C. R. Acad. Sci. Paris, Ser. I 354 (2016) 1055–1059

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Mathematical analysis/Partial differential equations

Existence of minimizers for the 2d stationary Griffith fracture model

Existence de déformations minimisant le modèle de Griffith des fractures 2d-stationnaires

Sergio Conti [a], Matteo Focardi [b], Flaviana Iurlano [a]

[a] *Institut für Angewandte Mathematik, Universität Bonn, 53115 Bonn, Germany*
[b] *DiMaI, Università di Firenze, 50134 Firenze, Italy*

ARTICLE INFO

Article history:
Received 4 March 2016
Accepted after revision 12 September 2016
Available online 28 September 2016

Presented by Haïm Brézis

ABSTRACT

We consider the variational formulation of the Griffith fracture model in two spatial dimensions and prove the existence of strong minimizers, that is deformation fields that are continuously differentiable outside a closed jump set and that minimize the relevant energy. To this aim, we show that minimizers of the weak formulation of the problem, set in the function space $GSBD^2$ and whose existence is well known, are actually strong minimizers following the approach developed by De Giorgi, Carriero, and Leaci in the corresponding scalar setting of the Mumford–Shah problem.

RÉSUMÉ

Nous considérons la formulation variationnelle du modèle de fracture de Griffith en dimension spatiale 2. Nous montrons l'existence de champs de déformation continûment différentiables hors d'un ensemble fermé de sauts, minimisant l'énergie relevante. Pour ce faire, nous montrons que les déformations minimisant la formulation faible du problème, dont l'existence est bien connue, placés dans l'espace des fonctions $GSBD^2$, minimisent de fait la formulation forte. Nous suivons l'approche développée par De Giorgi, Carriero et Leach dans le cadre scalaire correspondant du problème de Mumford–Shah.

1. Introduction

The study of brittle fracture in solids is based on the Griffith model, which combines elasticity with a term proportional to the surface area opened by the fracture. In its variational formulation, energy

E-mail addresses: sergio.conti@uni-bonn.de (S. Conti), focardi@math.unifi.it (M. Focardi), iurlano@iam.uni-bonn.de (F. Iurlano).

http://dx.doi.org/10.1016/j.crma.2016.09.003
1631-073X/

$$E[u,\Gamma] := \int_{\Omega\setminus\Gamma} \left(\frac{1}{2}\mathbb{C}e(u)\cdot e(u) + \kappa|u-g|^2\right)\mathrm{d}x + \beta\mathcal{H}^{n-1}(\Gamma) \tag{1}$$

is minimized over all closed sets $\Gamma \subset \Omega$ and all deformations $u \in C^1(\Omega\setminus\Gamma, \mathbb{R}^n)$ subject to suitable boundary and irreversibility conditions. Here $\Omega \subset \mathbb{R}^n$ is the reference configuration, the function $g \in L^\infty(\mathbb{R}^n;\mathbb{R}^n)$ represents external volume forces, $e(u) = (\nabla u + \nabla u^{\mathrm{T}})/2$ is the elastic strain, $\mathbb{C} \in \mathbb{R}^{(n\times n)\times(n\times n)}$ is the matrix of elastic coefficients, $\beta > 0$ the surface energy, $\kappa \geq 0$ a parameter. The evolutionary problem of fracture can be modelled as a sequence of variational problems, in which one minimizes (1) subject to varying loads with a kinematic restriction representing the irreversibility of fracture, see [6,19] and the references therein.

Mathematically, (1) is a vectorial free discontinuity problem. Much better understood is its scalar version, in which one replaces the elastic energy by the Dirichlet integral,

$$E_{\mathrm{MS}}[u,\Gamma] := \int_{\Omega\setminus\Gamma} \left(\frac{1}{2}|Du|^2 + \kappa|u-g|^2\right)\mathrm{d}x + \beta\mathcal{H}^{n-1}(\Gamma), \tag{2}$$

and one minimizes over all closed sets $\Gamma \subset \Omega$ and maps $u \in C^1(\Omega\setminus\Gamma, \mathbb{R})$. This scalar reduction coincides with the Mumford–Shah functional of image segmentation, which has been widely studied analytically and numerically [1,3,15,25]. By taking into account the structure of energy (2), it is natural to introduce the space $SBV(\Omega)$ of special functions of bounded variation, by imposing that the distributional derivative Du is a bounded measure, i.e. $u \in BV(\Omega)$, which can be written as $Du = \nabla u\mathcal{L}^n \llcorner \Omega + [u]\nu_u\mathcal{H}^{n-1} \llcorner J_u$, with ∇u the approximate gradient of u, $[u]$ the jump of u, J_u the $(n-1)$-rectifiable jump set of u, ν_u its normal. Therefore, the relaxation of (2) is

$$E^*_{\mathrm{MS}}[u] := \int_{\Omega} \left(\frac{1}{2}|\nabla u|^2\mathrm{d}x + \kappa|u-g|^2\right) + \beta\mathcal{H}^{n-1}(J_u), \tag{3}$$

and it is finite provided that u belongs to the subspace $SBV^2(\Omega)$ of functions in $SBV(\Omega)$ with approximate gradient $\nabla u \in L^2(\Omega;\mathbb{R}^n)$ and $\mathcal{H}^{n-1}(J_u) < \infty$. The existence of minimizers for the relaxed problem E^*_{MS} follows then from the general compactness properties of SBV^2, see [3] and the references therein.

The breakthrough in the quest for an existence theory for the Mumford–Shah functional (2) came with the proof by De Giorgi, Carriero, and Leaci in 1989 [16] of the fact that the jump set of minimizers u is essentially closed, in the sense that minimizers of the relaxed functional E^*_{MS} obey

$$\mathcal{H}^{n-1}((\overline{J_u}\setminus J_u)\cap\Omega) = 0, \quad \text{or equivalently} \quad \mathcal{H}^{n-1}(J_u\cap\Omega) = \mathcal{H}^{n-1}(\overline{J_u}\cap\Omega). \tag{4}$$

From this, elliptic regularity implies then that $(u, \overline{J_u}\cap\Omega)$ is a minimizer of the functional in the original formulation (2).

We address here the analogous existence issue for (1) in two spatial dimensions. We assume that $\mathbb{C}$ is a symmetric linear map from $\mathbb{R}^{n\times n}$ to itself with the properties

$$\mathbb{C}(z - z^{\mathrm{T}}) = 0 \quad \text{and} \quad \mathbb{C}z\cdot z \geq \alpha|z+z^{\mathrm{T}}|^2 \quad \text{for all } z \in \mathbb{R}^{n\times n} \tag{5}$$

for some $\alpha > 0$. Our main result is the following.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz set, $g \in L^\infty(\mathbb{R}^2;\mathbb{R}^2)$, $\mathbb{C}$ obeys the positivity condition (5), $\kappa \geq 0$. Then the functional (1) has a minimizer in the class*

$$\mathcal{A} := \{(u,\Gamma) : \Gamma \subset \overline{\Omega} \text{ closed}, u \in C^1(\Omega\setminus\Gamma, \mathbb{R}^2)\}. \tag{6}$$

The proof is sketched below and will be discussed in detail elsewhere [13]. In [13], we also consider generalizations of the basic model (1) with p-growth, $p \in (1,\infty)$, which may be appropriate for the study of materials with defects, such as damage or dislocations, and are obtained by replacing the quadratic volume energy density with

$$f_\mu(\xi) := \frac{1}{p}\left((\mathbb{C}\xi\cdot\xi + \mu)^{p/2} - \mu^{p/2}\right) \tag{7}$$

where $\mu \geq 0$ is a parameter.

This result is restricted to the two-dimensional case, because the approximation result in Proposition 2.2 below is only valid in two dimensions.

2. Outline of the proof

Following the ideas by De Giorgi, Carriero, and Leaci in the scalar case, the key point in obtaining Theorem 1.1 consists in establishing a one-sided Alfhors regularity for the jump set of (local) minimizers of the relaxed functional (3), also known in the literature as *density lower bound* estimate (for the precise formulation, see Theorem 2.1 below).

In this perspective, we start off by considering the weak formulation of (1). The functional setting is provided by $SBD^2(\Omega)$, the space of fields $u \in L^1(\Omega, \mathbb{R}^n)$ with symmetrized distributional derivative $Eu := (Du + Du^{\mathrm{T}})/2$, which is a bounded measure of the form $Eu = e(u)\mathcal{L}^n \llcorner \Omega + [u] \odot \nu_u \mathcal{H}^{n-1} \llcorner J_u$, with $[u]$ the jump of u, J_u the $(n-1)$-rectifiable jump set of u, ν_u its normal, and with the properties $e(u) \in L^2(\Omega, \mathbb{R}^{n\times n})$ and $\mathcal{H}^{n-1}(J_u) < \infty$. Here, $a \odot b = (a \otimes b + b \otimes a)/2$. $SBD^2(\Omega)$ is a subset of the space of functions with bounded deformation $BD(\Omega)$. The latter was introduced and investigated in [4,24,26–28] for the mathematical study of plasticity, damage, and fracture models in a geometrically linear framework. Instead, $SBD^2(\Omega)$ provides the natural function space in the modelling of fracture in linear elasticity [2,5]. In fracture models, where the energy does not control the amplitude of the jump, one naturally resorts to generalized special functions of bounded variation ($GSBV$) and to the recently introduced generalized special functions of bounded deformation ($GSBD$) [14]. The fine properties of BD, SBD^2 and $GSBD^2$ are much less understood than those of their scalar counterparts BV and SBV^2, respectively. Indeed, many standard technical tools are not available in this context, starting with basic ones such as truncation results and the coarea formula. Despite this, recently several contributions have improved the understanding of such spaces [7–9,11,18,20,21].

In view of the discussion above, the relaxation of (1) is

$$E^*_\kappa[u] := \int_\Omega \Big(\frac{1}{2}\mathbb{C}e(u)\cdot e(u) + \kappa|u-g|^2\Big)\mathrm{d}x + \beta\mathcal{H}^{n-1}(J_u) \tag{8}$$

for $u \in GSBD^2(\Omega)$. The density lower bound estimate for the jump set of minimizers in this setting is the content of the ensuing theorem.

Theorem 2.1 *(Density lower bound). If $u \in GSBD^2(\Omega)$ is a minimizer of the functional in (8), then there exist ϑ_0 and R_0, depending only on $\mathbb{C}$, g, κ, and β such that if $0 < \rho < R_0$, $x_0 \in \overline{J_u}\cap\Omega$, and $B_\rho(x_0) \subset\subset \Omega$, then*

$$\int_{B_\rho(x_0)} \Big(\frac{1}{2}\mathbb{C}e(u)\cdot e(u) + \kappa|u-g|^2\Big)\mathrm{d}x + \beta\mathcal{H}^1(J_u \cap B_\rho(x_0)) \geq \vartheta_0\rho. \tag{9}$$

Therefore,

$$\mathcal{H}^1((\overline{J_u} \setminus J_u)\cap\Omega) = 0. \tag{10}$$

Using this result, classical elliptic regularity yields that the minimizers u belong to $C^\infty(\Omega \setminus \overline{J_u}, \mathbb{R}^2)$ if g is smooth (see for instance [22]), so that $(u, \overline{J_u}\cap\Omega)$ is a minimizer of the strong formulation of the problem in (1). This leads directly to the proof of Theorem 1.1.

The density lower bound estimate is a mild regularity result for the jump set of a minimizer u, therefore it is natural to analyze the infinitesimal behaviour of u in points x_0 and, having selected a sequence $\rho_h \downarrow 0$, investigate the asymptotic of

$$u_h(x) := \rho_h^{-1/2} u(x_0 + \rho_h x).$$

We notice that the prefactor $\rho_h^{-1/2}$ is needed to balance the different scaling of the volume and surface term in the energy E^*_0. Indeed, we have

$$E^*_0[u_h; B_1] = \rho_h^{-1} E^*_0[u; B_{\rho_h}(x_0)];$$

in the formula above the domains of integration are indicated explicitly.

The original proof of the density lower bound in formula (9) in the scalar case is indirect [3,16] (see the more recent [17] for a direct proof in 2d). One first constructs truncations of the rescaled functions u_h and estimates them in SBV using a Poincaré–Wirtinger-type inequality. Then, using Ambrosio's SBV compactness theorem, one obtains the convergence of a subsequence, and shows that the limit is a local minimizer of the gradient term of the Mumford–Shah energy E^*_{MS} restricted to Sobolev spaces, i.e. it is an harmonic function, and in particular smooth. By a contradiction argument, one then shows that if in a ball the length of the jump set of a minimizer u of E^*_{MS} is sufficiently small and if u is not too far from being a (local) minimizer of the reduced functional

$$u \mapsto \int_\Omega \frac{1}{2}|\nabla u|^2\mathrm{d}x + \beta\mathcal{H}^{n-1}(J_u)\,,$$

then in the corresponding dyadic balls such an energy decays as fast as the Dirichlet integral for harmonic functions. From this, one deduces that the base point x_0 of the blow up process is not a jump point and the density lower bound follows at once.

The Poincaré–Wirtinger-type inequality proven by De Giorgi, Carriero, and Leaci states (in 2d) that if $u \in SBV^2(B_1)$ and $\mathcal{H}^1(J_u)$ is small, then there are $m \in \mathbb{R}$ and a modified function $\tilde{u} \in SBV^2(B_1)$ such that $\|\tilde{u} - m\|_{L^2(B_1)} \le c\|\nabla u\|_{L^2(B_1,\mathbb{R}^2)}$, $|D\tilde{u}|(B_1) \le 2\|\nabla u\|_{L^1(B_1,\mathbb{R}^2)}$, and $\tilde{u} = u$ on most of B_1, see [3, Th. 4.14] for details. The function $\tilde{u}$ is obtained from u by truncation, setting $\tilde{u}(x) = \max\{\tau^-, \min\{u(x), \tau^+\}\}$ for suitable $\tau^\pm \in \mathbb{R}$ chosen through the coarea formula, so that $\tilde{u}$ automatically fulfills $|D\tilde{u}| \le |Du|$. This procedure is not applicable to the vector-valued BD functions that appear in the Griffith model, since this space is not stable under truncation, and the coarea formula does not apply.

The key result to bypass such a problem is an approximation result for SBD^p functions, $p \in (1,\infty)$, with small jump set with $W^{1,p}$ functions, stated below in the case of interest $p = 2$ and established in [12]. This property yields an equivalent Poincaré–Wirtinger-type inequality for SBD^p functions, however restricted to two spatial dimensions.

Proposition 2.2 *(Approximation of SBD^2 fields). There exist universal constants $c, \eta > 0$ such that if $u \in SBD^2(B_\rho)$, $\rho > 0$, satisfies*

$$\mathcal{H}^1(J_u \cap B_\rho) < \eta\,(1-s)\frac{\rho}{2}$$

for some $s \in (0,1)$, then there are a countable family $\mathcal{F} = \{B\}$ of closed balls of radius $r_B < (1-s)\rho/2$ with finite overlap, $\cup_{\mathcal{F}} B \subset\subset B_\rho$ and a field $w \in SBD^2(B_\rho)$ such that

(i) $\rho^{-1}\sum_{\mathcal{F}} \mathcal{L}^2(B) + \sum_{\mathcal{F}} \mathcal{H}^1(\partial B) \le {}^{c}/_{\eta}\, \mathcal{H}^1(J_u \cap B_\rho)$;
(ii) $\mathcal{H}^1(J_u \cap \cup_{\mathcal{F}} \partial B) = \mathcal{H}^1((J_u \cap B_{s\rho}) \setminus \cup_{\mathcal{F}} B) = 0$;
(iii) $w = u$ $\mathcal{L}^2$*-a.e. on* $B_\rho \setminus \cup_{\mathcal{F}} B$;
(iv) $w \in W^{1,2}(B_{s\rho}, \mathbb{R}^2)$ *and* $\mathcal{H}^1(J_w \setminus J_u) = 0$;
(v) If $u \in L^\infty(B_\rho, \mathbb{R}^2)$*, then* $w \in L^\infty(B_\rho, \mathbb{R}^2)$ *with* $\|w\|_{L^\infty(B_\rho,\mathbb{R}^2)} \le \|u\|_{L^\infty(B_\rho,\mathbb{R}^2)}$;
(vi)

$$\int_B |e(w)|^2 \mathrm{d}x \le c \int_B |e(u)|^2 \mathrm{d}x \qquad \text{for each } B \in \mathcal{F}; \tag{11}$$

(vii) There is a skew-symmetric matrix A such that

$$\int_{B_{s\rho}\setminus \cup_{\mathcal{F}} B} |\nabla u - A|^2 \mathrm{d}x \le c \int_{B_\rho} |e(u)|^2 \mathrm{d}x. \tag{12}$$

Related results have been recently obtained in [9,20,21].

Proposition 2.2 holds for any exponent p; it is however restricted to two spatial dimensions. Its proof is based on covering the jump set of u with balls such that the total length of the jump set contained in each of them is comparable (but significantly smaller than) the radius. Clearly, these balls cover a small part of B_ρ, and u does not need to be modified outside them. In each one of the balls, then, a new function is constructed by a finite-element approximation, on a triangular grid that refines close to the boundary of the ball, as was done in [10] in the study of solid–solid phase transitions.

The key step is to show that one can choose such a grid with the property that all grid segments do not intersect the jump set of u. One then obtains an estimate of the oscillation of u along the segments, and hence on the corners of each of the triangles that constitute the grid. Linear interpolation gives then the desired extension. By convexity, the L^∞ norm of w inside each triangle equals the value of $|u|$ on one of the three vertices of the triangle; if all vertices are chosen as Lebesgue points for u, then one obtains $\|w\|_{L^\infty(T)} \le \|u\|_{L^\infty(T)}$ for every triangle T, and hence property (v). We refer to [12] for the details of the proof. In higher dimension, the same procedure would require finding a grid such that the edges do not intersect the jump set. This is however not possible, at least with the strategy of [10,12], as was explained in those papers.

The second key ingredient is an approximation of $GSBD^2$ functions by SBD^2 functions, which was proven in [23] generalizing a strategy by Chambolle [7].

Proposition 2.2 is used in the proof of Theorem 2.1 to replace the truncation procedure and the Poincaré–Wirtinger inequality. One then modifies the functions once more, subtracting not only a constant as in the BV case but also a linear function with skew-symmetric gradient, and obtains compactness. This permits to classify the blow-up limits of minimizers of (8) with vanishing length of the jump set and to show that they minimize a quadratic energy on Sobolev spaces, and therefore to conclude the proof of Theorem 2.1.

Acknowledgements

This work was partially supported by the Deutsche Forschungsgemeinschaft through the Sonderforschungsbereich 1060 *"The mathematics of emergent effects"*.

References

[1] L. Ambrosio, V.M. Tortorelli, Approximation of functionals depending on jumps by elliptic functionals via Γ-convergence, Commun. Pure Appl. Math. 43 (8) (1990) 999–1036.
[2] L. Ambrosio, A. Coscia, G. Dal Maso, Fine properties of functions with bounded deformation, Arch. Ration. Mech. Anal. 139 (3) (1997) 201–238.
[3] L. Ambrosio, N. Fusco, D. Pallara, Functions of Bounded Variation and Free Discontinuity Problems, Oxford University Press, Oxford, UK, 2000.
[4] G. Anzellotti, M. Giaquinta, Existence of the displacement field for an elastoplastic body subject to Hencky's law and von Mises yield condition, Manuscr. Math. 32 (1–2) (1980) 101–136.
[5] G. Bellettini, A. Coscia, G. Dal Maso, Compactness and lower semicontinuity properties in $SBD(\Omega)$, Math. Z. 228 (2) (1998) 337–351.
[6] B. Bourdin, G. Francfort, J.-J. Marigo, The Variational Approach to Fracture, Springer, New York, 2008.
[7] A. Chambolle, An approximation result for special functions with bounded deformation, J. Math. Pures Appl. (9) 83 (7) (2004) 929–954.
[8] A. Chambolle, A. Giacomini, M. Ponsiglione, Piecewise rigidity, J. Funct. Anal. 244 (1) (2007) 134–153.
[9] A. Chambolle, S. Conti, G. Francfort, Korn–Poincaré inequalities for functions with a small jump set, Indiana Univ. Math. J. 65 (4) (2016) 1373–1399.
[10] S. Conti, B. Schweizer, Rigidity and Γ convergence for solid–solid phase transitions with $SO(2)$-invariance, Commun. Pure Appl. Math. 59 (2006) 830–868.
[11] S. Conti, M. Focardi, F. Iurlano, Which special functions of bounded deformation have bounded variation? Proc. R. Soc. Edinb. A (2015), in press. arXiv:1502.07464.
[12] S. Conti, M. Focardi, F. Iurlano, Integral representation for functionals defined on SBD^p in dimension two, arXiv:1510.00145, 2015.
[13] S. Conti, M. Focardi, F. Iurlano, Existence result for the static Griffith fracture model in two dimensions, in preparation.
[14] G. Dal Maso, Generalised functions of bounded deformation, J. Eur. Math. Soc. 15 (5) (2013) 1943–1997.
[15] G. David, Singular Sets of Minimizers for the Mumford–Shah Functional, Prog. Math., vol. 233, Birkhäuser Verlag, Basel, Switzerland, 2005.
[16] E. De Giorgi, M. Carriero, A. Leaci, Existence theorem for a minimum problem with free discontinuity set, Arch. Ration. Mech. Anal. 108 (3) (1989) 195–218.
[17] C. De Lellis, M. Focardi, Density lower bound estimates for local minimizers of the 2*d* Mumford–Shah energy, Manuscr. Math. 142 (1–2) (2013) 215–232.
[18] G. De Philippis, F. Rindler, On the structure of $\mathcal{A}$-free measures and applications, Ann. Math. 184 (3) (2016) 1017–1039.
[19] G.A. Francfort, J.J. Marigo, Revisiting brittle fractures as an energy minimization problem, J. Mech. Phys. Solids 46 (1998) 1319–1342.
[20] M. Friedrich, A Korn type inequality in SBD for functions with small jump sets, arXiv:1505.00565, 2015.
[21] M. Friedrich, A Korn–Poincaré-type inequality for special functions of bounded deformation, arXiv:1503.06755, 2015.
[22] M. Giaquinta, L. Martinazzi, An Introduction to the Regularity Theory for Elliptic Systems, Harmonic Maps and Minimal Graphs, Appunti. Sc. Norm. Super. Pisa (N. S.), vol. 11, Edizioni della Normale, Pisa, 2012.
[23] F. Iurlano, A density result for GSBD and its application to the approximation of brittle fracture energies, Calc. Var. Partial Differ. Equ. 51 (2014) 315–342.
[24] R. Kohn, R. Temam, Dual spaces of stresses and strains, with applications to Hencky plasticity, Appl. Math. Optim. 10 (1) (1983) 1–35.
[25] D. Mumford, J. Shah, Optimal approximation by piecewise smooth functions and associated variational problems, Commun. Pure Appl. Math. 17 (1989) 577–685.
[26] P.-M. Suquet, Existence et régularité des solutions des équations de la plasticité, C. R. Acad. Sci. Paris, Sér. A–B 286 (1978) A1201–A1204.
[27] R. Temam, Problèmes mathématiques en plasticité, Méthodes Math. Inform. (Math. Methods Inf. Sci.), vol. 12, Gauthier–Villars, Montrouge, France, 1983.
[28] R. Temam, G. Strang, Functions of bounded deformation, Arch. Ration. Mech. Anal. 75 (1980/1981) 7–21.