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Homeomorphisms of a solenoid isotopic to the identity and its second cohomology groups

Homéomorphismes d'un solénoïde isotope à l'identité et ses deuxièmes groupes de cohomologie

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ABSTRACT

Some aspects of the study of the group of homeomorphisms of a one-dimensional solenoid which are isotopic to the identity are discussed in this paper. The maximal subgroup up to homotopy equivalence is described and the Euler class of the universal central extension of this group is calculated. This class being bounded gives an interpretation of the rotation element on the solenoid.

RÉSUMÉ

Certains aspects de l'étude du groupe d'homéomorphismes d'un solénoïde à une dimension qui sont isotopes à l'identité sont abordés dans la présente contribution. Son sous-groupe maximal, sauf classe d'équivalence homotopique, est décrit, et la classe d'Euler de l'extension centrale universelle de ce groupe est calculée. Cette classe, étant limitée, nous donne une interprétation de l'élément de rotation dans le solénoïde.

1. Introduction

At the end of the 19th century, Henri Poincaré [19] introduced an invariant of major importance for the study of the dynamics of homeomorphisms of the unit circle $\mathbb{S}^1$; this is the well-known *rotation number*. There are some equivalent ways of defining this topological invariant. Our interest lies in the works of É. Ghys [9] and [10]. At the beginning of this century, he found the rotation number using the language of cohomology of groups (see [6] for a detailed study of cohomology of groups).

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Consider the group $\mathrm{Homeo}_+(\mathbb{S}^1)$ of homeomorphisms of the circle that preserves the orientation and $\widetilde{\mathrm{Homeo}_+}(\mathbb{S}^1)$ the group of lifts with respect to the universal covering $\pi : \mathbb{R} \longrightarrow \mathbb{S}^1$. There is a surjective homomorphism $p : \widetilde{\mathrm{Homeo}_+}(\mathbb{S}^1) \longrightarrow \mathrm{Homeo}_+(\mathbb{S}^1)$ with kernel isomorphic to $\mathbb{Z}$. Moreover, p is a universal covering map and this is a central extension

$$0 \longrightarrow \mathbb{Z} \longrightarrow \widetilde{\mathrm{Homeo}_+}(\mathbb{S}^1) \xrightarrow{p} \mathrm{Homeo}_+(\mathbb{S}^1) \longrightarrow 1.$$

In fact, it is a universal central extension and the kernel is isomorphic to $\mathrm{H}_2(\mathrm{Homeo}_+(\mathbb{S}^1), \mathbb{Z})$.

Using the universal coefficient theorem, we can see that $\mathrm{H}^2(\mathrm{Homeo}_+(\mathbb{S}^1), \mathbb{Z}) \simeq \mathbb{Z}$. In particular, it can be shown that the Euler class of this extension $eu(\mathbb{S}^1)$ is bounded and represents a generator for the bounded cohomology group $\mathrm{H}_b^2(\mathrm{Homeo}_+(\mathbb{S}^1), \mathbb{Z})$. If $\phi : \mathbb{Z} \longrightarrow \mathrm{Homeo}_+(\mathbb{S}^1)$ is any homomorphism, the corresponding class $\phi^*(eu(\mathbb{S}^1)) \in \mathrm{H}_b^2(\mathbb{Z}, \mathbb{Z}) \simeq \mathbb{R}/\mathbb{Z}$ is the rotation number of the homeomorphism $\phi(1)$.

In this paper, we extend these results to the group of homeomorphisms of a one-dimensional solenoid S *that are isotopic to the identity.*

The solenoid S is a compact connected Abelian group that is constructed as the inverse limit of the system of coverings of the circle $\mathbb{S}^1 \longrightarrow \mathbb{S}^1$ given by $\omega \longmapsto \omega^n$ for $n \in \mathbb{N}$, and any two of these homomorphisms are related by divisibility of the exponents. The solenoid S can also be seen as a foliated space whose leaves are homeomorphic to $\mathbb{R}$ and each fiber is homeomorphic to the profinite completion of the integers $\widehat{\mathbb{Z}}$. The path-component of the identity element $0 \in \mathsf{S}$ is called the base leaf and it is denoted by $\mathcal{L}_0$.

It should be pointed out that S is not a manifold, but is closely related to $\mathbb{S}^1$, actually S is known as the algebraic universal covering space of the circle.

There is a well-defined covering map $\Pi : \mathbb{R} \times \widehat{\mathbb{Z}} \longrightarrow \mathsf{S}$, given as the quotient projection of a diagonal action of $\mathbb{Z}$ on the product space $\mathbb{R} \times \widehat{\mathbb{Z}}$. This exhibits S as the orbit space $\mathbb{R} \times_{\mathbb{Z}} \widehat{\mathbb{Z}}$. Denote by $\widetilde{\mathrm{Homeo}_+}(\mathsf{S})$ the group of lifts of elements in $\mathrm{Homeo}_+(\mathsf{S})$. Using this group of lifts, we show that the group $\mathrm{Homeo}_+(\mathsf{S})$ is homotopy equivalent to the subgroup of translations by elements on the base leaf $\mathcal{L}_0$ and consequently, every homotopy group $\pi_n(\mathrm{Homeo}_+(\mathsf{S}))$ is trivial.

Similar to the central extension for $\mathrm{Homeo}_+(\mathbb{S}^1)$, there is a central extension of the type

$$0 \longrightarrow \mathbb{Z} \longrightarrow \widetilde{\mathrm{Homeo}_+}(\mathsf{S}) \longrightarrow \mathrm{Homeo}_+(\mathsf{S}) \longrightarrow 1,$$

where $\mathbb{Z}$ is identified with the subgroup of deck transformations $\Delta(\mathbb{Z})$, which is contained in the center of $\widetilde{\mathrm{Homeo}_+}(\mathsf{S})$ (see section 3.3). We prove that this is a universal central extension. This can be done by using the fact that $\mathrm{Homeo}_+(\mathsf{S})$ is uniformly perfect, as shown in [3]. Consequently, we have the Schur multiplier $\mathrm{H}_2(\mathrm{Homeo}_+(\mathsf{S}), \mathbb{Z}) \simeq \mathbb{Z}$ and using the universal coefficient theorem, there is an Euler class

$$eu \in \mathrm{H}^2(\mathrm{Homeo}_+(\mathsf{S}), \mathbb{Z}) \simeq \mathbb{Z}.$$

We calculate directly the obstruction cocycle for this extension, which happens to be bounded. Therefore, it represents a generator of $\mathrm{H}_b^2(\mathrm{Homeo}_+(\mathsf{S}), \mathbb{Z}) \simeq \mathbb{Z}$. Consider the rotation element $\rho : \mathrm{Homeo}_+(\mathsf{S}) \longrightarrow \mathsf{S}$ as introduced in the work of A. Verjovsky and M. Cruz-López (see [8]). If $\varphi : \mathbb{Q} \longrightarrow \mathrm{Homeo}_+(\mathsf{S})$ is any homomorphism, we prove that the corresponding class

$$\varphi^*(eu) \in \mathrm{H}_b^2(\mathbb{Q}, \mathbb{Z}) \simeq \mathsf{S}$$

is the rotation element $\rho(\varphi(1))$ if and only if $\varphi(1)$ has ρ-bounded motion.

We would like to mention that the group of homeomorphisms of solenoids was studied in extension from the topological point of view by J. Keesling [13], his work contains the first decomposition of the group using translations and automorphisms. The idea of lifts to $\mathbb{R} \times \widehat{\mathbb{Z}}$ refers to the work of J. Kwapisz [14], who also describes the subgroup of automorphisms of the p-adic solenoid S_p, and proves some results in the context of isotopy classes. Finally, the study of the group of homeomorphisms of the 2-dimensional universal solenoid is contained in the work of C. Odden [18]; his results can be abstracted to the case of our interest

$$\mathrm{Homeo}(\mathsf{S}) \cong \mathrm{Homeo}_{\mathcal{L}_0}(\mathsf{S}) \times_{\mathbb{Z}} \widehat{\mathbb{Z}},$$

where $\mathrm{Homeo}_{\mathcal{L}_0}(\mathsf{S})$ denotes the subgroup of homeomorphisms that preserves the base leaf, and $\widehat{\mathbb{Z}}$ is identified with the subgroup of translations along the fiber.

The article is organized as follows: in section 2, we introduce the one-dimensional universal solenoid S. In section 3, we mention the main results about the group of homeomorphisms of the solenoid $\mathrm{Homeo}(\mathsf{S})$ that exist in the literature. Then, we concentrate on the study of $\mathrm{Homeo}_+(\mathsf{S})$. This section also contains the proof of the homotopy equivalence of $\mathrm{Homeo}_+(\mathsf{S})$. At the end, we calculate $\mathrm{H}^2(\mathrm{Homeo}_+(\mathsf{S}), \mathbb{Z})$, $\mathrm{H}_b^2(\mathrm{Homeo}_+(\mathsf{S}), \mathbb{Z})$ and relate the rotation element as we mentioned before with a cohomology class in $\mathrm{H}_b^2(\mathbb{Q}, \mathbb{Z})$.

2. The solenoid

For every integer $n \geq 1$, the circle $\mathbb{S}^1$ is identified as $\mathbb{R}/n\mathbb{Z}$, with covering map $\pi_n : \mathbb{R} \longrightarrow \mathbb{S}^1$ given by $x \longmapsto x + n\mathbb{Z}$. If $n, m \in \mathbb{N}$ and n divides m, there is a unique covering map $p_{nm} : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ such that $\pi_n = p_{nm} \circ \pi_m$,

$$\begin{tikzcd} & \mathbb{R} \arrow[dl, "\pi_n"'] \arrow[d, "\pi_m"] \\ \mathbb{S}^1 & \mathbb{S}^1. \arrow[l, "p_{nm}"'] \end{tikzcd}$$

Thus, $\{\mathbb{S}^1, p_{nm}\}$ defines an inverse system of compact Abelian groups and continuous homomorphisms. The inverse limit

$$\mathsf{S} := \varprojlim \mathbb{S}^1,$$

is called the *universal one-dimensional solenoid*. S is a compact connected Abelian group.

There is an injective and continuous map $P : \mathbb{R} \longrightarrow \mathsf{S}$ defined by $P(x) = (\pi_n(x))$, whose image is the path-component of the identity element $0 \in \mathsf{S}$ and it is a dense subset of S. We call this path-component the *base leaf* and use the notation $\mathcal{L}_0 = P(\mathbb{R})$. The projection onto the first coordinate $\mathsf{S} \longrightarrow \mathbb{S}^1$ defines a principal $\widehat{\mathbb{Z}}$-bundle, with $\widehat{\mathbb{Z}} := \varprojlim \mathbb{Z}/n\mathbb{Z}$ the profinite completion of $\mathbb{Z}$. $\widehat{\mathbb{Z}}$ is a Cantor group and admits a canonical inclusion of $\mathbb{Z}$ whose image is dense.

S can also be identified with the quotient space $\mathbb{R} \times_{\mathbb{Z}} \widehat{\mathbb{Z}}$, where $\mathbb{Z}$ acts by covering transformations on the first coordinate and by translations on the second. Explicitly, for every $\gamma \in \mathbb{Z}$,

$$\gamma \cdot (x, k) := (x + \gamma, k - \gamma)$$

defines a properly discontinuous free action on the product space $\mathbb{R} \times \widehat{\mathbb{Z}}$ whose orbit space is $\mathsf{S} \cong \mathbb{R} \times_{\mathbb{Z}} \widehat{\mathbb{Z}}$.

Observe that the base leaf $\mathcal{L}_0$ coincides with the image of $\mathbb{R} \times \mathbb{Z}$ under the canonical projection $\mathbb{R} \times \widehat{\mathbb{Z}} \longrightarrow \mathsf{S}$. It can also be proved that S is a one-dimensional foliated space whose leaves are homeomorphic to $\mathbb{R}$, and every fiber is homeomorphic to $\widehat{\mathbb{Z}}$.

3. Homeomorphisms of the solenoid

Let $\mathrm{Homeo}(\mathsf{S})$ be the group of homeomorphisms of the solenoid. J. Keesling [13] shows a topological decomposition of this group. Two fundamental examples of homeomorphisms of solenoids are given by translations along the leaves and translations along the fiber; explicitly, let $y \in \mathcal{L}_0$ and $d_y \in \mathrm{Homeo}(\mathsf{S})$ be the *translation along the leaf* $\mathcal{L}_0$ defined by $d_y(z) = z + y$; observe that d_y is an homeomorphism that preserves $\mathcal{L}_0$. For any $w \in \widehat{\mathbb{Z}}$ define the *translation along the fiber* $s_w \in \mathrm{Homeo}(\mathsf{S})$ as $s_w(z) = z + w$; that is, there is a canonical inclusion $\widehat{\mathbb{Z}} \hookrightarrow \mathrm{Homeo}(\mathsf{S})$.

The example of translations along the leaves is an example of an important kind of homeomorphisms: let $\mathrm{Homeo}_{\mathcal{L}_0}(\mathsf{S})$ be the subgroup of homeomorphisms that preserves the base leaf $\mathcal{L}_0$. It is easy to see that

$$\mathrm{Homeo}_{\mathcal{L}_0}(\mathsf{S}) \cap \widehat{\mathbb{Z}} = \mathbb{Z},$$

that is for every $\gamma \in \mathbb{Z}$, $d_\gamma \equiv s_\gamma$. Then, directly from the work of C. Odden [18] the complete group of homeomorphisms of S can be described as

$$\mathrm{Homeo}(\mathsf{S}) \cong \mathrm{Homeo}_{\mathcal{L}_0}(\mathsf{S}) \times_{\mathbb{Z}} \widehat{\mathbb{Z}};$$

where the action of $\mathbb{Z}$ is defined by

$$\gamma \cdot (h, s_w) = (d_\gamma \circ h, s_w \circ s_\gamma) = (d_\gamma \circ h, s_{w+\gamma}) \qquad (\gamma \in \mathbb{Z}).$$

If $\mathrm{Homeo}_+(\mathsf{S})$ is the subgroup of *homeomorphisms of the solenoid that are isotopic to the identity*, the mapping class group of homeomorphisms that preserves the base leaf $\Gamma_{\mathcal{L}_0} := \mathrm{Homeo}_{\mathcal{L}_0}(\mathsf{S})/\mathrm{Homeo}_+(\mathsf{S})$ is isomorphic to the subgroup of automorphisms $\mathrm{Aut}(\mathsf{S}) \cong \mathrm{Aut}(\mathbb{Q}) \cong \mathbb{Q}^*$. J. Kwapisz [14] shows explicitly the form of every automorphism of the solenoid. He also proves that every homeomorphism in $\mathrm{Homeo}_{\mathcal{L}_0}(\mathsf{S})$ is isotopic to an automorphism and that every two translations along the fiber are isotopic if and only if they differ by an integer translation. The case of lifts of elements in $\mathrm{Homeo}_+(\mathsf{S})$ to the product space $\mathbb{R} \times \widehat{\mathbb{Z}}$ is considered in the following.

3.1. Limit-periodic displacements

Recall that S is the orbit space of $\mathbb{R} \times \widehat{\mathbb{Z}}$ under the $\mathbb{Z}$-action

$$\gamma \cdot (x, k) = (x + \gamma, k - \gamma) \qquad (\gamma \in \mathbb{Z}).$$

Let $\Pi : \mathbb{R} \times \widehat{\mathbb{Z}} \longrightarrow \mathsf{S}$ denote the canonical projection. Then, Π is an infinite cyclic covering [17].

Let $\Delta(\mathbb{Z})$ be the group of *deck transformations* on $\mathbb{R}\times\mathbb{Z}$; namely,

$$\Delta(\mathbb{Z}) := \left\{ (x,k) \overset{\Delta(\gamma)}{\longmapsto} (x-\gamma, k+\gamma) \,:\, \gamma\in\mathbb{Z} \right\} \simeq \mathbb{Z}.$$

Assume that $\Phi:\mathbb{R}\times\widehat{\mathbb{Z}}\longrightarrow\mathbb{R}$ is a bounded continuous function that is $\Delta(\mathbb{Z})$-periodic; *i.e.* for every $\gamma\in\mathbb{Z}$, $\Phi(x-\gamma,k+\gamma)=\Phi(x,k)$. For a fixed $k\in\widehat{\mathbb{Z}}$ we define $\Phi_k:=\Phi(\cdot,k):\mathbb{R}\longrightarrow\mathbb{R}$. We are interested in the case when Φ_k is a limit-periodic function whose convex hull is isomorphic to S, with respect to the compact-open topology in the Banach space $\mathcal{C}(\mathbb{R})$ (see [4]).

According to Bohr (see for example [5] or [7]), the mean value of the function Φ_k exists and is equal to

$$\mathrm{M}\{\Phi_k\} := \lim_{X\longrightarrow\infty}\frac{1}{X}\int_{-X/2}^{X/2}\Phi_k(x)\mathrm{d}x.$$

If $y\in\mathbb{R}$, define the translation $\mathrm{T}_y:\mathbb{R}\longrightarrow\mathbb{R}$ by $\mathrm{T}_y(x)=x+y$, then $\mathrm{M}\{\Phi_k\circ\mathrm{T}_y\}=\mathrm{M}\{\Phi_k\}$. In particular, given $\gamma\in\mathbb{Z}$ we have that $\mathrm{M}\{\Phi_k\circ\mathrm{T}_\gamma\}=\mathrm{M}\{\Phi_k\}$. Using the equivariance of Φ_k with respect to the $\mathbb{Z}$-action, for every $k\in\widehat{\mathbb{Z}}$, $\gamma\in\mathbb{Z}$ and $x\in\mathbb{R}$, we have the relation $\Phi_k(x+\gamma)=\Phi_{k+\gamma}(x)$. In consequence, for each $k\in\widehat{\mathbb{Z}}$,

$$\mathrm{M}\{\Phi_{k+\gamma}\}=\mathrm{M}\{\Phi_k\},\quad \forall\gamma\in\mathbb{Z}.$$

Lemma 3.1. *For every* $k\in\widehat{\mathbb{Z}}$, $\mathrm{M}\{\Phi_k\}=\mathrm{M}\{\Phi_0\}$.

Proof. We have that $\mathrm{M}\{\Phi_0\}=\mathrm{M}\{\Phi_\gamma\}$ for every $\gamma\in\mathbb{Z}$. If $k\in\widehat{\mathbb{Z}}$ is given, let $\{\gamma_n\}_{n\in\mathbb{Z}}\subset\widehat{\mathbb{Z}}$ be a sequence that converges uniformly to k. Thus, the collection of functions $\{\Phi_{\gamma_n}\}$ converges uniformly to the function Φ_k and consequently

$$\mathrm{M}\{\Phi_0\}=\lim\mathrm{M}\{\Phi_{\gamma_n}\}=\mathrm{M}\{\Phi_k\}.\qquad\square$$

As a consequence of this result, the behavior of the original function $\Phi:\mathbb{R}\times\widehat{\mathbb{Z}}\longrightarrow\mathbb{R}$ is completely determined by its mean value over the marked leaf $\mathbb{R}\times\{0\}$; that is, Φ can be written in a unique way as

$$\Phi(x,k)=\mathrm{M}\{\Phi_0\}+\Psi(x,k);$$

where $\Psi:\mathbb{R}\times\widehat{\mathbb{Z}}\longrightarrow\mathbb{R}$ is a $\Delta(\mathbb{Z})$-periodic, bounded continuous function, $\Psi\equiv\Phi-\mathrm{M}\{\Phi_0\}$ and for every $k\in\widehat{\mathbb{Z}}$, $\mathrm{M}\{\Psi_k\}=0$.

3.2. Lifts of the identity component

Let $F:\mathbb{R}\times\widehat{\mathbb{Z}}\longrightarrow\mathbb{R}\times\widehat{\mathbb{Z}}$ be a lift of $f\in\mathrm{Homeo}_+(\mathsf{S})$ to $\mathbb{R}\times\widehat{\mathbb{Z}}$. Then, following [14] F has the form

$$F(x,k)=(x+\Phi(x,k),k+\alpha);$$

where $\Phi:\mathbb{R}\times\widehat{\mathbb{Z}}\longrightarrow\mathbb{R}$ is a $\Delta(\mathbb{Z})$-periodic, bounded continuous function and $\alpha\in\mathbb{Z}\hookrightarrow\widehat{\mathbb{Z}}$; that is, for every $k\in\widehat{\mathbb{Z}}$, the function $\mathbb{R}\longrightarrow\mathbb{R}$ defined by $x\longmapsto x+\Phi_k(x)$ has limit-periodic displacement and the function $\widehat{\mathbb{Z}}\to\widehat{\mathbb{Z}}$ defined by $k\longmapsto k+\alpha$ is a minimal translation. Denote by $\widetilde{\mathrm{Homeo}}_+(\mathsf{S})$ the set of all lifts of homeomorphisms in $\mathrm{Homeo}_+(\mathsf{S})$; it is easy to see that $\widetilde{\mathrm{Homeo}}_+(\mathsf{S})$ is a group.

Using similar ideas as in the proof of the homotopy equivalence of $\mathrm{Homeo}_+(\mathbb{S}^1)$ by É. Ghys (see [9]), we obtain the following result.

Proposition 3.2. *The inclusion by translations on the base leaf*

$$\mathbb{R}\times_{\mathbb{Z}}\mathbb{Z}\hookrightarrow\mathrm{Homeo}_+(\mathsf{S})$$

is a homotopy equivalence.

Proof. Take $f\in\mathrm{Homeo}_+(\mathsf{S})$ and $F\in\widetilde{\mathrm{Homeo}}_+(\mathsf{S})$ a lift of f of the form

$$F(x,k)=(x+\Phi(x,k),k+\alpha)\qquad(\alpha\in\mathbb{Z}).$$

By the above remarks Φ can be written as $\Phi\equiv\mathrm{M}\{\Phi_0\}+\Psi$, where $\Psi_k:\mathbb{R}\longrightarrow\mathbb{R}$ has zero mean value for each $k\in\widehat{\mathbb{Z}}$.

We can define an homotopy through the functions

$$F_s(x,k)=(x+\mathrm{M}\{\Phi_0\}+(1-s)\Psi(x,k),k+\alpha),\quad 0\le s\le 1.$$

That is, for every $s \in [0,1]$, we have that $F_s \in \widetilde{\mathrm{Homeo}_+}(\mathsf{S})$; or equivalently, the set consisting of the $\Delta(\mathbb{Z})$-periodic, bounded continuous functions $\Psi : \mathbb{R} \times \widehat{\mathbb{Z}} \longrightarrow \mathbb{R}$, satisfying that for each $k \in \widehat{\mathbb{Z}}$, $\mathrm{M}\{\Psi_k\} = 0$, is a convex set.

Observe also that $F_0 \equiv F$, $F_1(x,k) = (x + \mathrm{M}\{\Phi_0\}, k + \alpha)$ and we obtain a continuous retraction of $\widetilde{\mathrm{Homeo}_+}(\mathsf{S})$ onto the subgroup of translations isomorphic to $\mathbb{R} \times \mathbb{Z}$. Moreover, for each $s \in [0,1]$, F_s commutes with the elements of the group of deck transformations $\Delta(\mathbb{Z})$. Therefore, we can define a continuous deformation of the quotient

$$\widetilde{\mathrm{Homeo}_+}(\mathsf{S})/\Delta(\mathbb{Z}) \simeq \mathrm{Homeo}_+(\mathsf{S})$$

on the subgroup of translations over the base leaf $\mathcal{L}_0 \cong \mathbb{R} \times_{\mathbb{Z}} \mathbb{Z}$. □

Remark 3.3. The last argument shows that we have a topological homeomorphism

$$\mathrm{Homeo}_+(\mathsf{S}) \simeq \mathcal{L}_0 \times \mathrm{H}_{\mathrm{id}}(\mathsf{S});$$

where $\mathrm{H}_{\mathrm{id}}(\mathsf{S})$ is a contractible convex set. Moreover, every homotopy group $\pi_n(\mathrm{Homeo}_+(\mathsf{S}))$ is trivial.

3.3. The isotopy component of the identity

Recall that for any $F \in \widetilde{\mathrm{Homeo}_+}(\mathsf{S})$, there is a $\Delta(\mathbb{Z})$-periodic, bounded continuous function $\Phi : \mathbb{R} \times \widehat{\mathbb{Z}} \longrightarrow \mathbb{R}$ (*i.e.* Φ satisfies the relation

$$\Phi(x+\gamma, k-\gamma) = \Phi(x,k) \qquad (\gamma \in \mathbb{Z})),$$

and an element $\alpha \in \mathbb{Z}$ such that F can be written as

$$F(x,k) = (x + \Phi(x,k), k + \alpha).$$

In particular, there is a continuous function $\tilde{\phi} : \mathsf{S} \longrightarrow \mathbb{R}$ such that the following diagram commutes

$$\begin{tikzcd} \mathbb{R} \times \widehat{\mathbb{Z}} \arrow[r, "\Phi"] \arrow[d, "\Pi"'] & \mathbb{R} \\ \mathsf{S} \arrow[ur, "\tilde{\phi}"'] & \end{tikzcd}.$$

Hence, there is a continuous function $\phi : \mathsf{S} \longrightarrow \mathsf{S}$, satisfying that $\phi(\mathsf{S}) \subset \mathcal{L}_0$ and ϕ completes the diagram

$$\begin{tikzcd} \mathbb{R} \times \widehat{\mathbb{Z}} \arrow[r, "\Phi"] \arrow[d, "\Pi"'] & \mathbb{R} \arrow[d, "\mathrm{P}"] \\ \mathsf{S} \arrow[ur, "\tilde{\phi}"'] \arrow[r, "\phi"'] & \mathsf{S}. \end{tikzcd}$$

Consequently, we have a well-defined map

$$\begin{aligned} \mathfrak{p} : \widetilde{\mathrm{Homeo}_+}(\mathsf{S}) &\longrightarrow \mathrm{Homeo}_+(\mathsf{S}) \\ F &\longmapsto s_\alpha \circ (\mathrm{id} + \phi). \end{aligned}$$

Moreover, $\mathfrak{p}$ is an onto homomorphism whose kernel is identified with the group of deck transformations $\Delta(\mathbb{Z}) \simeq \mathbb{Z}$. For every $\gamma \in \mathbb{Z}$ and every $F \in \widetilde{\mathrm{Homeo}_+}(\mathsf{S})$ we have that for each $(x,k) \in \mathbb{R} \times \widehat{\mathbb{Z}}$:

$$\begin{aligned} \Delta(\gamma)(F(x,k)) &= \Delta(\gamma)(x + \Phi(x,k), k + \alpha) \\ &= (x + \Phi(x,k) - \gamma, k + \alpha + \gamma) \\ &= (x - \gamma + \Phi(x,k), k + \gamma + \alpha) \\ &= (x - \gamma + \Phi(x-\gamma, k+\gamma), k + \gamma + \alpha) \\ &= F(x-\gamma, k+\gamma) = F(\Delta(\gamma)(x,k)). \end{aligned}$$

Equivalently, $\Delta(\mathbb{Z})$ is contained in the center of $\widetilde{\mathrm{Homeo}_+}(\mathsf{S})$ and therefore, we have arrived at the central extension

$$0 \longrightarrow \mathbb{Z} \longrightarrow \widetilde{\mathrm{Homeo}_+}(\mathsf{S}) \xrightarrow{\mathfrak{p}} \mathrm{Homeo}_+(\mathsf{S}) \longrightarrow 1.$$

From the work of J. Aliste-Prieto and S. Petite (see [3]), we know that $\mathrm{Homeo}_+(\mathsf{S})$ is uniformly perfect; in fact, every element of $\mathrm{Homeo}_+(\mathsf{S})$ can be written as the product of two commutators. Thus, there exists a universal central extension and the kernel of such an extension can be mapped surjectively with $\mathbb{Z}$ (for a detailed study of universal central extensions see for example [16] and the second chapter of [20]).

We focus our attention on lifts F of the form $(x,k) \longmapsto (x + \Phi(x,k), k)$. As we already noted, for every fixed $k \in \widehat{\mathbb{Z}}$, the map $F_k(x) := x + \Phi_k(x)$, has limit-periodic displacement and is strictly increasing. Moreover, F_k is a real-valued homeomorphism and, using the equivariant condition of Φ_k, for every $\gamma \in \mathbb{Z}$

$$F_{k+\gamma}(x) + \gamma = F_k(x+\gamma).$$

Equivalently, F_k and $F_{k+\gamma}$ are conjugated to a translation by γ in $\mathbb{R}$.

If $k \in \widehat{\mathbb{Z}}$, consider the following subgroup of real valued homeomorphisms:

$$\widetilde{\mathrm{Homeo}_+}(\mathrm{S})_k = \left\{ F_k : \mathbb{R} \longrightarrow \mathbb{R} : F \in \widetilde{\mathrm{Homeo}_+}(\mathrm{S}) \right\}.$$

Lemma 3.4. *For every $k \in \widehat{\mathbb{Z}}$, $\widetilde{\mathrm{Homeo}_+}(\mathrm{S})_k$ is homeomorphic to an immersed subgroup of the compactly supported homeomorphisms group* $\mathrm{Homeo}_C(\mathbb{R})$.

Proof. From the proof of $\mathrm{Homeo}_+(\mathrm{S})$ being uniformly perfect (see Theorem 6.1. in [3]), we observe that every $f \in \mathrm{Homeo}_+(\mathrm{S})$ can be written as a product of two commutators, say $f = f_1 \circ f_2$, with f_1 and f_2 in $\mathrm{Homeo}_+(\mathrm{S})$. Moreover, there is a finite collection of boxes $\{[0, \mathrm{t_i}] \times_{\mathbb{Z}} K_i\}_{i=1}^{l}$; where $\{K_i\}$ is a finite partition of the fiber $\widehat{\mathbb{Z}}$, such that this collection forms a cover of the support of f_2 and from the partition property

$$f_1 = g_1 \circ \ldots \circ g_l,$$

with $\mathrm{supp}(g_i) \subset [0, \mathrm{t_i}] \times_{\mathbb{Z}} K_i$.

Hence, take $F(x,k) = (x + \Phi(x,k), k)$ a lift of f. Using the diagonal action, for every fixed $k \in \widehat{\mathbb{Z}}$, there is at most a finite collection of compact sets

$$\{[\gamma_i, \mathrm{t_i} + \gamma_i] : \gamma_i \in \mathbb{Z}\}_{i=1}^{l};$$

with

$$\mathrm{supp}(F_k) \subset \bigcup_{i=1}^{l} [\gamma_i, \mathrm{t_i} + \gamma_i].$$

That is, taking the lifts of the boxes $[0, \mathrm{t_i}] \times_{\mathbb{Z}} K_i$ to $\mathbb{R} \times \widehat{\mathbb{Z}}$, we only consider the closed intervals that intersect the leaf $\mathbb{R} \times \{k\}$ as the ones that support the homeomorphism $F_k : \mathbb{R} \longrightarrow \mathbb{R}$. □

In particular, for every $k \in \widehat{\mathbb{Z}}$, following Mather [15] we have that $\mathrm{H}_n\left(\widetilde{\mathrm{Homeo}_+}(\mathrm{S})_k, \mathbb{Z}\right)$ vanish for all $n \geq 1$.

Theorem 3.5. *The exact sequence*

$$0 \longrightarrow \mathbb{Z} \longrightarrow \widetilde{\mathrm{Homeo}_+}(\mathrm{S}) \xrightarrow{\mathfrak{p}} \mathrm{Homeo}_+(\mathrm{S}) \longrightarrow 1,$$

is the universal central extension.

Proof. For every $k \in \widehat{\mathbb{Z}}$, let Φ_k be a bounded continuous limit-periodic function of the real line. Consider the map $\widehat{\mathbb{Z}} \longrightarrow \mathrm{C}_b^{lp}(\mathbb{R})$ defined as $k \longmapsto \Phi_k$. Using the exponential map, we know that this is a continuous correspondence.

Hence, the map $\widehat{\mathbb{Z}} \longrightarrow \mathrm{Homeo}(\mathbb{R})$, $k \longmapsto F_k$ is also continuous. Moreover, F_k has constant mean value for the displacement Φ_k; that is $\mathrm{M}\{\Phi_k\} = \mathrm{M}\{\Phi_0\}$, with Φ_0 the displacement of F_0.

Thus, we have a continuous map $\widehat{\mathbb{Z}} \longrightarrow \widetilde{\mathrm{Homeo}_+}(\mathrm{S})_k$ such that

$$\mathrm{H}_n\left(\widetilde{\mathrm{Homeo}_+}(\mathrm{S})_k, \mathbb{Z}\right) = 0, \qquad (n \geq 1).$$

Consequently, using that $F_{k+\gamma}$ and F_k are conjugated via a translation by $\gamma \in \mathbb{Z}$, we can assert that $\mathrm{H}_n\left(\widetilde{\mathrm{Homeo}_+}(\mathrm{S}), \mathbb{Z}\right) = 0$ for every $n \geq 1$.

In particular,

$$\mathrm{H}_1\left(\widetilde{\mathrm{Homeo}_+}(\mathrm{S}), \mathbb{Z}\right) = \mathrm{H}_2\left(\widetilde{\mathrm{Homeo}_+}(\mathrm{S}), \mathbb{Z}\right) = 0,$$

which is equivalent to say that

$$0 \longrightarrow \mathbb{Z} \longrightarrow \widetilde{\mathrm{Homeo}_+}(\mathsf{S}) \xrightarrow{\mathfrak{p}} \mathrm{Homeo}_+(\mathsf{S}) \longrightarrow 1$$

is the universal central extension. That is, we know that $\widetilde{\mathrm{Homeo}_+}(\mathsf{S})$ is perfect and that every central exact sequence of $\widetilde{\mathrm{Homeo}_+}(\mathsf{S})$ by $\mathbb{Z}$ must split. □

As a consequence, we have the Schur multiplier

$$\mathrm{H}_2(\mathrm{Homeo}_+(\mathsf{S}), \mathbb{Z}) \simeq \mathbb{Z}.$$

Moreover, using the universal coefficient theorem we have that

$$\mathrm{H}^2(\mathrm{Homeo}_+(\mathsf{S}), \mathbb{Z}) \simeq \mathrm{Hom}(\mathrm{H}_2(\mathrm{Homeo}_+(\mathsf{S}), \mathbb{Z}), \mathbb{Z}) \simeq \mathbb{Z}.$$

Let eu be a generator of this group, that is eu is the Euler class of the given extension.

3.4. The second bounded cohomology

Consider the obstruction cocycle

$$c(f, g) = \sigma(f \circ g)^{-1} \circ \sigma(f) \circ \sigma(g).$$

Note that the lifts $\sigma(f \circ g)$ and $\sigma(f) \circ \sigma(g)$ differ by a deck transformation $\Delta_\gamma \in \Delta(\mathbb{Z})$ for some $\gamma \in \mathbb{Z}$, because both are lifts of the same element $f \circ g \in \mathrm{Homeo}_+(\mathsf{S})$. That is,

$$\sigma(f \circ g) \circ \Delta_\gamma \equiv \sigma(f) \circ \sigma(g).$$

Define $eu : \mathrm{Homeo}_+(\mathsf{S})^2 \longrightarrow \mathbb{Z}$ as $eu(f, g) = \gamma$.

Consider the specific normalized section $\sigma : \mathrm{Homeo}_+(\mathsf{S}) \longrightarrow \widetilde{\mathrm{Homeo}_+}(\mathsf{S})$, such that for every $f \in \mathrm{Homeo}_+(\mathsf{S})$,

$$\sigma(f)(0, 0) \in [0, 1) \times \widehat{\mathbb{Z}}.$$

In particular, $\sigma(f \circ g)(\gamma, -\gamma) \in [\gamma, \gamma + 1) \times \widehat{\mathbb{Z}}$. Thus, $\sigma(g)(0, 0) \in [0, 1) \times \widehat{\mathbb{Z}}$ and because of $\sigma(f)(1, -1) \in [1, 2) \times \widehat{\mathbb{Z}}$, we conclude that $\sigma(f) \circ \sigma(g)(0, 0) \in [0, 2) \times \widehat{\mathbb{Z}}$. Therefore γ is equal to 0 or 1.

Consequently, the Euler class $eu \in \mathrm{H}^2(\mathrm{Homeo}_+(\mathsf{S}), \mathbb{Z})$ associated with $c(f, g)$ is bounded and

$$\mathrm{H}^2_b(\mathrm{Homeo}_+(\mathsf{S}), \mathbb{Z}) \simeq \mathbb{Z}.$$

Define this class as $eu_b = [c(f, g)] \in \mathrm{H}^2_b(\mathrm{Homeo}_+(\mathsf{S}), \mathbb{Z})$.

From the work of A. Verjovsky and M. Cruz-López [8], we know that there is a rotation element

$$\rho : \mathrm{Homeo}_+(\mathsf{S}) \longrightarrow \mathsf{S},$$

which can be defined using the notion of asymptotic cycle of Schwartzmann.

Specifically, let $f \in \mathrm{Homeo}_+(\mathsf{S})$ and define the suspension space of f as

$$\Sigma_f(\mathsf{S}) := \mathsf{S} \times [0, 1]/(z, 1) \sim (f(z), 0).$$

Thus, $\Sigma_f(\mathsf{S}) \cong \mathsf{S} \times \mathbb{S}^1$ is a compact Abelian topological group whose character group is isomorphic to $\mathbb{Q} \times \mathbb{Z}$.

For any character, it is obtained a 1-cocyle and by Birkhoff's ergodic theorem, there is a well-defined homomorphism $H_f : \mathbb{Q} \times \mathbb{Z} \longrightarrow \mathbb{R}$, such that its natural projection to $\mathbb{S}^1$ determines an element in the character group of $\mathbb{Q} \times \mathbb{Z}$. That is, using Pontryagin's duality, there is a well-defined element $\rho(f) \in \mathsf{S} \times \mathbb{S}^1$. Moreover, this element does not depend on the second coordinate.

Furthermore, if $f = \mathrm{id} + \phi$, with displacement function ϕ, the element $\rho(f)$ is identified with the element $\int_{\mathsf{S}} \phi \, \mathrm{d}\mu \in \mathsf{S}$, determined by the character of $\mathbb{Q}$ given by

$$q \longmapsto \exp\left(2\pi \mathrm{i} q \int_{\mathsf{S}} \phi \mathrm{d}\mu\right);$$

where μ is an f-invariant measure (see Remark 3.3 in [8]).

Remark 3.6. This element was also found in the work of J. Aliste-Prieto [1] using that for any f-invariant measure μ,

$$\int_{\mathsf{S}} \phi \mathrm{d}\mu = \lim_{n \longrightarrow \infty} \frac{1}{n} \sum_{i=0}^{n} \phi(f^i(z))$$

for μ-almost every point $z \in \mathsf{S}$. However, our approach seems more direct and without the use of specific dynamics. Even so, our results are still valid for this definition of the rotation element.

It turns out that this element can be stated in the language of bounded cohomology.

Lemma 3.7. $\mathrm{H}_b^2(\mathbb{Q}, \mathbb{Z}) \simeq \mathsf{S}$.

Proof. Let $0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{R} \longrightarrow \mathbb{S}^1 \longrightarrow 1$ be the exponential exact sequence and construct the associated long exact sequence in bounded cohomology

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{R} \longrightarrow \mathbb{S}^1 \longrightarrow \mathrm{H}_b^1(\mathbb{Q}, \mathbb{Z}) \longrightarrow \mathrm{H}_b^1(\mathbb{Q}, \mathbb{R}) \longrightarrow \mathrm{H}_b^1(\mathbb{Q}, \mathbb{S}^1) \longrightarrow$$
$$\longrightarrow \mathrm{H}_b^2(\mathbb{Q}, \mathbb{Z}) \longrightarrow \mathrm{H}_b^2(\mathbb{Q}, \mathbb{R}) \longrightarrow \mathrm{H}_b^2(\mathbb{Q}, \mathbb{S}^1) \longrightarrow \cdots .$$

Using that $\mathbb{Q}$ is Abelian and therefore is an amenable group $\mathrm{H}_b^n(\mathbb{Q}, \mathbb{R}) = 0$ for every $n \geq 1$ (see [11] or [12] for details). Calculating these groups for $n = 1$ and $n = 2$, we obtain that

$$\mathrm{H}_b^2(\mathbb{Q}, \mathbb{Z}) \simeq \mathrm{H}_b^1(\mathbb{Q}, \mathbb{S}^1) \simeq \mathrm{Hom}(\mathbb{Q}, \mathbb{S}^1) \simeq \mathsf{S}. \qquad \square$$

Finally, we say that f has *ρ-bounded motion (or bounded mean variation)* if there exists a constant $C > 0$ such that $|f^n(z) - z - n\rho(f)| < C$, for every $z \in \mathsf{S}$ and $n \geq 0$. From both [8] and [1], we know that a necessary and sufficient condition in order to have $f \in \mathrm{Homeo}_+(\mathsf{S})$ semi-conjugated to the translation by $\rho(f)$, is that f has ρ-bounded motion (see also [2] for more details on the semi-conjugacy problem). Thus, we have proved the following.

Theorem 3.8. *Let $\varphi : \mathbb{Q} \longrightarrow \mathrm{Homeo}_+(\mathsf{S})$ be a homomorphism. The class $\varphi^*(eu_b) \in \mathrm{H}_b^2(\mathbb{Q}, \mathbb{Z})$ is the rotation element $\rho(\varphi(1))$ if and only if $\varphi(1)$ has ρ-bounded motion.*

Remark 3.9. The natural question of finding this rotation element as in the case of quasimorphisms is postponed for a work in progress.

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