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The infinite differentiability of the speed for excited random walks

La vitesse d'une marche aléatoire excitée est infiniment différentiable

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ABSTRACT

We prove that the speed of the excited random walk is infinitely differentiable with respect to the bias parameter in $(0, 1)$ for the dimension $d \geqslant 2$.

RÉSUMÉ

Nous montrons que la vitesse d'une marche aléatoire excitée sur $\mathbf{Z}^d$, $d \geqslant 2$, est infiniment différentiable par rapport au paramètre de biais dans $(0, 1)$.

1. Excited random walk

Excited random walk (ERW) was introduced in 2003 by I. Benjamini and D. Wilson [2]. This model is described as follows: an excited random walk (ERW) with bias parameter $\beta \in [0, 1]$ is a discrete time nearest-neighbor random walk $(Y_n)_{n\geqslant 0}$ on the lattice $\mathbb{Z}^d$ obeying the following rule: when the walk visits a site for the first time, it jumps with probability $(1+\beta)/2d$ to the right, with probability $(1-\beta)/2d$ to the left, and with probability $1/(2d)$ to the other nearest-neighbor sites. When the walk is at a visited site, it jumps uniformly at random to one of the $2d$ neighboring sites. Let $(e_i : 1 \leqslant i \leqslant d)$ denote the canonical generators of the group $\mathbb{Z}^d$. Denote by $\{Y_n \notin\}$ the event $[\#\{i \leqslant n : Y_i = Y_n\} = 1]$. Let $\mathcal{F}_n$ be the σ-algebra $\sigma(Y_0, Y_1, ..., Y_n)$ generated by the random walk up to time n. From the description above of the ERW, the law $\mathbb{P}_\beta$ of ERW, which is the probability on the path space $(\mathbb{Z}^d)^{\mathbb{N}}$, is defined by:

$$\mathbb{P}_\beta(Y_0 = 0) = 1,$$

$$\mathbb{P}_\beta[Y_{n+1} - Y_n = \pm e_i | \mathcal{F}_n] = \begin{cases} \frac{1}{2d} & \text{for } 2 \leqslant i \leqslant d, \\ \frac{1 \pm \beta 1_{Y_n \notin}}{2d} & \text{for } i = 1. \end{cases}$$

Denote by $\mathbb{E}_\beta$ the expectations respectively of the law of the ERW.

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In 2007, J. Bérard and A. Ramírez [3] proved a law of large numbers and a central limit theorem for the excited random walk with $d \geqslant 2$, namely:

- (Law of large numbers.) There exists $v = v(\beta, d)$, $0 < v < +\infty$ such that a.s.

$$\lim_{n\to\infty} n^{-1} Y_n \cdot e_1 = v.$$

- (Central limit theorem.) There exists $\sigma = \sigma(\beta, d)$, $0 < \sigma < +\infty$ such that $(n^{-1/2}(Y_{\lfloor nt\rfloor} \cdot e_1 - v\lfloor nt\rfloor), t \geqslant 0)$ converges in law as $n \to +\infty$ to a Brownian motion with variance σ^2.

A law of large numbers and a d-dimensional result of the central limit theorem for the excited random walk in random environment is also proved by another approach, see Theorem 1.2 of [7].

Our main result about regularity for the ERW is the following.

Theorem 1.1. *For $d \geqslant 2$, $\beta \in (0, 1)$, let $v(\beta)$ be the speed of the ERW then the speed $v(\beta)$ is infinitely differentiable on $(0, 1)$ i.e. $v(\beta) \in C^\infty(0, 1)$.*

Using the lace expansion technique, it is shown in [5], Theorem 2.3, that the speed is in an appropriate sense continuous in the drift parameter β if $d \geqslant 6$ and even differentiable if $d \geqslant 8$. Using cut times as in [4], we also proved the differentiability of the speed of the ERW for $d \geqslant 6$ (see [8]). Actually, for 1-dimensional multi-excited random walks (including RWRE), the continuity of the speed was considered in [9,1]. In our paper, we prove that the speed is infinitely differentiable on $(0, 1)$ for all $d \geqslant 2$ using renewal times and Girsanov's transform.

2. Idea of the proof of Theorem 1.1

2.1. The renewal structure

We define the renewal times for an ERW. Let $\{Y_n\}_{n\geqslant 0}$ be an ERW on $\mathbb{Z}^d$.

Definition 2.1. We present the definition based on the definition given in [3] and [7]. With the convention that $\inf\{\emptyset\} = \infty$, all random times in the Definition take values on $[0, +\infty]$. For every $u > 0$ let:

$$T_u = \min\{k \geqslant 1 : Y_k \cdot e_1 \geqslant u\}.$$

Define

$$\overline{D} = \inf\{m \geqslant 0 : Y_m \cdot e_1 < Y_0 \cdot e_1\}.$$

For ERW, it has been proved in [3] that $\mathbb{P}_\beta(\overline{D} = \infty) > 0$. Therefore, we can define the conditional probability $\hat{\mathbb{P}}_\beta(\cdot) = \mathbb{P}_\beta(\cdot|\overline{D} = \infty)$. Let $\hat{\mathbb{E}}_\beta$ be the expectation with respect to $\hat{\mathbb{P}}_\beta$. Furthermore, define two sequences of $\mathcal{F}^Y_n$-stopping times $\{S_n : n \geqslant 0\}$ and $\{D_n : n \geqslant 0\}$ as follows: we let $S_0 = 0$, $R_0 = Y_0 \cdot e_1$ and $D_0 = 0$. Next, define by induction on $k \geqslant 0$

$$\begin{aligned} S_{k+1} &= T_{R_k+1} \\ D_{k+1} &= \overline{D} \circ \theta_{S_{k+1}} + S_{k+1} \\ R_{k+1} &= \sup\{Y_i \cdot e_1 : 0 \leqslant i \leqslant D_{k+1}\}, \end{aligned}$$

where θ is the canonical shift on the space of trajectories. Let

$$\kappa = \inf\{n \geqslant 0 : S_n < \infty, D_n = \infty\}.$$

We define the first renewal time as follows:

$$\tau_1 = S_\kappa.$$

We then define, by induction on $n \geqslant 1$, the sequence of renewal times $\tau_1, \tau_2, \ldots$ as follows:

$$\tau_{n+1} = \tau_n + \tau_1(Y_{\tau_n+\cdot}).$$

Next, we define $D^{(0)}_i = D_i$ and $S^{(0)}_i = S_i$ and for every $k \geqslant 1$ two sequences $D^{(k)}_i$ and $S^{(k)}_i$ w.r.t. the trajectory $(Y_{\tau_k+\cdot})$, in the same way that the sequences D_i and S_i are defined w.r.t. $(Y_\cdot)$. For example, $S^{(1)}_0$, $R^{(1)}_0 = Y_{\tau_1} \cdot e_1$, $D^{(1)}_0 = 0$ and we define by induction on $i \geqslant 0$,

$$S_{i+1}^{(1)} = T_{R_i^{(1)}+1}$$
$$D_{i+1}^{(1)} = \overline{D} \circ \theta_{S_{i+1}^{(1)}} + S_{i+1}^{(1)}$$
$$R_{i+1}^{(1)} = \sup\{Y_i \cdot e_1 : 0 \leqslant i \leqslant D_{i+1}^{(1)}\}.$$

For every $k \geqslant 1$ and $j \geqslant 0$ such that $S_j^{(k)} < \infty$, we need to introduce the σ-algebra $\mathcal{G}_j^{(k)}$ of the events up to $S_j^{(k)}$ as the smallest σ-algebra containing all of the sets of the form $\{\tau_1 \leqslant n_1\} \cap \{\tau_2 \leqslant n_2\} \cap \ldots \{\tau_k \leqslant n_k\} \cap A$, where $n_1 < n_2 < \ldots < n_k$ are integers and $A \in \mathcal{F}_{n_k + S_j^{(0)} \circ \theta_{n_k}}$. By convention, let $\tau_0 = 0$ and $\mathcal{G}_0^{(0)}$ be trivial.

On the existence of renewal times and the existence of the moments of all orders for ERW, we have the following key lemma proved in [3,7].

Lemma 2.2. *Consider an ERW with bias β, let $(\tau_k, k \geqslant 1)$ be the associated renewal times. Then, there exist $C, \alpha > 0$ depending on β and such that for every $n \geqslant 1$,*

$$\sup_{k \geqslant 0} \mathbb{P}_\beta[\tau_{k+1} - \tau_k > n | \mathcal{G}_0^{(k)}] \leqslant C e^{-n^\alpha} \ a.s.$$

In particular, for every $k \geqslant 0$ and $p \geqslant 1$, then $\tau_k < \infty$, a.s. and $\mathbb{E}_\beta[(\tau_{k+1} - \tau_k)^p] < \infty$.

A property very important of renewal times is that they cut a trajectory of the random walk into the independent increments as the following lemma (see [3] and [7]).

Lemma 2.3. *Under the probability $\mathbb{P}_\beta$, the random variables $(X_{\tau_{k+1}} - X_{\tau_k}, \tau_{k+1} - \tau_k)_{k \geqslant 1}$ and (X_{τ_1}, τ_1) are independent and $(X_{\tau_{k+1}} - X_{\tau_k}, \tau_{k+1} - \tau_k)_{k \geqslant 1}$ have the same law as (X_{τ_1}, τ_1) under the probability $\mathbb{P}_\beta$ conditionally on $\overline{D} = \infty$. We use the notation $\hat{\mathbb{P}}_\beta(\cdot) = \mathbb{P}_\beta(\cdot | \overline{D} = \infty)$.*

2.2. Girsanov's transform

In this section, we prove the smoothness of the speed using Girsanov's transform. First, we need a lemma as follows.

Lemma 2.4. *For all $c \in (0, 1]$, let $\tau := \tau_1$ then*

$$\sup_{t \in [c,1]} \mathbb{P}_t(\tau > n) \leqslant C' e^{n^{-\alpha}},$$
$$\sup_{t \in [c,1]} \hat{\mathbb{P}}_t(\overline{D} = \infty) \geqslant \varphi > 0,$$
$$\sup_{t \in [c,1]} \hat{\mathbb{P}}_t(\tau > n) \leqslant C e^{n^{-\alpha}},$$

where C', C, φ, α are positive constants depending only on c.

This lemma is proved by repeating with a minor change of the proof on the estimation of renewal times in [7,6].

Let $\beta_0, \beta \in (0, 1]$; we have Girsanov's transform:

Lemma 2.5.

$$\frac{\mathrm{d}\mathbb{P}_\beta}{\mathrm{d}\mathbb{P}_0}|_{\mathcal{F}_n} = \prod_{i=0}^{n-1} (1 + \beta \mathcal{E}_i 1_{Y_i \notin})$$
$$\frac{\mathrm{d}\mathbb{P}_\beta}{\mathrm{d}\mathbb{P}_{\beta_0}}|_{\mathcal{F}_n} = \prod_{i=0}^{n-1} \left(\frac{1 + \beta \mathcal{E}_i 1_{Y_i \notin}}{1 + \beta_0 \mathcal{E}_i 1_{Y_i \notin}} \right).$$

We denote

$$M_n(\beta) := \prod_{i=0}^{n-1} (1 + \beta \mathcal{E}_i 1_{Y_i \notin}) \text{ and } M_n(\beta, \beta_0) := \prod_{i=0}^{n-1} \left(\frac{1 + \beta \mathcal{E}_i 1_{Y_i \notin}}{1 + \beta_0 \mathcal{E}_i 1_{Y_i \notin}} \right). \tag{1}$$

Moreover, Girsanov's transform for renewal times is as follows.

Lemma 2.6. *Consider a σ-algebra $\mathcal{F}_\tau$ that is defined by*

$$\mathcal{F}_\tau = \{A \in \mathcal{F} : \forall n,\ \exists B_n \in \mathcal{F}_n \text{ such that } A \cap \{\tau = n\} = \mathcal{B}_n \cap \{\tau = n\}\}.$$

Then τ is $\mathcal{F}_\tau$-measurable, $(\overline{D} = \infty) \in \mathcal{F}_\tau$ and

$$\frac{\mathrm{d}\mathbb{P}_\beta}{\mathrm{d}\mathbb{P}_{\beta_0}}|_{\mathcal{F}_\tau} = M_\tau(\beta, \beta_0).\frac{\mathbb{P}_\beta(\overline{D} = \infty)}{\mathbb{P}_0(\overline{D} = \infty)}. \tag{2}$$

To prove the infinite differentiability of the speed, we also need the following lemma.

Lemma 2.7. *Let $I = (a, b)$ be an open interval of $\mathbb{R}$, $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $H(x, \omega)$ be a mapping*

$$H : I \times \Omega \to \mathbb{R}$$
$$(x, \omega) \mapsto H(x, \omega)$$

such that for every $x \in I$, $H(x, \omega)$ is a random variable, and for every $\omega \in \Omega$, $H(x, \omega)$ is a smooth function on I. Moreover, suppose that for every $n \geqslant 0$

$$\sup_{x\in I} \mathbb{E}\left(\left|\frac{\partial^n H}{\partial x^n}(x, \omega)\right|\right) < +\infty.$$

Then $\mathbb{E}[H(x, \omega)]$ is a smooth function and for every $k \geqslant 1$

$$\frac{\partial^k}{\partial x^k}[\mathbb{E}(H(x, \omega))] = \mathbb{E}\left(\left|\frac{\partial^k H}{\partial x^k}(x, \omega)\right|\right).$$

This lemma can be proved by the induction in k and using Fubini's Theorem.

2.3. Sketch of proof of Theorem 1.1

In this paper, we prove the regularity of the speed using the renewal times. These times exist and have the important property as in Lemma 2.4 for all dimensions $d \geqslant 2$ and all bias $\beta \in (0, 1)$. The law of large number gives the expression of the speed of the ERW in the first renewal time (see [3]) as follows

$$v(\beta) = \frac{\hat{\mathbb{E}}_\beta X_\tau}{\hat{\mathbb{E}}_\beta \tau}.$$

By Lemma 2.6, we get the formula of the speed,

$$v(\beta) = \frac{\hat{\mathbb{E}}_\beta X_\tau}{\hat{\mathbb{E}}_\beta \tau} = \frac{\hat{\mathbb{E}}_{\beta_0}[X_\tau M_\tau(\beta, \beta_0)]}{\hat{\mathbb{E}}_{\beta_0}[\tau M_\tau(\beta, \beta_0)]},$$

where β_0 fixed in $(0, 1)$. Next,

$$\frac{\partial}{\partial\beta}[M_\tau(\beta, \beta_0)] = \frac{\partial}{\partial\beta}\left[\prod_{i=0}^{\tau-1}\left(\frac{1 + \beta\mathcal{E}_i 1_{Y_i\notin}}{1 + \beta_0\mathcal{E}_i 1_{Y_i\notin}}\right)\right] = \left[\sum_{i=0}^{\tau-1}\left(\frac{\mathcal{E}_i 1_{Y_i\notin}}{1 + \beta\mathcal{E}_i 1_{Y_i\notin}}\right)\right] M_\tau(\beta, \beta_0). \tag{3}$$

Set $V_\tau = \sum_{i=0}^{\tau-1}\left(\frac{\mathcal{E}_i 1_{Y_i\notin}}{1+\beta\mathcal{E}_i 1_{Y_i\notin}}\right)$. From (3), we get

$$\frac{\partial^{n+1}}{\partial\beta^{n+1}}[M_\tau(\beta, \beta_0)] = \frac{\partial^n}{\partial\beta^n}[V_\tau(\beta)M_\tau(\beta, \beta_0)] = \sum_{k=0}^{n} C_n^k \frac{\partial^k}{\partial\beta^k}[V_\tau(\beta)]\frac{\partial^{n-k}}{\partial\beta^{n-k}}[M_\tau(\beta, \beta_0)], \tag{4}$$

where

$$C_n^k = \frac{n!}{k!(n-k)!}.$$

We have, for all $k \geqslant 0$ and $I = (\beta_0 - \delta, \beta_0 + \delta)$,

$$\sup_{\beta\in I}\left|\frac{\partial^k}{\partial\beta^k}[V_\tau(\beta)]\right| = \sup_{\beta\in I}\left|(-1)^k k! \sum_{i=0}^{\tau-1}\left(\frac{(\mathcal{E}_i 1_{Y_i\notin})^{k+1}}{(1 + \beta\mathcal{E}_i 1_{Y_i\notin})^{k+1}}\right)\right| \leqslant \frac{k!\tau}{(1 - \beta_0 - \delta)^{k+1}}. \tag{5}$$

We will prove by induction in n that

$$\left|\frac{\partial^n}{\partial\beta^n}[M_\tau(\beta,\beta_0)]\right| \leqslant \sum_{k=0}^{n} c_{kn}\tau^k M_\tau(\beta,\beta_0), \tag{6}$$

where c_{kn} are non-negative constants depending only on n, β_0, δ. For $n=0$, it is true with $c_{00}=1$. Suppose that it is true up to $n \geqslant 0$. For $n+1$ then by induction hypothesis combined with (4), (5) we have

$$\left|\frac{\partial^{n+1}}{\partial\beta^{n+1}}[M_\tau(\beta,\beta_0)]\right| \leqslant \sum_{k=0}^{n} C_n^k \frac{k!\tau}{(1-\beta_0-\delta)^{k+1}} \sum_{i=0}^{n-k} c_{i,n-k}\tau^i M_\tau(\beta,\beta_0) = \sum_{i=0}^{n+1} c_{i,n+1}\tau^i M_\tau(\beta,\beta_0),$$

where $c_{(i+1)(n+1)} = \sum_{k=0}^{n} C_n^k \frac{k!}{(1-\beta_0-\delta)^{k+1}} c_{i,n-k}$ for $i=1,\dots,n$ and $c_{0,n+1}=0$. This proves (6).

On $I=(\beta_0-\delta,\beta_0+\delta)$, then

$$\sup_{\beta\in I}\hat{\mathbb{E}}_{\beta_0}\left[\left|\frac{\partial^n}{\partial\beta^n}[X_\tau M_\tau(\beta,\beta_0)]\right|\right] = \sup_{\beta\in I}\hat{\mathbb{E}}_{\beta_0}\left[\left|X_\tau\frac{\partial^n}{\partial\beta^n}[M_\tau(\beta,\beta_0)]\right|\right].$$

Since $|X_\tau| \leqslant \tau$ then

$$\sup_{\beta\in I}\hat{\mathbb{E}}_{\beta_0}\left[\left|\frac{\partial^n}{\partial\beta^n}[X_\tau M_\tau(\beta,\beta_0)]\right|\right] \leqslant \sup_{\beta\in I}\sum_{k=0}^{n} c_{kn}\hat{\mathbb{E}}_{\beta}\left[\tau^{k+1}\right] < +\infty. \tag{7}$$

The last inequality follows from

$$\sup_{t\in(\beta_0-\delta,\beta_0+\delta)} \mathbb{E}_t(\tau^n) < +\infty \text{ for all } n \geqslant 1,$$

where we used the fact (see Lemma 2.4) that

$$\sup_{t\in(\beta_0-\delta,\beta_0+\delta)} \mathbb{P}_t(\tau > n) \leqslant Ce^{-n^\alpha}.$$

Combining (7) with Lemma 2.7, we get the smoothness of $\hat{\mathbb{E}}_{\beta_0}[X_\tau M_\tau(\beta,\beta_0)]$ and similarly for $\hat{\mathbb{E}}_{\beta_0}[\tau M_\tau(\beta,\beta_0)]$. This implies the smoothness of the speed $v(\beta)$.

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