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Kobayashi measure hyperbolicity for singular directed varieties of general type

Hyperbolicité au sens de la mesure de Kobayashi pour les variétés dirigées singulières de type général

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A R T I C L E   I N F O

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A B S T R A C T

In this note, we prove the non-degeneracy of the Kobayashi–Eisenman volume measure of a singular directed varieties (X, V), i.e. the Kobayashi measure hyperbolicity of (X, V), as long as the canonical sheaf $\mathcal{K}_V$ of V is big in the sense of Demailly.

R É S U M É

Dans cette note, nous démontrons la non-dégénérescence de la mesure de volume au sens de Kobayashi–Eisenman pour une variété dirigée singulière (X, V), c'est-à-dire l'hyperbolicité de la mesure au sens de Kobayashi de (X, V) lorsque le faisceau canonique $\mathcal{K}_V$ de V est gros au sens de Demailly.

1. Introduction

Let (X, V) be a *complex directed manifold*, i.e X is a complex manifold equipped with a holomorphic subbundle $V \subset T_X$. The philosophy behind the introduction of directed manifolds, as initially suggested by J.-P. Demailly, is that there are certain fonctorial constructions that work better in the category of directed manifolds (ref. [3]). This is so even in the "absolute case", i.e. in the case $V = T_X$. In general, singularities of V cannot be avoided, even after blowing-up, and V can be seen as a coherent subsheaf of T_X such that T_X/V is torsion free. Such a sheaf V is a subbundle of T_X outside of an analytic subset of codimension at least 2, which we denote here by $\mathrm{Sing}(V)$. The Kobayashi–Eisenman volume measure can also be defined for such (singular) directed pair (X, V).

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Definition 1.1. Let (X, V) be a directed manifold with $\dim(X) = n$ and let $\text{rank}(V) = r$. Then the Kobayashi–Eisenman volume measure of (X, V) is the pseudometric defined on any $\xi \in \Lambda^r V_x$ for $x \notin \text{Sing}(V)$, by

$$\mathbf{e}^r_{X,V}(\xi) := \inf\{\lambda > 0;\ \exists f : \mathbb{B}_r \to X,\ f(0) = x,\ \lambda f_*(\tau_0) = \xi,\ f_*(T_{\mathbb{B}_r}) \subset V\},$$

where $\mathbb{B}_r$ is the unit ball in $\mathbb{C}^r$ and $\tau_0 = \frac{\partial}{\partial t_1} \wedge \cdots \wedge \frac{\partial}{\partial t_r}$ is the unit r-vector of $\mathbb{C}^r$ at the origin. One says that (X, V) is *Kobayashi measure hyperbolic* if $\mathbf{e}^r_{X,V}$ is generically positive definite, i.e. positive definite on a Zariski open set.

In [3], the author also introduced the concept of *canonical sheaf* $\mathcal{K}_V$ for any singular directed variety (X, V), and he showed that the "bigness" of $\mathcal{K}_V$ implies that all non-constant entire curves $f : \mathbb{C} \to (X, V)$ must satisfy certain global algebraic differential equations. In this note, we study the Kobayashi–Eisenman volume measure of the singular directed variety (X, V), and give another geometric consequence of the bigness of $\mathcal{K}_V$. Our main theorem is as follows.

Theorem 1.2. *Let (X, V) be a compact complex directed variety (where V is possibly singular), and let $\text{rank}(V) = r$, $\dim(X) = n$. If V is of general type (see Definition 2.1 below), with a base locus $\text{Bs}(V) \subsetneq X$ (see also Definition 2.1), then (X, V) is Kobayashi measure hyperbolic.*

Remark 1. In the absolute case, Theorem 1.2 is proved in [6] and [7]; for a smooth directed variety it is proved in [3].

2. Proof of the main theorem

Proof. Since the singular set $\text{Sing}(V)$ of V is an analytic set of codimension ≥ 2, the top exterior power $\Lambda^r V$ of V is a coherent sheaf of rank 1, and it admits a generically injective morphism to its bidual $(\Lambda^r V)^{**}$, which is an invertible sheaf (and therefore, can be seen as a line bundle). We give below an explicit construction of the multiplicative cocycle that represents the line bundle $(\Lambda^r V)^{**}$.

Since $V \subset T_X$ is a coherent sheaf, we can take a covering by open coordinate balls $\{U_\alpha\}$ satisfying the following property: on each U_α, there exist sections $e^{(\alpha)}_1, \ldots, e^{(\alpha)}_{k_\alpha} \in \Gamma(U_\alpha, T_X|_{U_\alpha})$ that generate the coherent sheaf V on U_α. Thus the sections $e^{(\alpha)}_{i_1} \wedge \cdots \wedge e^{(\alpha)}_{i_r} \in \Gamma(U_\alpha, \Lambda^r T_X|_{U_\alpha})$ with $(i_1, \ldots, i_r)$ varying among all r-tuples of $(1, \ldots, k_\alpha)$ generate the coherent sheaf $\Lambda^r V|_{U_\alpha}$, which is a subsheaf of $\Lambda^r T_X|_{U_\alpha}$. Denote $v^{(\alpha)}_I := e^{(\alpha)}_{i_1} \wedge \cdots \wedge e^{(\alpha)}_{i_r}$. Then, since $\text{codim}(\text{Sing}(V)) \geq 2$ we know that the common zero set of the family of sections $v^{(\alpha)}_I$ is contained in $\text{Sing}(V)$, and thus all tensors $v^{(\alpha)}_I$ are proportional via meromorphic factors. By simplifying in a given section $v^{(\alpha)}_{I_0}$ the common zero divisor of the various meromorphic quotients $v^{(\alpha)}_{I_0}/v^{(\alpha)}_I$, one obtains a section $v_\alpha \in \Gamma(U_\alpha, \Lambda^r T_X|_{U_\alpha})$ (uniquely defined up to an invertible factor), and holomorphic functions $\{\lambda_I \in \mathcal{O}(U_\alpha)\}$ that do not have common factors, such that $v^{(\alpha)}_I = \lambda_I v_\alpha$ for all I. From this construction, we can see that on $U_\alpha \cap U_\beta$, v_α and v_β coincide up to multiplication by a nowhere-vanishing holomorphic function, i.e.

$$v_\alpha = g_{\alpha\beta} v_\beta$$

on $U_\alpha \cap U_\beta \neq \emptyset$, where $g_{\alpha\beta} \in \mathcal{O}^*_X(U_\alpha \cap U_\beta)$. This multiplicative cocycle $\{g_{\alpha\beta}\}$ defines the line bundle $(\Lambda^r V)^{***}$. If we take a Kähler metric ω on X, it induces a smooth Hermitian metric H_r on $\Lambda^r T_X$, and from the natural inclusion $\Lambda^r V \to \Lambda^r T_X$, ω also induces a *singular* Hermitian metric h_s of $(\Lambda^r V)^{***}$ whose local weight φ_α is equal to $\log|v_\alpha|^2_{H_r}$. It is easy to show that h_s has *analytic singularities*, and that its set of singularities satisfies $\text{Sing}(h_s) \subset \text{Sing}(V)$. Indeed, we have $\text{Sing}(h_s) = \bigcup_\alpha \{p \in U_\alpha | v_\alpha(p) = 0\}$. Now, one gives the following definition.

Definition 2.1. With the notations above, (X, V) is said to be *of general type* if there exists a singular Hermitian metric h on the invertible sheaf $(\Lambda^r V)^{***}$ with analytic singularities satisfying the following two conditions:

(1) the curvature current $\Theta_h \geq \epsilon\omega$, i.e., it is a Kähler current;
(2) h is more singular than h_s, that is, there exists a globally defined quasi-psh function χ which is bounded from above such that

$$e^\chi \cdot h = h_s.$$

Since h and h_s have both analytic singularities, χ also has analytic singularities, and thus e^χ is a continuous function. Moreover, $e^{\chi(p)} > 0$ if $p \notin \text{Sing}(h)$. We define *the base locus of V* to be

$$\text{Bs}(V) := \bigcap_h \text{Sing}(h),$$

where h varies among all the singular metrics on $(\Lambda^r V)^{***}$ satisfying Properties (1) and (2) above.

Now fix a point $p \notin \mathrm{Bs}(V) \cup \mathrm{Sing}(V)$; then by Definition 2.1 we can find a singular metric h on $(\Lambda^r V)^{***}$ with analytic singularities satisfying Properties (1) and (2) above, and $p \notin \mathrm{Sing}(h)$. Let f be any holomorphic map from the unit ball $\mathbb{B}_r \subset \mathbb{C}^r$ to (X, V) such that $f(0) = p$, then on each $f^{-1}(U_\alpha)$ we have

$$f_*\left(\frac{\partial}{\partial t_1} \wedge \cdots \wedge \frac{\partial}{\partial t_r}\right) = a^{(\alpha)}(t) \cdot v_\alpha|_f,$$

where $a^{(\alpha)}(t)$ is meromorphic function, with poles contained in $f^{-1}\big(\mathrm{Sing}(V) \cap U_\alpha\big)$, and satisfies

$$\left| f_*\left(\frac{\partial}{\partial t_1} \wedge \cdots \wedge \frac{\partial}{\partial t_r}\right)\right|^2_{H_r} = |a^{(\alpha)}(t)|^2 \cdot |v_\alpha|^2_{H_r} = |a^{(\alpha)}(t)|^2 \cdot \mathrm{e}^{\varphi_\alpha \circ f},$$

which is bounded on any relatively compact set.

Therefore, $\frac{\partial}{\partial t_1} \wedge \cdots \wedge \frac{\partial}{\partial t_r}$ can be seen as a (meromorphic!) section of $f^*(\Lambda^r V)^{**}$, and thus we set

$$\delta(t) := \left|\frac{\partial}{\partial t_1} \wedge \cdots \wedge \frac{\partial}{\partial t_r}\right|^2_{f^* h^{-1}} = |a^{(\alpha)}(t)|^2 \cdot \mathrm{e}^{\phi_\alpha \circ f}, \tag{1}$$

where ϕ_α is the local weight of h. By Property (2) above, there exists a globally defined quasi-psh function χ on X which is bounded from above such that

$$\delta(t) = \mathrm{e}^{\chi \circ f} \cdot \left| f_*\left(\frac{\partial}{\partial t_1} \wedge \cdots \wedge \frac{\partial}{\partial t_r}\right)\right|^2_{H_r}. \tag{2}$$

Now we define a semi-positive metric $\widetilde{\gamma}$ on $\mathbb{B}_r$ by putting $\widetilde{\gamma} := f^*\omega$, then we have

$$\frac{\left| f_*(\frac{\partial}{\partial t_1} \wedge \cdots \wedge \frac{\partial}{\partial t_r})\right|_{H_r}}{\det \widetilde{\gamma}} \le C_0(f(t)) \le C_1, \tag{3}$$

where $C_0(z)$ is a bounded function on X which does not depend on f, and we take C_1 to be its upper bound. One can find a conformal factor $\lambda(t)$ so that $\gamma := \lambda \widetilde{\gamma}$ satisfies

$$\det \gamma = \delta(t)^{\frac{1}{2}}.$$

Combining (2) and (3) together, we obtain

$$\lambda \le C_1^{\frac{1}{r}} \cdot \mathrm{e}^{\frac{\chi \circ f}{2r}}.$$

Since $\Theta_h \ge \epsilon \omega$, by (1) and (2) we have

$$\mathrm{dd}^c \log \det \gamma \ge \frac{\epsilon}{2} f^*\omega = \frac{\epsilon}{2\lambda}\gamma \ge \frac{\epsilon}{2C_1^{\frac{1}{r}}} \mathrm{e}^{-\frac{\chi \circ f}{2r}} \gamma.$$

By Property (2) in Definition 2.1 applied to h, there exists a constant $C_2 > 0$ such that

$$\mathrm{e}^{-\frac{\chi}{2r}} \ge C_2.$$

Denote $A := \frac{\epsilon C_2}{2C_1^{\frac{1}{r}}}$, and it is a universal constant that does not depend on f. Then by the Ahlfors–Schwarz Lemma (see Lemma 2.2 below), we have

$$\delta(0) \le \left(\frac{r+1}{A}\right)^{2r}.$$

Since $p \notin \mathrm{Sing}(h) \cup \mathrm{Sing}(V)$, then we have $\mathrm{e}^{\chi(p)} > 0$, and thus

$$\left| f_*\left(\frac{\partial}{\partial t_1} \wedge \cdots \wedge \frac{\partial}{\partial t_r}\right)\right|^2_{H_r}(0) \le \mathrm{e}^{-\chi(p)}\delta(0) = \mathrm{e}^{-\chi(p)} \cdot \left(\frac{r+1}{A}\right)^{2r}.$$

Since f is taken to be arbitrary, we conclude by Definition 1.1 that the Kobayashi–Eisenman volume measure $\mathbf{e}^r_{X,V}$ is positive definite outside of $\mathrm{Bs}(V) \cup \mathrm{Sing}(V)$, and therefore, (X, V) is Kobayashi measure hyperbolic. □

Lemma 2.2 *(Ahlfors-Schwarz). Let* $\gamma = \sqrt{-1}\sum \gamma_{jk}(t)\mathrm{d}t_j \wedge \mathrm{d}t_k$ *be an almost everywhere positive hermitian form on the ball* $B(0,R) \subset \mathbb{C}^r$ *of radius* R*, such that*

$$-\mathrm{Ricci}(\gamma) := \sqrt{-1}\partial\overline{\partial}\log\det\gamma \geq A\gamma$$

in the sense of currents, for some constant $A > 0$*. Then*

$$\det(\gamma)(t) \leq \left(\frac{r+1}{AR^2}\right)^r \frac{1}{(1-\frac{|t|^2}{R^2})^{r+1}}.$$

Remark 2. If V is regular, then V is of general type if and only if $\Lambda^r V^*$ is a big line bundle. In this situation, the base locus $\mathrm{Bs}(V) = \mathbf{B}_+(\Lambda^r V)$, where $\mathbf{B}_+(\Lambda^r V^*)$ is the *augmented base locus* for the big line bundle $\Lambda^r V^*$ (ref. [8]).

With the notations above, we define the coherent ideal sheaf $\mathcal{I}(V)$ to be germ of holomorphic functions which is locally bounded with respect to h_s, i.e., $\mathcal{I}(V)$ is the integral closure of the ideal generated by the coefficients of v_α in some local trivialization of $\Lambda^r T_X$. We denote by $K_V := \Lambda^r V^{***}$ and $\mathcal{K}_V := K_V \otimes \mathcal{I}(V)$ the sheaf $\mathcal{K}_V$ is defined in [3] to be the *canonical sheaf of* (X,V). It is easy to show that the zero scheme of $\mathcal{I}(V)$ is equal to $\mathrm{Sing}(h_s) = \mathrm{Sing}(V)$. The sheaf $\mathcal{K}_V$ is said to be a *big sheaf* iff for some log-resolution $\mu : \widetilde{X} \to X$ of $\mathcal{I}(V)$ with $\mu^*\mathcal{I}(V) = \mathcal{O}_{\widetilde{X}}(-D)$, the invertible sheaf $\mu^*K_V - D$ is big in the usual sense. Now we have the following statement.

Proposition 2.1. *V is of general type if and only if $\mathcal{K}_V$ is big. Moreover, we have*

$$\mathrm{Bs}(V) \subset \mu(\mathbf{B}_+(\mu^*K_V - D)) \cup \mathrm{Sing}(h_s) \subset \mu(\mathbf{B}_+(\mu^*K_V - D)) \cup \mathrm{Sing}(V).$$

Proof. By Definition 2.1, the condition that $\mathcal{K}_V$ is a big sheaf implies that K_V and $\mu^*K_V - D$ are both big line bundles. For $m \gg 0$, we have an isomorphism

$$\mu^* : H^0(X, (mK_V - A) \otimes \mathcal{I}(V)^m) \xrightarrow{\approx} H^0(\widetilde{X}, m\mu^*K_V - \mu^*A - mD). \tag{4}$$

Let us fix a very ample divisor A. Then for $m \gg 0$, the base locus (in the usual sense) $\mathbf{B}(m\mu^*K_V - mD - \mu^*A)$ is stably contained in $\mathbf{B}_+(\mu^*K_V - D)$ [8]. Thus we can take a $m \gg 0$ to choose a basis $s_1, \ldots, s_k \in H^0(\widetilde{X}, m\mu^*K_V - mD - \mu^*A)$, whose common zero is contained in $\mathbf{B}_+(\mu^*K_V - D)$. By the isomorphism (4), there exists $\{e_i\}_{1\leq i\leq k} \subset H^0(X, (mK_V - A) \otimes \mathcal{I}(V)^m)$ such that

$$\mu^*(e_i) = s_i.$$

We define a singular metric h_m on $mK_V - A$ by putting

$$|\xi|^2_{h_m} := \frac{|\xi|^2}{\sum_{i=1}^k |e_i|^2} \quad \text{for} \quad \xi \in (mK_V - A)_x.$$

Choose a smooth metric h_A on A such that the curvature $\Theta_A \geq \epsilon\omega$ is a smooth Kähler form. Then $h := (h_m h_A)^{\frac{1}{m}}$ defines a singular metric on K_V with analytic singularities, such that its curvature current $\Theta_h \geq \frac{1}{m}\Theta_A \geq \frac{\epsilon}{m}\omega$. From the construction we know that h is more singular than h_s, and $\mathrm{Sing}(h) \subset \mu(\mathbf{B}_+(\mu^*K_V - D)) \cup \mathrm{Sing}(h_s)$. □

Remark 3. Thanks to Proposition 2.1, we could have taken Definition 2.1 as another equivalent definition of the bigness of $\mathcal{K}_V$, one that is more analytic. By Theorem 1.2, we can replace the condition that V is of general type by the bigness of $\mathcal{K}_V$, and we see in this way that the definition of the canonical sheaf of a singular directed variety is very natural.

A direct consequence of Theorem 1.2 is the following corollary, which was suggested in [5].

Corollary 2.3. *Let (X,V) be directed varieties with $\mathrm{rank}(V) = r$, and f be a holomorphic map from $\mathbb{C}^r$ to (X,V) with generic maximal rank. Then if $\mathcal{K}_V$ is big, the image of f is contained in $\mathrm{Bs}(V) \subsetneq X$.*

The famous conjecture by Green–Griffiths states that, in the absolute case, the converse of Theorem 1.2 should be true. It is natural to ask whether we have similar results for arbitrary directed varieties. A result by Marco Brunella [2] gives a weak converse of Theorem 1.2 for every 1-directed variety:

Theorem 2.4. *Let X be a compact Kähler manifold equipped with a singular holomorphic foliation $\mathcal{F}$ by curves. Suppose that $\mathcal{F}$ contains at least one leaf that is hyperbolic, then the canonical bundle $K_{\mathcal{F}}$ is pseudoeffective.*

Indeed, Brunella proved more than the results stated in the above theorem. By putting on $K_{\mathcal{F}}$ precisely the Poincaré metric of hyperbolic leaves, he constructed a singular Hermitian metric h on $K_{\mathcal{F}}$ (possibly not with analytic singularities), such that the set of points where h is locally unbounded is the polar set $\mathrm{Sing}(\mathcal{F}) \cup \mathrm{Parab}(\mathcal{F})$, where $\mathrm{Parab}(\mathcal{F})$ is the union of parabolic leaves, and such that the curvature Θ_h of the metric h is a positive current. In this vein, a natural question is:

Question 2.5. Can Brunella's theorem be strengthened by stating that when a foliation $(X, \mathcal{F})$ admits a hyperbolic leaf, then not only $K_{\mathcal{F}}$ is pseudo-effective, but also the canonical sheaf $\mathcal{K}_{\mathcal{F}} = K_{\mathcal{F}} \otimes \mathcal{I}(\mathcal{F})$ is pseudo-effective? In other words, can we find a singular Hermitian metric h on $K_{\mathcal{F}}$ with the curvature Θ_h is a positive current, and h is more singular than h_s? (Recall that h_s is the singular metric on $K_{\mathcal{F}}$ induced by a Hermitian metric on T_X.)

Remark 4. In [10], the author introduces the definition of *canonical singularities* for foliations, in dimension 2 this definition is equivalent to *reduced singularities* in the sense of Seidenberg. The generic foliation by curves of degree d in $\mathbb{C}P^n$ is another example of canonical singularities. In this situation, one cannot expect to improve the "bigness" of the canonical sheaf $\mathcal{K}_{\mathcal{F}}$ by blowing-up. Indeed, this birational model is "stable" in the sense that $\pi_* \mathcal{K}_{\widetilde{\mathcal{F}}} = \mathcal{K}_{\mathcal{F}}$ for any birational model $\pi : (\widetilde{X}, \widetilde{\mathcal{F}}) \to (X, \mathcal{F})$. However, on a complex surface endowed with a foliation $\mathcal{F}$ with reduced singularities, if f is an entire curve tangent to the foliation, and $T[f]$ is the Ahlfors current associated with f, then in [9] it is shown that the lower bound for $T[f] \cdot c_1(T_{\mathcal{F}})$ can be improved by an infinite sequence of blowing-ups. Indeed, for certain singularities, the separatrices containing them are rational curves, and thus the lifted entire curve will not pass through these singularities. In the literature (*e.g.*, [1,9]) this type of singularity is sometimes called "*small*", i.e. the lifted entire curve will not pass through these singularities. Since $T[f] \cdot c_1(T_{\mathcal{F}})$ is related to value distribution, these "small" singularities do not have any negative contribution to the lower bound for $T[f] \cdot c_1(T_{\mathcal{F}})$, which will be substantially increased by the effect of performing blow-ups. In [4], this "Diophantine approximation" idea has been generalized to higher dimensions; for more details, we refer the reader to the above-mentioned papers.

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