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Horofunctions on graphs of linear growth



Horofonctions sur les graphes à croissance linéaire

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ABSTRACT

We prove that a linear growth graph has finitely many horofunctions. This provides a short and simple proof that any finitely generated infinite group of linear growth is virtually cyclic.

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R É S U M É

Nous montrons qu'un graphe à croissance linéaire admet un nombre fini d'horofonctions. Cela donne une preuve courte et simple que chaque groupe infini de type fini à croissance linéaire est virtuellement cyclique.

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1. Introduction

Let Γ be an infinite, connected, locally finite graph, and denote by $d(\cdot, \cdot)$ the graph metric on Γ . Let $o \in \Gamma$. Then Γ is said to have *linear volume growth* if the balls about o for the metric d grow at most linearly in the radius. The graph Γ is said to have *polynomial volume growth* if these balls grow at most polynomially.

Given an element $z \in \Gamma$ we define the *Busemann function* $b_z : \Gamma \rightarrow \mathbb{Z}$ via $b_z(y) = d(z, y) - d(z, o)$. Given a geodesic ray $\omega = (z_1, z_2, \dots)$ in Γ we define the *horofunction* $f_\omega : G \rightarrow \mathbb{Z}$ by $f_\omega(y) = \lim_{n \rightarrow \infty} b_{z_n}(y)$. It is a well-known and simple fact that this limit exists. Note that f_ω is not constant, and in fact $f_\omega(z_n) = -n$, which shows that f_ω is unbounded.

Theorem 1.1. *Let Γ be an infinite, connected, locally finite graph of linear volume growth. Then the set of horofunctions on Γ is finite.*

A finitely generated group G is said to have polynomial (respectively, linear) volume growth if some (and hence every) Cayley graph of G has polynomial (respectively, linear) volume growth. A remarkable theorem of Gromov's states that a finitely generated group of polynomial volume growth contains a nilpotent subgroup of finite index [1]. As an application of Theorem 1.1, we give a short argument to prove the linear-growth case of Gromov's theorem, as follows.

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Theorem 1.2. *Let G be a finitely generated infinite group of linear volume growth. Then G contains a cyclic subgroup of finite index.*

In fact, Gromov's theorem implies that a group of subquadratic growth is virtually cyclic, and this has also been proved by elementary methods by Justin [4], van den Dries & Wilkie [6], and Imrich & Seifter [3], the last two of these giving bounds on the index of the cyclic subgroup in terms of the volume growth. Nonetheless, the present proof is completely different to all of those and rather short, so we record it here.

We prove [Theorem 1.1](#) in [Section 2](#). The proof of [Theorem 1.2](#) is in [Section 3](#).

Let us mention a related (probably much more difficult) question.

Conjecture 1.3. *Let G be a Cayley graph of polynomial volume growth. Then the set of horofunctions on G is countable.*

Proving this conjecture would provide an alternative proof to Gromov's theorem (by using a variant of [Lemma 3.1](#) below). This has been suggested by Karlsson [5]. One method to prove this conjecture could be using the structure of finitely generated nilpotent groups, and relying on Gromov's theorem, but that would somehow miss the point. (For example, in [7] Walsh shows that nilpotent groups always have a finite orbit in the space of horofunctions. It seems that this can be extended to virtually nilpotent groups as well.) It would be interesting to prove this conjecture even in the quadratic growth case without using Gromov's theorem, since that would imply a new proof of the characterization of recurrent groups (which are finite extensions of \mathbb{Z} or \mathbb{Z}^2).

2. Horofunctions on a graph of linear growth

For a graph Γ , we say that Γ is \mathbb{N} -partite on a sequence $(\Gamma_n)_{n \in \mathbb{N}}$ of disjoint sets Γ_n if Γ has vertex set $\bigcup_{n \in \mathbb{N}} \Gamma_n$ and the neighbours of every $x \in \Gamma_n$ lie in $\Gamma_{n-1} \cup \Gamma_{n+1}$. We call the sets Γ_n the *partite sets* of Γ . We call a path in Γ *monotone* if it has at most one vertex in each Γ_n .

The proof of [Theorem 1.1](#) essentially rests on the following graph-theoretic result.

Proposition 2.1. *Let Γ be an \mathbb{N} -partite graph whose partite sets all have cardinality $k \geq 1$. Then there exist monotone paths $\gamma_1, \dots, \gamma_k$ in Γ such that every infinite monotone path in Γ has infinite intersection with some γ_j .*

Recall that in a bipartite graph on two finite sets X_1, X_2 of equal cardinality, a *matching* is a subgraph in which each element of X_1 is connected to precisely one element of X_2 , and vice versa. Hall's Marriage Theorem [2] states that if there is no matching then there exists some subset $Y \subset X_1$ such that the neighbourhood of Y in X_2 has strictly smaller cardinality than Y itself.

Lemma 2.2. *If for each $n \in \mathbb{N}$ there is a matching in Γ between Γ_n and Γ_{n+1} then Γ satisfies [Proposition 2.1](#).*

Proof. It is easy to see that the existence of such matchings implies that the vertices of Γ may be partitioned into k monotone paths, and that this is sufficient to satisfy the proposition. \square

Given an \mathbb{N} -partite graph Γ and a sequence $N = (n_j)_j \subset \mathbb{N}$, we may define a new graph on $\Gamma_N := \bigcup_j \Gamma_{n_j}$ by placing an edge between $x \in \Gamma_{n_j}$ and $x' \in \Gamma_{n_{j+1}}$ if and only if there exists a monotone path between x and x' in Γ . (Note that Γ_N is an \mathbb{N} -partite graph with partite sets Γ_{n_j} .) The following is then immediate.

Lemma 2.3. *If there exists a sequence $N = (n_j)_j$ such that Γ_N satisfies [Proposition 2.1](#), then the conclusion of [Proposition 2.1](#) holds for Γ as well.*

Proof of [Proposition 2.1](#). We proceed by induction on k . The base case $k = 1$ is easy, so we assume that $k > 1$. We may also delete every element of Γ that does not lie in any infinite monotone path; the only potential problem with this is that the Γ_n may no longer all have the same cardinality, but, using [Lemma 2.3](#), we may fix this by passing to a subsequence.

Let $N = (n_j)_j$ be a sequence and consider the \mathbb{N} -partite graph Γ_N . If for every j there exists a matching (in the graph Γ_N) between Γ_{n_j} and $\Gamma_{n_{j+1}}$, then we are done by combining [Lemmas 2.2 and 2.3](#).

Thus, we assume that a sequence as above does not exist. Specifically, by Hall's Marriage Theorem, there exists n such that for any $m > n$, there exist $U_m \subset \Gamma_n$ and $V_m \subset \Gamma_m$ such that $1 \leq |V_m| < |U_m| \leq k$, and such that every monotone path from U_m to Γ_m ends in V_m . Without loss of generality we assume that the sets V_m are minimal with respect to these properties, and hence that every element $v \in V_m$ lies in some monotone path from U_m to Γ_m . By passing to a subsequence of $m > n$, we have an infinite sequence $n < m_1 < m_2 < \dots$ such that $|V_{m_j}| = |V_{m_1}|$ all have the same size and $U_{m_j} = U_{m_1} = U$ are all the same fixed subset.

Let $M = (m_j)_j$ and consider the graph Γ_M . We claim that Γ_M satisfies [Proposition 2.1](#), which will suffice by [Lemma 2.3](#). We move to proving this claim.

Every monotone path in Γ starting in U and ending in Γ_{m_j} must end in V_{m_j} . Thus, by minimality of V_{m_j} , any monotone path in Γ starting in V_{m_j} and ending in $\Gamma_{m_{j+1}}$ must end in $V_{m_{j+1}}$. In particular, in the graph Γ_M , any infinite monotone path γ must satisfy the following dichotomy: either $\gamma \cap V_{m_j} = \emptyset$ for all j , or there exists j_0 such that for all $j > j_0$ we have $\gamma \cap V_{m_j} \neq \emptyset$.

Let Γ_A be the induced subgraph of Γ_M on the vertex set $\bigcup_j V_{m_j}$, and Γ_B the induced subgraph on the vertex set $\bigcup_j (\Gamma_{m_j} \setminus V_{m_j})$. Note that Γ_A is \mathbb{N} -partite with partite sets $(\Gamma_A)_j = V_{m_j}$, all of size $v = |V_{m_1}| < k$. Also, Γ_B is \mathbb{N} -partite with partite sets $(\Gamma_B)_j = \Gamma_{m_j} \setminus V_{m_j}$, which all have size $w = k - v < k$. Thus, any infinite monotone path in Γ_M induces an infinite monotone path in either Γ_A or in Γ_B . By induction, there exist infinite monotone paths $\alpha_1, \dots, \alpha_v$ in Γ_A such that any infinite monotone path in Γ_A must intersect one of these infinitely many times. Similarly, there are such paths β_1, \dots, β_w in Γ_B . Thus, any infinite monotone path in Γ_M must intersect one of $\alpha_1, \dots, \alpha_v, \beta_1, \dots, \beta_w$ infinitely many times. This completes the proof. \square

Proof of Theorem 1.1. Note that if Γ has linear growth then, writing B_r for the ball of radius r about o , there is some $k \in \mathbb{N}$ and an infinite increasing sequence m_1, m_2, \dots such that $|B_{m_n} \setminus B_{m_{n-1}}| = k$ for every n . Define $\Gamma_n = B_{m_n} \setminus B_{m_{n-1}}$, and define an \mathbb{N} -partite graph $\hat{\Gamma}$ on $\Gamma_1, \Gamma_2, \dots$ by joining $x \in \Gamma_n$ to $x' \in \Gamma_{n+1}$ if and only if there is a path in Γ from x to x' of length $m_{n+1} - m_n$.

Define a map α from the set of geodesic rays in Γ starting at o to the set of monotone paths in $\hat{\Gamma}$ (in the sense of Proposition 2.1) in the natural way. Specifically, if ω is a geodesic ray in Γ starting at o then $\alpha(\omega)$ is the unique monotone path in $\hat{\Gamma}$ passing through the same elements of $\bigcup_n \Gamma_n$ as ω . Note that α is surjective onto the set of monotone paths in $\hat{\Gamma}$, and also that if $\alpha(\omega)$ and $\alpha(\omega')$ have infinite intersection then so do ω and ω' .

Let $\gamma_1, \dots, \gamma_k$ be as given by Proposition 2.1, and pick, using the surjectivity of α , geodesic rays $\omega_1, \dots, \omega_k$ in Γ starting at o such that $\alpha(\omega_i) = \gamma_i$. If β is a geodesic ray in Γ , the tail of β coincides with the tail of some geodesic ray β' in Γ starting at o (see Lemma 2.4 below). However, $\alpha(\beta')$ has infinite intersection with some γ_i by Proposition 2.1, and so β has infinite intersection with ω_i . This implies in particular that $f_\beta = f_{\omega_i}$, and so $f_{\omega_1}, \dots, f_{\omega_k}$ is a complete set of horofunctions. \square

For completeness we include a short argument for the following standard lemma.

Lemma 2.4. *If $\gamma = (x_0, x_1, x_2, \dots)$ is a geodesic ray starting at x_0 then there exists some N such that (x_N, x_{N+1}, \dots) coincides with the tail of a geodesic ray ω starting at o .*

Proof. The sequence $d(x_n, o) - d(x_n, x_0)$ is non-increasing in n , since

$$d(x_{n+1}, x_0) = d(x_n, x_0) + 1 \tag{1}$$

and $|d(x_{n+1}, o) - d(x_n, o)| \leq 1$ for every n . The triangle inequality also implies that $d(x_n, o) - d(x_n, x_0)$ is bounded below by $-d(o, x_0)$. The sequence $(d(x_n, o) - d(x_n, x_0))_{n=1}^\infty$ is therefore eventually constant, say for $n \geq N$. Combined with (1), this implies that $d(x_{n+1}, o) = d(x_n, o) + 1$ for $n \geq N$. The infinite path ω having initial segment some geodesic path from o to x_N , followed by x_{N+1}, x_{N+2}, \dots , is therefore a geodesic ray starting at o . \square

3. The linear-growth case of Gromov's theorem

A group G acts on the space $\{f : G \rightarrow \mathbb{R} \mid f(1) = 0\}$ by $x \cdot f(y) = f(x^{-1}y) - f(x^{-1})$. Note that for a Busemann function b_z we have $x \cdot b_z = b_{xz}$, and hence for a horofunction f_ω we have $x \cdot f_\omega = f_{x\omega}$.

The following observation we learned from Anders Karlsson.

Lemma 3.1. *If the set of horofunctions on a group G contains a finite orbit then G has a finite-index subgroup admitting a surjective homomorphism onto \mathbb{Z} .*

Proof. Letting G act on the finite orbit, G contains a finite-index subgroup H that fixes some element f_ω of the orbit. Thus for $h \in H, g \in G$ we have $f_\omega(g) = h^{-1} \cdot f_\omega(g) = f_\omega(hg) - f_\omega(h)$, which implies that f_ω is a homomorphism $H \rightarrow \mathbb{Z}$ and that $f_\omega(Hg) = f_\omega(H) + f_\omega(g)$ for every $g \in G$. In particular, if $f_\omega(H) = \{0\}$ then f_ω is constant on the finitely many cosets of H , contradicting the fact that horofunctions are unbounded. We conclude that the image $f_\omega(H)$ is a non-trivial subgroup of \mathbb{Z} , and thus admits a surjective homomorphism onto \mathbb{Z} . \square

Remark 3.2. Essentially the same argument shows more generally that if Γ is a graph of linear growth and $G < \text{Aut}(\Gamma)$ acts transitively on the vertices of Γ then G has a finite-index subgroup admitting a surjective homomorphism onto \mathbb{Z} .

Proof of Theorem 1.2. Theorem 1.1 implies that G has a finite set of horofunctions. The set of horofunctions is invariant, so in this case it contains a finite orbit. Lemma 3.1 therefore implies that there exists $N \triangleleft G$ of finite index such that N

admits a surjective homomorphism onto \mathbb{Z} . Let $K \triangleleft N$ be the kernel of this homomorphism. Since N is finite index in G it is finitely generated of linear growth. Since $N/K \cong \mathbb{Z}$, it must be that K is finite. Hence, N is finite-by- \mathbb{Z} , which by standard methods implies that N is also \mathbb{Z} -by-finite. Thus, G contains a finite-index infinite cyclic subgroup. \square

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