same occurs for the map $\mu \in \mathcal{K} \to R(\mu) \in \mathscr{B}(H^{-1}(\Omega), H_0^1(\Omega))$. The same can be said about the C^k, C^∞ or analytic dependence. On the other hand, the compactness of the set $\mathcal{K}$ together with the continuous dependence on μ ensures the compactness of $\mathscr{R}$ in $\mathscr{B}(H^{-1}(\Omega), H_0^1(\Omega))$.

The goal is then to approximate the compact set of resolvents $\mathscr{R}$ of the Banach space $\mathscr{B}(H^{-1}(\Omega), H_0^1(\Omega))$ by a sequence of finite-dimensional subspaces V_n of dimension $n \ge 1$. The weak greedy algorithms yield the subspaces V_n that approximate the set $\mathscr{R}$ in the best possible manner, in the sense of the Kolmogorov n-width. The subspaces V_n are defined as the span of the most distinguished resolvent operators $R(\mu_1), \ldots, R(\mu_n)$ with $\mu_1, ..., \mu_n$ chosen as follows.

Fix a constant $\gamma \in (0, 1)$. Choose $\mu_1 \in \mathcal{K}$ such that

$$||R(\mu_1)||_{-1,1} \ge \gamma \max_{\mu \in \mathcal{K}} ||R(\mu)||_{-1,1}. \tag{6.2}$$

We then proceed in a recursive manner. Having found $\mu_1, \ldots, \mu_n$, denote $V_n = \text{span}\{R(\mu_1), \ldots, R(\mu_n)\}$ and choose the next element μ_{n+1} such that

$$\text{dist}(R(\mu_{n+1}), V_n) \ge \gamma \max_{\mu \in \mathcal{K}} \text{dist}(R(\mu), V_n). \tag{6.3}$$

From the previous existing theory (see [9] and [7]) this algorithm is well known to yield nearly optimal approximation rates of the parameterized set of resolvents $\mathscr{R}$. As indicated in the previous references because, now, we are in a Banach space setting, $\mathscr{B}(H^{-1}(\Omega), H_0^1(\Omega))$, a loss of the order $1/2$ on the approximation rate of the Kolmogorov width has to be expected.

However, the difficulty on its implementation consists in computing, for each step, the distance $\text{dist}(R(\mu), V_n)$. This would require, in particular, computing $R(\mu)$ for all values of μ, and this is unfeasible and, precisely, what we want to avoid.

But the existing theory has also developed a means of bypassing this difficulty. In fact, it is well known that the same algorithm yields optimal approximation rates if it is implemented, as indicated above, but replacing $\text{dist}(R(\mu), V_n)$ by a "surrogate", i.e. a different distance function, easier to be computed, and giving a uniform bound from below for the true distance $\text{dist}(R(\mu), V_n)$.

Theorem 1.1 is a first attempt in that direction, ensuring that the L^1-distance between two coefficients provides a lower bound on the distance between the resolvents. But the issue of finding true surrogates for the distance of a resolvent to the subspace generated by a finite number of others is open in the general multi-dimensional case. The results in Subsection 5.3 yield such a surrogate in dimension $n = 1$, see (5.8).

Accordingly, in one space dimension, $n = 1$, the implementation of the weak greedy algorithm for the approximation of the parameter set of resolvents can be done as above, by modifying the recursive step as follows: fix some $0 < \gamma < 1$. Having found $\mu_1, \ldots, \mu_n$, with the corresponding diffusivity coefficients $\sigma_1(x), \ldots, \sigma_N(x)$, to choose the next element μ_{n+1} such that the corresponding diffusivity coefficient σ_{N+1} satisfies

$$\text{dist}_{L^\infty(0,1)} \left(\frac{1}{\sigma_{N+1}}, \text{span}\left[\frac{1}{\sigma_i}; i, \ldots, N \right] \right) \geq \gamma \max_{\mu \in \mathscr{K}} \text{dist}_{L^\infty(0,1)} \left(\frac{1}{\sigma(\mu)}, \text{span}\left[\frac{1}{\sigma_i}; i = 1, \ldots, N \right] \right). \tag{6.4}$$

The important consequence of this fact is that, for the identification of the most relevant parameter values μ_n, we do not need to solve the elliptic equation, but simply deal with the family of coefficients $\sigma(x, \mu)$, solving a by now classical L^∞-minimization problem in an approximated manner as indicated in (6.4) by a multiplicative factor $(0 < \gamma < 1)$. Once this choice of μ_n is done, this readily allows identifying the most relevant resolvent or elliptic problem, for all values of the right-hand-side term $f \in H^{-1}(\Omega)$, contrarily to previous developments where the choice of these snapshots was f-dependent.

The choice of the parameters μ_n that we achieve in this manner is optimal from the point of view of the approximation of the resolvents and can then be applied to any $f \in H^{-1}(\Omega)$, as mentioned above. But, of course, for a specific value of $f \in H^{-1}(\Omega)$, the ad-hoc application of the weak greedy method will lead to better approximations. But for this to be done one has to afford the cost of implementing the weak greedy method for each $f \in H^{-1}(\Omega)$ again and again. The advantage of the method developed in this paper is that it leads to uniform, robust approximation results, valid for all $f \in H^{-1}(\Omega)$ and can be used as a preconditioner to later use further greedy arguments, adapted to each right hand side term f.

In practice, given an arbitrary value of μ, the resolvent can be identified with the corresponding multiplier $1/\sigma(\mu)$. Therefore, it can be approximated by a suitable linear combination of the weak-greedy offline choices $1/\sigma_i$. This gives an easy and computationally inexpensive way of approximating the resolvent associated with μ.

Note, however, that this program was only fully developed in the one-dimensional case, since the multi-dimensional analogue of the surrogate in terms of the coefficients is not known. In the next section, we address, in the multi-dimensional context, the simpler problem of variables density functions.

Remark 6.1. It is worthwhile mentioning that the program we carried out for the conductivity coefficient in the one-dimensional case can be extended to the problem of identifying the density coefficient from the corresponding resolvent in an arbitrary dimension.

For sake of simplicity we assume in this remark that Ω is $C^{1,1}$-smooth.

Consider, for $f \in L^2(\Omega)$ and $\rho \in L^\infty(\Omega)$, the problem of finding $u \in H_0^1(\Omega)$ so that

$$-\Delta u = \rho f \ \text{in} \ \Omega.$$

The corresponding variational formulation consists in seeking $u \in H_0^1(\Omega)$ satisfying:

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega \rho f v \, dx, \ \ v \in H_0^1(\Omega). \tag{6.5}$$

By Lax–Milgram lemma, this problem has a unique solution $u_\rho := R_\rho f$.

As Ω is $C^{1,1}$-smooth, R_ρ maps $L^2(\Omega)$ into $\mathcal{H} = H_0^1(\Omega) \cap H^2(\Omega)$. When $\rho \equiv 1$, we denote R_ρ by R. Then R is nothing but the inverse of the bounded operator $A : \mathcal{H} \to L^2(\Omega)$ given by $Au = -\Delta u$.

Using once again Lebesgue's differentiation theorem, we can prove that M_ρ, the multiplication operator by $\rho \in L^\infty(\Omega)$, acting as an operator on $L^2(\Omega)$, satisfies $\|M_\rho\| := \|M_\rho\|_{\mathscr{B}(L^2(\Omega))} = \|\rho\|_{L^\infty(\Omega)}$.

Since $R_\rho = R M_\rho$ or equivalently $M_\rho = A R_\rho$, we derive

$$\|R\|^{-1} \|R_\rho\| \leq \|\rho\|_{L^\infty(\Omega)} \leq \|A\| \|R_\rho\|.$$

Fix $\rho_1, \ldots, \rho_N \in L^\infty(\Omega)$, and let $\rho \in V_N = \text{span}\{\rho_1, \ldots \rho_N\}$ and $\widetilde{\rho} \in L^\infty(\Omega)$. In light of the linearity of the mapping $\rho \to R_\rho$ we get

$$\|R\|^{-1} \|R_\rho - R_{\widetilde{\rho}}\| \leq \|\rho - \widetilde{\rho}\|_{L^\infty(\Omega)} \leq \|A\| \|R_\rho - R_{\widetilde{\rho}}\|.$$

Accordingly

$$\mathbf{d}(R_{\widetilde{\rho}}, \mathbf{R}_N) = \text{dist}_{L^\infty(\Omega)}(\widetilde{\rho}, V_N),$$

where $\mathbf{R}_N = \text{span}\{R_{\rho_1}, \ldots, R_{\rho_N}\}$ and

$$\mathbf{d}(R_\rho, R_{\widetilde{\rho}}) = \|\rho - \widetilde{\rho}\|_{L^\infty(\Omega)}$$

yields an appropriate surrogate between resolvents.

This allows the full application of the weak greedy algorithm described in the previous section in this case of multi-dimensional density-dependent elliptic equations.

7. Extensions and further comments

The results of this paper constitute a first contribution on a topic that is rich in open interesting problems. We mention here some of them:

- *Surrogates in the multi-dimensional case.* In Subsection 5.3, we have found surrogates for the elliptic problem with variable diffusivity in dimension $n = 1$. This problem is totally open in the multi-dimensional case. The case of the variable density was solved in the previous section.
- *Elliptic matrices.* In dimensions $n \geq 2$ the same problems can be formulated in the context of elliptic problems involving coefficients $\sigma_{ij}(x, \mu)$, i.e. to equations of the form

$$-\sum_j \partial_j (\sigma_{ij}(x, \mu) \partial_i u) = f.$$

Of course, the problem is much more complex in this case, since there is not only one coefficient σ to be identified, but rather all the family σ_{ij} with $i, j = 1, ..., N$.
- *Elliptic systems.* The same problems arise also in the context of elliptic systems, such as, for instance, the system of elasticity.
- *Evolution equations.* The problems addressed in the present paper make also sense for evolution problems and, in particular, parabolic, hyperbolic and Schrödinger equations.
  Let us consider for instance the heat equation:

$$\begin{cases} u_t - \operatorname{div}(\sigma \nabla u) = 0 \text{ in } \Omega \times (0, \infty) \\ \quad u = 0 \text{ on } \Gamma \times (0, \infty) \\ \quad u(x, 0) = f(x) \text{ in } \Omega. \end{cases} \tag{7.1}$$

The same questions we have addressed in the elliptic context arise also in the parabolic one. In this case, the question can be formulated as follows: *Does the resolvent $f \in L^2(\Omega) \to C([0, \infty); L^2(\Omega))$ determine the diffusivity coefficient in an unique manner? Is the map from resolvent to diffusion coefficient Lipschitz in suitable norms?*
The way the question has been formulated is easy to solve. In fact it is sufficient to observe that the elliptic equation is subordinated to the parabolic one, so that, the time integral of the parabolic solution, namely,

$$v(x) = \int_0^\infty u(x, t)\, dt,$$

solves the elliptic equation

$$\begin{cases} -\operatorname{div}(\sigma \nabla v) = f \text{ in } \Omega \\ \quad v = 0 \text{ on } \Gamma. \end{cases} \tag{7.2}$$

This can be easily seen integrating the parabolic equation in time and using the fact that the solutions to the heat equation tend to zero as $t \to \infty$.
Therefore, a full knowledge of the parabolic resolvent yields, in particular, fully, the elliptic resolvent as well. According to the previous results in this paper, the diffusivity coefficient can be determined, and the dependence is Lipschitz. Our comments on the explicit representation of solutions in $1 - d$ and their possible use for the development of greedy algorithms apply as well.
This simple observation however raises many other interesting problems: can the same Lipschitz identification result be achieved if the parabolic solution is only known in a finite time interval $[0, T]$? What about diffusivity coefficients depending also on time $\sigma = \sigma(x, t)$? What happens when, rather than the solution for the initial value problem, one considers those with non-homogeneous source terms?
Similar questions arise for wave-like equations and the same answer can be expected if the models under consideration are dissipative. This allows us to integrate the equations under consideration in the time interval $(0, \infty)$ and employing the decay of solutions as $t \to \infty$. Of course, the expected behavior is completely different in the absence of damping.
- *Control problems.* Greedy and weak greedy methods have been implemented in the context of controllability of finite and infinite-dimensional Ordinary Differential Equations (ODEs) in [10]. But this has been done for fixed specific data to be controlled. It would be interesting to analyze whether the results of this paper can be extended to these controllability problems so as to achieve approximations, independent of the data to be controlled. For this to be done, one would need to adapt the Lipschitz stability estimate in (1.3) to control problems, getting upper bounds on the distance between the generators of the semigroups in terms of the distance between the corresponding control maps. This is an issue that needs significant further investigation.

Acknowledgements

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