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Algebraic geometry

The boundary of the orbit of the 3-by-3 determinant polynomial ☆

La frontière de l'orbite du polynôme déterminant 3 par 3

Jesko Hüttenhain, Pierre Lairez

Technische Universität Berlin, Straße des 17. Juni 136, 10623 Berlin, Germany

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ABSTRACT

We consider the 3×3 determinant polynomial and we describe the limit points of the set of all polynomials obtained from the determinant polynomial by linear change of variables. This answers a question of Joseph M. Landsberg.

RÉSUMÉ

Nous étudions le polynôme donné par le déterminant 3×3 et décrivons l'adhérence de l'ensemble des polynômes obtenus par changements de variables linéaires à partir de ce déterminant, ce qui répond à une question de Joseph M. Lansberg.

0. Introduction

Mulmuley and Sohoni [11] propose, in their *geometric complexity theory*, to study the geometry of the orbit closure of some polynomials under linear change of variables, and especially, the determinant polynomial. Yet, very few explicit results describing the geometry are known in low dimension. The purpose of this work is to describe the boundary of the orbit of the 3×3 determinant, that is, the set of limit points of the orbit that are not in the orbit.

Let $\det_3$ be the polynomial

$$\det_3 \stackrel{\text{def}}{=} \det \begin{pmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{pmatrix} \in \mathbb{C}[x_1, \dots, x_9],$$

which we consider as a homogeneous form of degree 3 on the space $\mathbb{C}^{3\times 3}$ of 3×3 matrices, denoted W. Let $\mathbb{C}[W]_3$ denote the 165-dimensional space of all homogeneous forms of degree 3 on W. The group $\mathrm{GL}(W)$ acts on $\mathbb{C}[W]_3$ by right

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E-mail addresses: jesko@math.tu-berlin.de (J. Hüttenhain), pierre@lairez.fr (P. Lairez).

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composition. For a nonzero $P \in \mathbb{C}[W]_3$, let $\Omega(P)$ denote the (projective) orbit of P, namely the set of all $[P \circ a] \in \mathbb{P}(\mathbb{C}[W]_3)$, with $a \in \mathrm{GL}(W)$. The *boundary* of the orbit of P, denoted $\partial\Omega(P)$, is $\overline{\Omega(P)} \setminus \Omega(P)$, where $\overline{\Omega(P)}$, denoted also $\overline{\Omega}(P)$, is the Zariski closure of the orbit in $\mathbb{P}(\mathbb{C}[W]_3)$.

Our main result is a description of $\partial\Omega(\det_3)$ that answers a question of Landsberg [10, Problem 5.4]: the two known components are the only ones. In §1, we explain the construction of the two components. Our contribution lies in §2 where we show that there is no other component.

Theorem 1. *The boundary $\partial\Omega(\det_3)$ has exactly two irreducible components:*

- *The orbit closure of the determinant of the generic traceless matrix, namely*

$$P_1 \stackrel{\text{def}}{=} \det \begin{pmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & -x_1 - x_5 \end{pmatrix};$$

- *The orbit closure of the universal homogeneous polynomial of degree two in three variables, namely*

$$P_2 \stackrel{\text{def}}{=} x_4 \cdot x_1^2 + x_5 \cdot x_2^2 + x_6 \cdot x_3^2 + x_7 \cdot x_1 x_2 + x_8 \cdot x_2 x_3 + x_9 \cdot x_1 x_3.$$

The two components are different in nature: the first one is the orbit closure of a polynomial in only eight variables and is included in the orbit of $[\det_3]$ under the action of $\mathrm{End}\, W$; the second is more subtle and is not included in the $\mathrm{End}\, W$-orbit of $[\det_3]$. Both components have analogues in higher dimension and some results are known about them [9].

1. Construction of two components of the boundary

For $P \in \mathbb{C}[W]_3 \setminus \{0\}$, let $H(P) \subset \mathrm{GL}(W)$ denote its stabilizer, that is

$$H(P) \stackrel{\text{def}}{=} \{a \in \mathrm{GL}(W) \mid P \circ a = P\}.$$

The stabilizer $H(\det_3)$ is generated by the transposition map $A \mapsto A^{\mathsf{T}}$ and the maps $A \mapsto UAV$, with U and V in $\mathrm{SL}(\mathbb{C}^3)$ [3].

Lemma 2. *For any $P \in \mathbb{C}[W]_3$, $\dim \Omega(P) = 80 - \dim H(P)$. In particular, $\dim \Omega(\det_3) = 64$ and $\dim \Omega(P_1) = \dim \Omega(P_2) = 63$.*

Proof. An easy application of the fiber dimension theorem to the map $a \in \mathrm{GL}(W) \mapsto P \circ a \in \mathbb{C}[W]_3$ gives that the dimension of the orbit of P in $\mathbb{C}[W]_3$ is $81 - \dim H(P)$. Since the projective orbit in $\mathbb{P}(\mathbb{C}[W]_3)$ has one dimension less, the first claim follows.

The stabilizer $H(\det_3)$ has dimension 16, hence $\dim \Omega(\det_3) = 64$. To compute the dimension of $H(P_i)$, $1 \leqslant i \leqslant 2$, one can compute the dimension of its Lie algebra defined as

$$T_1 H(P_i) = \left\{ a \in \mathrm{End}(W) \;\middle|\; P(x + ta(x)) = P(x) + O(t^2) \right\}.$$

It amounts to computing the nullspace of a 165×81 matrix, which is easy using a computer. $\square$

Lemma 3. *The boundary $\partial\Omega(\det_3)$ is pure of dimension* 63.

Proof. Let $\Omega'(\det_3)$ be the affine orbit of $\det_3$ in $\mathbb{C}[W]_3$ under the action of $\mathrm{GL}(W)$. It is isomorphic to $\mathrm{GL}(W)/H(\det_3)$, which is an affine variety because $H(\det_3)$ is reductive [12, §4.2]. Therefore $\Omega'(\det_3)$ is an affine open subset of its closure, it follows that the complement of $\Omega'(\det_3)$ in its closure is pure of codimension 1 [7, Corollaire 21.12.7], and the same holds true after projectivization. $\square$

Let φ be the rational map

$$\varphi : [a] \in \mathbb{P}(\mathrm{End}\, W) \dashrightarrow [\det_3 \circ a] \in \overline{\Omega}(\det_3). \tag{1}$$

Let also Z be the irreducible hypersurface of $\mathbb{P}(\mathrm{End}\, W)$

$$Z \stackrel{\text{def}}{=} \{[a] \in \mathbb{P}(\mathrm{End}\, W) \mid \det(a) = 0\}.$$

Note the difference between $\det_3 \circ a$, which is a regular function of W, and $\det(a)$, which is a scalar. The indeterminacy locus of φ is a strict subset of Z. By definition, $\Omega(\det_3) = \varphi(\mathbb{P}(\mathrm{End}\, W) \setminus Z)$. Let $\varphi(Z)$ denote the image of the set of the points of Z where φ is defined.

Lemma 4. *The closure $\overline{\varphi(Z)}$ is an irreducible component of $\partial\Omega(\det_3)$. Furthermore $\overline{\varphi(Z)} = \overline{\Omega}(P_1)$.*

Proof. The closure $\overline{\varphi(Z)}$ is clearly contained in $\overline{\Omega}(\det_3)$ since $\mathrm{GL}(W)$ is dense in $\mathrm{End}(W)$. The image $\varphi(Z)$ does not intersect $\Omega(\det_3)$: to show this, let us consider the function $\nu : \mathbb{C}[W]_3 \to \mathbb{N}$, which associates with P the dimension of the linear subspace of $\mathbb{C}[W]_2$ spanned by the partial derivatives $\frac{\partial P}{\partial x_1}, \dots, \frac{\partial P}{\partial x_9}$. The function ν is invariant under the action of $\mathrm{GL}(W)$. Because every form in $\varphi(Z)$ can be written as a polynomial in at most eight linear forms, $\nu(P) \leqslant 8$ for all $P \in \varphi(Z)$. On the other hand, $\nu(\det_3) = 9$ and so $\nu(P) = 9$ for any $P \in \Omega(\det_3)$. This shows that $\varphi(Z) \cap \Omega(\det_3) = \varnothing$. Thus $\overline{\varphi(Z)}$ is contained in the boundary $\partial\Omega(\det_3)$. Moreover, $\overline{\varphi(Z)}$ is irreducible because Z is.

Clearly $P_1 \in \varphi(Z)$ and by Lemma 2, $\Omega(P_1)$ has dimension 63. Since

$$\overline{\Omega}(P_1) \subset \overline{\varphi(Z)} \subset \partial\Omega(\det_3),$$

they all three have dimension 63 and $\overline{\Omega}(P_1) = \overline{\varphi(Z)}$ because the latter is irreducible. This gives a component of $\partial\Omega(\det_3)$. □

Lemma 5. *The orbit closure $\overline{\Omega}(P_2)$ is an irreducible component of $\partial\Omega(\det_3)$ and is distinct from $\overline{\Omega}(P_1)$.*

Proof. We first prove that $[P_2] \in \partial\Omega(\det_3)$. Let

$$A = \begin{pmatrix} 0 & x_1 & -x_2 \\ -x_1 & 0 & x_3 \\ x_2 & -x_3 & 0. \end{pmatrix} \text{ and } S = \begin{pmatrix} 2x_6 & x_8 & x_9 \\ x_8 & 2x_5 & x_7 \\ x_9 & x_7 & 2x_4 \end{pmatrix}.$$

By Jacobi's formula, $\det(A + tS) = \det A + \mathrm{Tr}(\mathrm{adj}(A)S)t + o(t)$, where $\mathrm{adj}(A)$ is the adjugate matrix of A, which equals $u^{\mathrm{T}}u$ with $u = (x_3, x_2, x_1)$. Since $\det(A) = 0$, the projective class of the polynomial $\det(A+tS)$ tends to $[\mathrm{Tr}(\mathrm{adj}(A)S)]$ when $t \to 0$, and by construction, this limit is a point in $\overline{\Omega}(\det_3)$. Besides

$$\mathrm{Tr}(\mathrm{adj}(A)S) = uSu^T = 2P_2,$$

thus $[P_2] \in \overline{\Omega}(\det_3)$. Yet $[P_2]$ is not in $\Omega(\det_3)$, because its orbit has dimension 63, by Lemma 2, whereas the orbit of every point of $\Omega(\det_3)$ is $\Omega(\det_3)$ itself. Therefore $[P_2]$ is in the boundary $\partial\Omega(\det_3)$. Since $\Omega(P_2)$ has dimension 63, this gives a component of $\partial\Omega(\det_3)$. It remains to show that $[P_2]$ is not in $\Omega(P_1)$, and indeed $\nu(P_2) = 9$ whereas $\nu(P_1) = 8$, where ν is the function introduced in the proof of Lemma 4. □

Note that Lemma 5 generalizes to higher dimensions: the limit of the determinant on the space of skew-symmetric matrices always leads to a component of the boundary of the orbit of $\det_n$, when $n \geqslant 3$ is odd, as shown by Landsberg, Manivel, and Ressayre [9, Prop. 3.5.1].

2. There are only two components

Let E denote $\mathrm{End}(W)$ and recall the rational map $\varphi : \mathbb{P}(E) \dashrightarrow \overline{\Omega}(\det_3)$ defined in (1). Let $B \subset \mathbb{P}(E)$ denote the indeterminacy locus of φ, that is, the set of all $[a] \in \mathbb{P}(E)$ whose image $a(W) \subset W$ contains only singular matrices. The locus B is a subset of Z because every a not in Z is surjective and thus has invertible matrices in its image. One way to describe the orbit closure $\overline{\Omega}(\det_3)$ is to give a resolution of the indeterminacies of the rational map φ, that is a, projective birational morphism $\rho : X \to \mathbb{P}(E)$ such that $\varphi \circ \rho$ is a regular map. In this case, the regular map $\varphi \circ \rho$ is projective and therefore its image is closed and equals $\overline{\Omega}(\det_3)$. As we will see, it is actually enough to resolve the indeterminacies of φ on some open subset of $\mathbb{P}(E)$.

Let $H = H(\det_3) \subset \mathrm{GL}(W)$ denote the stabilizer of $\det_3$ described above. The group H acts on $\mathbb{P}(E)$ by left multiplication and the rational map φ is H-invariant: for $a \in \mathrm{End}(W)$ and $h \in H$, $\varphi([ha]) = [\det_3 \circ h \circ a] = \varphi([a])$. Let $\mathbb{P}(E)^{\mathrm{ss}}$ be the open subset of all semistable points in $\mathbb{P}(E)$ under the action of H, that is the set of all $[a] \in \mathbb{P}(E)$ such that there exists a non-constant homogeneous H-invariant regular function $f \in \mathbb{C}[E]^H$ on E such that $f(a) \neq 0$. Equivalently [12, §4.6], the complement of $\mathbb{P}(E)^{\mathrm{ss}}$ is the set of all $[a] \in \mathbb{P}(E)$ such that 0 is in the closure of Ha in E. Let X be the closure in $\mathbb{P}(E)^{\mathrm{ss}} \times \overline{\Omega}(\det_3)$ of the graph of the rational map φ, namely

$$X \overset{\text{def}}{=} \mathrm{Closure}\left\{([a], [P]) \in \mathbb{P}(E)^{\mathrm{ss}} \times \overline{\Omega}(\det_3) \,\middle|\, [P] = [\det_3 \circ a]\right\}.$$

Let $\rho : X \to \mathbb{P}(E)^{\mathrm{ss}}$ denote the first projection. By construction, it is the blowup of $\mathbb{P}(E)^{\mathrm{ss}}$ along the ideal sheaf defined by the condition $\det_3 \circ a = 0$, whose support is the indeterminacy locus $B \cap \mathbb{P}(E)^{\mathrm{ss}}$. (The condition $\det_3 \circ a = 0$ expands into 165 homogeneous polynomials of degree 3 in the 81 coordinates of a.)

The variety X also carries a regular map $\psi : X \to \overline{\Omega}(\det_3)$ given by the second projection. By construction, it resolves the indeterminacies of φ on $\mathbb{P}(E)^{\mathrm{ss}}$: the rational map $\varphi \circ \rho : X \to \overline{\Omega}(\det_3)$ extends to a regular map which equals ψ.

Lemma 6. $\psi(X) = \overline{\Omega}(\det_3)$.

Proof. The image of φ, which is $\Omega(\det_3)$, is included in $\psi(X)$ and $\psi(X) \subset \overline{\Omega}(\det_3)$. Thus, it is enough to show that $\psi(X)$ is closed.

Let T be the projective variety $\mathbb{P}(E) \times \mathbb{P}(\mathbb{C}[W]_3)$. The group H acts on T by $h \cdot (a, P) = (h \cdot a, P)$. Let T^{ss} the open subset of semi-stable points for this action; clearly $T^{ss} = \mathbb{P}(E)^{ss} \times \mathbb{P}(\mathbb{C}[W]_3)$. The GIT quotient $T^{ss}//H$ is a projective variety and the canonical morphism $\pi : T^{ss} \to T^{ss}//H$ maps H-invariant closed subsets to closed subsets (e.g., [12, §4.6]); in particular, $\pi(X)$ is closed. Moreover, the map ψ is H-invariant, so it factors as $\psi' \circ \pi$ for some regular map $\psi' : T^{ss}//H \to \mathbb{P}(\mathbb{C}[W]_3)$. The image $\pi(X)$ is closed in the projective variety $T^{ss}//H$, thus $\psi'(\pi(X))$ is closed. This proves the claim, since the latter is just $\psi(X)$. □

The construction of X follows a general method to resolve the indeterminacies of a rational map, and as such, it gives little information. In fact X is a blowup of $\mathbb{P}(E)^{ss}$ along a smooth variety.

First of all, the indeterminacy locus B is precisely known, thanks to the classification of the maximal linear subspaces of E containing only singular matrices [1,6,4]. Let H^0 denote the connected component of 1 in H — due to the transposition map, H has two components. For every $[a] \in B$, there is a $h \in H^0$ such that $(ha)(W)$ is a subset of one of the following spaces of singular matrices:

$$\begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ * & * & 0 \end{pmatrix}, \begin{pmatrix} * & * & * \\ * & 0 & 0 \\ * & 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & \alpha & -\beta \\ -\alpha & 0 & \gamma \\ \beta & -\gamma & 0 \end{pmatrix}, \ \alpha, \beta, \gamma \in \mathbb{C}.$$

The first three are called *compression spaces*, and the fourth is the space of 3×3 skew-symmetric matrices, denoted Λ_3. They give four components of B. Let B_1, B_2, B_3 and B_{skew} denote them, respectively. For example

$$B_{\text{skew}} = \left\{ [a] \in \mathbb{P}(E) \;\middle|\; \exists U, V \in \mathrm{SL}(\mathbb{C}^3) : \forall p \in W : Ua(p)V \in \Lambda_3 \right\}.$$

Lemma 7. *We have $B \cap \mathbb{P}(E)^{ss} = B_{\text{skew}} \cap \mathbb{P}(E)^{ss} \neq \varnothing$.*

Proof. It is easy to check that the three matrices

$$\begin{pmatrix} t & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & t^{-2} \end{pmatrix} \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ * & * & 0 \end{pmatrix} \begin{pmatrix} t & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & t^{-2} \end{pmatrix}, \begin{pmatrix} t^2 & 0 & 0 \\ 0 & t^{-1} & 0 \\ 0 & 0 & t^{-1} \end{pmatrix} \begin{pmatrix} * & * & * \\ * & 0 & 0 \\ * & 0 & 0 \end{pmatrix} \begin{pmatrix} t^2 & 0 & 0 \\ 0 & t^{-1} & 0 \\ 0 & 0 & t^{-1} \end{pmatrix}$$

all tend to 0 when $t \to 0$, for any constants $*$. This proves that B_1, B_2 and B_3 do not meet $\mathbb{P}(E)^{ss}$.

To show that $B \cap \mathbb{P}(E)^{ss}$ is not empty, pick any three points p_1, p_2 and p_3 in W. The function

$$\tau : a \in E \mapsto \operatorname{Tr}\left(a(p_1) \cdot \operatorname{adj}(a(p_2)) \cdot a(p_3) \cdot \operatorname{adj}(a(p_1 + p_2 + p_3))\right) \in \mathbb{C},$$

is H^0-invariant: if $h \in H$ is the map $A \mapsto UAV$, for some $U, V \in \mathrm{SL}(\mathbb{C}^3)$, then

$$\tau(ha) = \operatorname{Tr}\left(Ua(p_1)V \cdot V^{-1} \operatorname{adj}(a(p_2))U^{-1} \cdot Ua(p_3)V \cdot V^{-1} \operatorname{adj}(a(p_1 + p_2 + p_3))U^{-1}\right),$$

which equals $\tau(a)$. It follows that the function $a \mapsto \tau(a) + \tau(Ta)$ is H-invariant, where $T : A \mapsto A^{\mathrm{T}}$ is the transposition map. Consider the function $b : W \to W$ defined by

$$b = \begin{pmatrix} 0 & x_1 & -x_2 \\ -x_1 & 0 & x_3 \\ x_2 & -x_3 & 0 \end{pmatrix}, \tag{2}$$

where the x_i's are linear forms $W \to \mathbb{C}$. This gives a point $[b]$ in B_{skew}. If the points p_i's are generic, then a simple computation shows that $\tau(b) + \tau(Tb) \neq 0$. □

Lemma 8. *The subvariety $B_{\text{skew}} \cap \mathbb{P}(E)^{ss}$ is smooth and $\rho : X \to \mathbb{P}(E)^{ss}$ is the blowup of $\mathbb{P}(E)^{ss}$ along it.*

Proof. Let $\mathcal{I}$ be the ideal sheaf generated by the condition $\det_3 \circ a = 0$. Its support is clearly $B \cap \mathbb{P}(E)^{ss}$, which is also $B_{\text{skew}} \cap \mathbb{P}(E)^{ss}$, by Lemma 7. By definition, X is the blowup of $\mathbb{P}(E)^{ss}$ along $\mathcal{I}$. By contrast, the blowup of $\mathbb{P}(E)^{ss}$ along $B_{\text{skew}} \cap \mathbb{P}(E)^{ss}$ is defined to be the blowup of the *reduced* ideal sheaf whose support is $B_{\text{skew}} \cap \mathbb{P}(E)^{ss}$. Thus, it is enough to check that $\mathcal{I}$ is smooth (which implies reduced). Let $[b] \in B_{\text{skew}}$ be the point defined in (2).

We first observe that $B_{\text{skew}} \cap \mathbb{P}(E)^{ss} = [H \cdot b \cdot \mathrm{GL}(W)]$, the orbit of $[b]$ under the left action of H and the right action of $\mathrm{GL}(W)$ by multiplication. The right-to-left inclusion is clear because the left-hand side is invariant under both actions and contains $[b]$. Conversely, let $[a] \in B_{\text{skew}} \cap \mathbb{P}(E)^{ss}$. By definition of B_{skew}, we may assume that the image of a is included in Λ_3, up to replacing a by another point in its orbit Ha. If the image of a had dimension 2 or less, then a would also lie in some of the B_i's, $1 \leqslant i \leqslant 3$ [2].[1] Since $[a] \in \mathbb{P}(E)^{\mathrm{ss}}$, Lemma 7 ensures that a is not in one of the B_i's, thus a has rank 3 and its image is Λ_3. Then there is a $g \in \mathrm{GL}(W)$ such that $a = bg$, and thus $a \in H \cdot b \cdot \mathrm{GL}(W)$.

Regarding the smoothness, since $\mathcal{I}$ is invariant under the action of H and $\mathrm{GL}(W)$ and since $B_{\mathrm{skew}} \cap \mathbb{P}(E)^{\mathrm{ss}}$ is an orbit under the same action, it is enough to check that $\mathcal{I}$ is smooth at one point, say $[b]$. By the Jacobian criterion [5, §V.3], it is enough to check that the dimension of the tangent space

$$T = \left\{ c \in T_{[b]}\mathbb{P}(E) \;\middle|\; \forall p \in W, \det(b(p) + tc(p)) = O(t^2) \right\},$$

equals the dimension of B_{skew} at $[b]$. The dimension of T is easily computed using a computer: it is equal to 34. To compute the dimension of B_{skew}, we use again the fact that it is an orbit under a group action: it is smooth and the tangent space at $[b]$ equals

$$\begin{aligned} T_{[b]}B_{\mathrm{skew}} &= \{mb + bc \mid m \in T_1 H,\ c \in T_1 \mathrm{GL}(W)\} \subset T_{[b]}\mathbb{P}(E) \\ &= \{p \in W \mapsto Mb(p) + b(p)N + b(c(p)) \in W \mid M, N \in W,\ c \in \mathrm{End}(W)\}. \end{aligned}$$

Using a computer, we find that this space has also dimension 34, which terminates the proof. □

Proof of Theorem 1. Let D be the inverse image of the hypersurface Z by the blowup ρ. D is a hypersurface with exactly two irreducible components because $\mathbb{P}(E)$ is smooth and because the center of the blowup ρ is also smooth and included in Z [8, Lecture 7]. Respectively, the two components are the exceptional divisor $\rho^{-1}(B_{\mathrm{skew}})$ and the strict transform of Z, i.e. the closure of $\rho^{-1}(Z \setminus B_{\mathrm{skew}})$.

On the other hand $\psi(X \setminus D) = \varphi(\mathrm{GL}(W)) = \Omega(\det_3)$, thus $\partial\Omega(\det_3) \subset \psi(D)$, by Lemma 6. This proves that $\partial\Omega(\det_3)$ has at most two components: The components found in §1 are the only ones.[2] This finishes the proof of Theorem 1. □

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[1] Bürgin and Draisma [2, Theorem 2 and the discussion above it], states that a subspace of E of dimension 2 containing only singular matrices is contained in a compression space.

[2] Though it is not necessary, we check easily that the image of the exceptional divisor is $\overline{\Omega}(P_2)$ while the image of the strict transform of Z gives $\overline{\Omega}(P_1)$.