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Mathematical problems in mechanics

Dynamic behavior of convergent rapid granular flows

Comportement dynamique d'un flux granulaire rapide convergent

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A R T I C L E   I N F O

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A B S T R A C T

We consider a dilute flow of granular particles passing through a nozzle under gravity. This setting is an analogue to high-speed nozzle flows, which is a classical problem in the study of compressible gases. Contrary to the widely held belief that the behavior of very dilute granular systems is qualitatively similar to that of gases, we show that dilute granular systems can exhibit a type of intermittency that has no analogue in gas dynamics.

R É S U M É

Nous considérons un flux dilué de particules granulaires passant par un tube d'injection sous l'action de la gravité. Ce problème est similaire à celui d'un écoulement de flux à grande vitesse, ce qui constitue un problème classique dans l'étude des gaz compressibles. Contrairement à l'idée très répandue que le comportement des systèmes granulaires très dilué est qualitativement semblable à celui des gaz, nous montrons que les systèmes dilués granulaires peuvent présenter un type d'intermittences qui n'a pas d'analogue dans la dynamique des gaz.

1. Introduction and formulation

Granular flows exhibit rich phenomena in their dynamic behaviors due to unique and complicated particle interactions. While the complication of dense granular systems has been attracting more attention, the behavior of dilute granular flows is under less investigation. Here we show that random collisional events between particles in dilute granular nozzle flows can lead to intermittency, the irregular alternation of phases in dynamical systems. This is of great interest in many applications where a steady flux of particles through the nozzle is required.

In this study, we will investigate the intermittent behavior of dilute granular flows. To be specific, we will study dilute flows of inelastic particles that are injected into a nozzle and fall under gravity g. To illustrate the mechanism in the simplest

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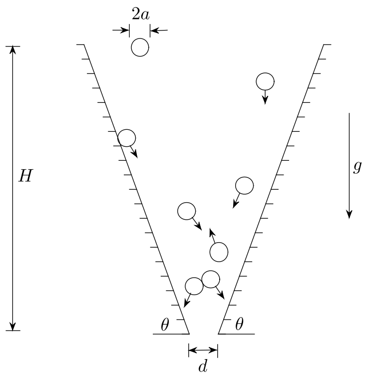

Fig. 1. Sketch of particles falling through a nozzle under gravity.

possible setting, we consider identical frictionless inelastic particles of radius a injected into a symmetric two-dimensional nozzle composed of rigid flat walls aligned at an angle θ to the horizontal. We denote the height of the portion of the nozzle above the bottleneck as H and the width of the bottleneck as d (see Fig. 1). The particles are injected from the top of the nozzle with zero velocity and injection frequency f. The locations of particle injection are randomly selected from a given distribution. We will focus on the simple case of a uniform distribution. We note that the fundamental mechanism for intermittency reported here also exists for much more general conditions.

In our system of dilute granular flows, particles travel relatively long distances between collisions and the principal mechanism for momentum transport between particles is via instantaneous collisional interactions. For simplicity, we will neglect frictional effects. Two different types of collisional interactions occur in this system: (1) collisions between a particle and a wall, i.e. particle–wall collisions, and (2) collisions between particles, i.e. interparticle collisions. Both types of collisions are dissipative. To quantify the energy losses inherent in these collisions, we introduce the coefficient of restitution e, which is defined as the ratio of relative speeds along the line of impact after and before a collision. Specifically, we have e_{w} for particle–wall collisions and e_{p} for interparticle collisions.

Each particle, between its entrance into the nozzle and its passage through the bottleneck, experiences a sequence of events, including instantaneous events and free-fall motion. Instantaneous events are the ones that occur exactly at a point of time t, including (1) entry into the nozzle, (2) collisions with the right wall, (3) collisions with the left wall, (4) collisions with other particles, and (5) exit through the bottleneck. Each event applies a change to the particle's velocity and/or location, which are denoted by $P^{(i)}(t) = (u^{(i)}(t), v^{(i)}(t), x^{(i)}(t), y^{(i)}(t))^{\mathrm{T}}$. Here $u^{(i)}(t)$ ($v^{(i)}(t)$) is the horizontal (vertical) velocity at time t, $x^{(i)}(t)$ ($y^{(i)}(t)$) is the horizontal (vertical) location at time t, and the superscript i is the index of the particle based on the order in which it was injected. For particle i, the time of the k-th collisional event occurring is denoted by $t_k^{(i)}$.

Each particle is released with zero velocity at the top of the nozzle, the width of which is $2D$ with $D = \frac{d}{2} + H\cot\theta$. We denote the time when particle i enters the system by $t_{\mathrm{in}}^{(i)}$, namely, $P_0^{(i)} = P^{(i)}(t_{\mathrm{in}}^{(i)}) = (0, 0, X^{(i)}, H)^{\mathrm{T}}$, where $X^{(i)}$ is a random variable under the probability density function

$$p(x) = \begin{cases} 1/(2D - 2a\csc\theta), & \text{when } |x| < D - a\csc\theta, \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

When particle i hits the right nozzle wall, i.e. when $y^{(i)}/(x^{(i)} - d/2 + a\csc\theta) = \tan\theta$, its velocity component normal to the wall is reduced due to the inelastic collision, while the velocity component tangential to the wall remains unaffected. Therefore, its location and velocity are updated using the operator R defined by

$$P^{(i)}(t_k^{(i)+}) = RP^{(i)}(t_k^{(i)-}) = \begin{pmatrix} W(\theta) & 0_{2\times2} \\ 0_{2\times2} & I_{2\times2} \end{pmatrix} P^{(i)}(t_k^{(i)-}), \tag{2}$$

where $I_{2\times2}$ is a 2×2 identity matrix, $0_{2\times2}$ is a 2×2 zero matrix, and

$$W(\theta) = \frac{1}{2}\begin{pmatrix} 2 - 2(1+e_{\mathrm{w}})\sin^2\theta & (1+e_{\mathrm{w}})\sin 2\theta \\ (1+e_{\mathrm{w}})\sin 2\theta & 2 - 2(1+e_{\mathrm{w}})\cos^2\theta \end{pmatrix}. \tag{3}$$

Similarly, when particle i collides with the left nozzle wall, i.e. when $y^{(i)}/(-x^{(i)} - d/2 + a\csc\theta) = \tan\theta$, its location and velocity are updated using the operator L defined by

$$P^{(i)}(t_k^{(i)+}) = LP^{(i)}(t_k^{(i)-}) = \begin{pmatrix} W(-\theta) & 0_{2\times2} \\ 0_{2\times2} & I_{2\times2} \end{pmatrix} P^{(i)}(t_k^{(i)-}). \tag{4}$$

When particles i and j collide, i.e. when $(x^{(i)} - x^{(j)})^2 + (y^{(i)} - y^{(j)})^2 = 4a^2$, their velocities along the line of impact are determined by the conservation of momentum and the coefficient of restitution e_{p}, while the components of velocity perpendicular to the line of impact remain unchanged. Hence, when particle i collides with particle j, the changes of its state can be described by the operator $C^{(j)}$, which is defined by

$$P^{(i)}(t_k^{(i)+}) = C^{(j)} P^{(i)}(t_k^{(i)-}) = \begin{pmatrix} C_1(\alpha) & 0_{2\times2} \\ 0_{2\times2} & I_{2\times2} \end{pmatrix} P^{(i)}(t_k^{(i)-}) + \begin{pmatrix} C_2(\alpha) & 0_{2\times2} \\ 0_{2\times2} & 0_{2\times2} \end{pmatrix} P^{(j)}(t_k^{(i)-}), \quad (5)$$

where $\alpha = \tan^{-1}[(y^{(i)}(t_k^{(i)-}) - y^{(j)}(t_k^{(i)-}))/(x^{(i)}(t_k^{(i)-}) - x^{(j)}(t_k^{(i)-}))]$ and

$$C_1(\alpha) = \frac{1}{4}\begin{pmatrix} 4 - 2(1+e_\mathrm{p})\cos^2\alpha & -(1+e_\mathrm{p})\sin 2\alpha \\ -(1+e_\mathrm{p})\sin 2\alpha & 4 - 2(1+e_\mathrm{p})\sin^2\alpha \end{pmatrix}, C_2(\alpha) = \frac{1+e_\mathrm{p}}{4}\begin{pmatrix} 2\cos^2\alpha & \sin 2\alpha \\ \sin 2\alpha & 2\sin^2\alpha \end{pmatrix}. \quad (6)$$

Every particle eventually exits the system by going through the bottleneck. When particle i passes through the bottleneck, i.e. when $y^{(i)} = 0$, it is removed from the system, which is denoted by the event E at the exit time $t = t_\mathrm{out}^{(i)}$.

Between instantaneous events, each particle experiences free-fall motion with a parabolic trajectory. During $[t, t+\Delta t]$ in which no instantaneous events occur, the changes of particle i can be described by an operator M:

$$P^{(i)}(t+\Delta t) = M(\Delta t)P^{(i)}(t) = \left[I_{4\times4} + \Delta t\begin{pmatrix} 0_{2\times2} & 0_{2\times2} \\ I_{2\times2} & 0_{2\times2} \end{pmatrix}\right] P^{(i)}(t) + (0, -g\Delta t, 0, -g\Delta t^2/2)^\mathrm{T}. \quad (7)$$

Therefore, for each particle, we will obtain a sequence of events as follows:

$$S^{(i)} = EM(t_\mathrm{out}^{(i)} - t_{n^{(i)}}^{(i)})Q_{n^{(i)}}^{(i)} M(t_{n^{(i)}}^{(i)} - t_{n^{(i)}-1}^{(i)}) \dots Q_1^{(i)} M(t_1^{(i)} - t_\mathrm{in}^{(i)})P_0^{(i)}, \qquad \text{for particle } i, \quad (8)$$

where $Q_k^{(i)} \in \{R, L, C\}$ for all k. The trajectory of each particle can be determined from its event sequence. Now we describe how to determine the event sequence. Consider any particle in the system, e.g., particle i, whose next instantaneous event can only be one of the four possible events: colliding with the right wall R, colliding with the left wall L, colliding with another particle C, and exiting the system E. From the aforementioned criteria for different types of instantaneous events, we can calculate the time increments for the next occurrences of all possible events:

$$\tau_R^{(i)} = \frac{1}{g}(v^{(i)} - u^{(i)}\tan\theta) + \frac{1}{g}\left[(v^{(i)} - u^{(i)}\tan\theta)^2 - 2g[\tan\theta(x^{(i)} - d/2) + a\sec\theta - y^{(i)}]\right]^{1/2}, \text{ for event R}, \quad (9)$$

$$\tau_L^{(i)} = \frac{1}{g}(v^{(i)} + u^{(i)}\tan\theta) + \frac{1}{g}\left[(v^{(i)} + u^{(i)}\tan\theta)^2 - 2g[\tan\theta(-x^{(i)} - d/2) + a\sec\theta - y^{(i)}]\right]^{1/2}, \text{ for event L}, \quad (10)$$

$$\tau_E^{(i)} = \frac{1}{g}\left[v^{(i)} + \left(v^{(i)2} + 2gy\right)^{1/2}\right], \text{ for event E}, \quad (11)$$

and $\tau_j^{(i)}$ for event $C^{(j)}$ ($j \in \{j \mid \text{particle } j \text{ in the system and } j \neq i\}$), which is the smaller positive solution for

$$[x^{(i)} - x^{(j)} + (u^{(i)} - u^{(j)})\tau_j^{(i)}]^2 + [y^{(i)} - y^{(j)} + (v^{(i)} - v^{(j)})\tau_j^{(i)}]^2 = 4a^2, \quad (12)$$

if it exists. Therefore, the time increment for the next instantaneous event is $\tau^{(i)} = \min\{\tau_R^{(i)}, \tau_L^{(i)}, \tau_E^{(i)}, \min_j\{\tau_j^{(i)}\}\}$ for particle i, and the time increment for the next instantaneous event of the whole system is $\tau = \min_i\{\tau^{(i)}\}$. This allows us to determine which particles are involved in the next instantaneous event of the system. Accordingly, we propagate the system by the time increment τ and update the event sequences of the particles. By repeating this process, we can determine the event sequences and therefore obtain the trajectories of all particles.

2. Main results

The procedure described above gives us an event-driven algorithm for determining the dynamic behavior of the system and studying its phenomena. We chose $H/a = 100$, $d/a = 10$, $e_\mathrm{p} = e_\mathrm{w} = 0.9$, and varied the dimensionless injection frequency $F = f\sqrt{2H/g}$ as well as the angle of the nozzle θ. Note that all quantities in the system are scaled by the particle radius a and the characteristic time $\sqrt{2H/g}$ (the time it takes for a particle to fall freely from the inlet to the bottleneck). The systems we considered were sufficiently dilute that clustering effects were negligible [1] and inelastic collapse [5] did not occur in our studies. We only examined the event sequences after a certain time t_0, from which the transient effects were negligible and the system reached a statistically steady state. Then we studied this steady state over a period of T. Any particle, e.g., particle i, stays in the system for a time period $[t_\mathrm{in}^{(i)}, t_\mathrm{out}^{(i)}]$. Therefore, the number of particles in the system at time t is $N(t) = \sum_i \chi_{[t_\mathrm{in}^{(i)}, t_\mathrm{out}^{(i)}]}(t)$, where χ is the indicator function, and the average number of particles in the system over the whole time interval $[t_0, t_0 + T]$ is $\langle N\rangle = \frac{1}{T}\int_{t_0}^{t_0+T} N(t)\mathrm{d}t = \frac{1}{T}\sum_i (t_\mathrm{out}^{(i)} - t_\mathrm{in}^{(i)})$.

In Fig. 2, we show how $\langle N\rangle$ varies with the dimensionless injection frequency F at different angles. Intuitively, one would expect that the changes in F would lead to gradual changes in $\langle N\rangle$. This is indeed the case for shallow nozzles (e.g., $\theta = 30°$

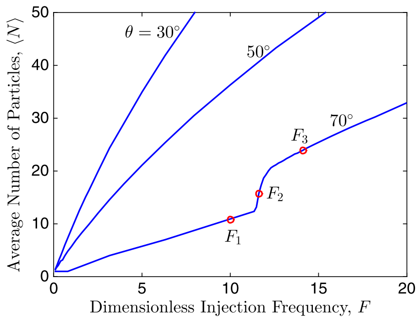

Fig. 2. The time-averaged number of particles in the system $\langle N\rangle$ is plotted against the dimensionless injection frequency $F = f\sqrt{\frac{2H}{g}}$ for nozzles with various angles θ. For nozzles with $\theta = 70°$, there is a dramatic increase in $\langle N\rangle$ over a relatively small range of frequency (near $F = 12$). Three representative frequencies ($F_1 = 10.0$, $F_2 = 11.6$, and $F_3 = 14.1$) are labeled for $\theta = 70°$.

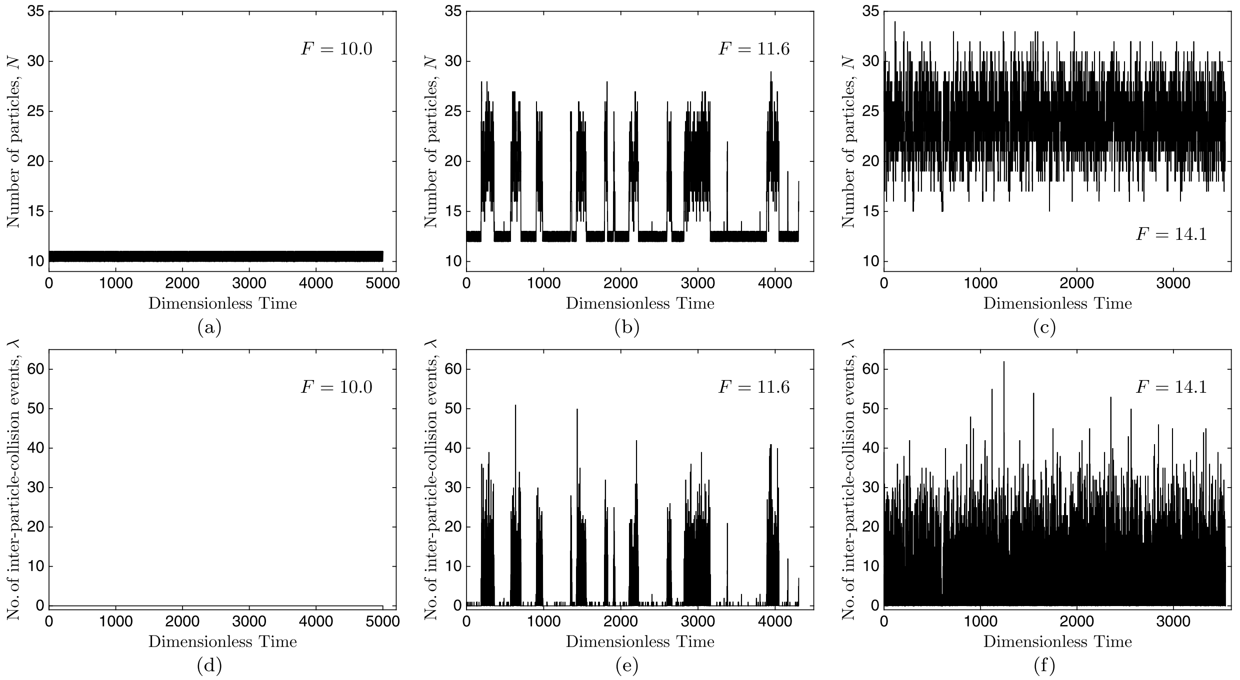

Fig. 3. Time sequences of the number of particles in the system $N(t)$ ((a), (b) and (c)) and the number of interparticle-collision events $\lambda(t)$ ((d), (e) and (f)) at three representative frequencies ($F = 10.0$, 11.6, and 14.1) for $\theta = 70°$. Intermittency is observed in Figs. (b) and (e) for $F = 11.6$.

and 50° in Fig. 2). However, steep nozzles (e.g., $\theta = 70°$ in Fig. 2) have surprisingly different behavior. Fig. 2 shows that $\langle N\rangle$ increases dramatically over a small range of F (near $F = 12$ for $\theta = 70°$).

To investigate this rapid change that occurs for $\theta = 70°$, we plot $N(t)$ for three representative injection frequencies $F = 10.0$, 11.6, and 14.1 (denoted by F_1, F_2 and F_3 in Fig. 2) in Figs. 3(a), 3(b), and 3(c), respectively. We denote by $\Theta(S^{(i)})$ the number of interparticle collision events in the event sequence $S^{(i)}$, the definition of which is given by Eq. (8). The time sequences of this random number, $\lambda(t) = \sum_i \Theta(S^{(i)})\chi_{\{t^{(i)}_{\text{out}}\}}(t)$, for $F = 10.0$, 11.6, and 14.1 are plotted in Figs. 3(d), 3(e), and 3(f), respectively.

Figs. 3(a) and 3(d) show that, at $F = 10.0$ (low injection frequency), the nozzle has relatively few particles and these particles seldom collide with each other. Moreover, the number of particles remains approximately constant. Meanwhile, Figs. 3(c) and 3(f) show that, at $F = 14.1$ (high injection frequency), the number of particles in the system becomes much larger and interparticle collisions occur more frequently. Note that $N(t)$ shown in Fig. 3(c) and $\lambda(t)$ shown in Fig. 3(f) for $F = 14.1$ have much larger fluctuations than those shown in Figs. 3(a) and 3(d) for $F = 10.0$. Therefore, the system is in two very different states at these two frequencies, and the state in which the system remains does not change over time.

One would expect that the system at $F = 11.6$ (intermediate injection frequency) just resembles those at $F = 10.0$ and 14.1, only with quantitative differences in $N(t)$ and $\lambda(t)$. However, Figs. 3(b) and 3(e) demonstrate that the flow intermittently switches between two completely different states. In one state, the system behaves almost identically to that at $F = 10.0$. The number of particles in the system $N(t)$ remains relatively small when few interparticle collisions occur. In the other state, the system exhibits qualitative similarity with that at $F = 14.1$. Both $N(t)$ in Fig. 3(b) and $\lambda(t)$ in Fig. 3(e) show that the flow repeatedly switches from one state to the other, and vice versa. Therefore, the phenomenon of intermittency occurs in this dilute granular flow.

This phenomenon is very surprising since gas dynamics contains no analogue of this type of intermittency. It is widely believed that, in the absence of cluster formation, highly dilute granular flows exhibit qualitatively similar behavior to compressible gases [7]. For high-speed gas flows, nozzle flows readily form shocks, but nevertheless, within the nozzle, the flow patterns tend to be relatively simple. Particularly, intermittent flow within the nozzle does not occur. It is therefore natural to expect that the flow of a highly dilute granular material through a nozzle would be relatively simple and would not exhibit intermittency. However, we have shown here, even in a simple setting, intermittency can exist in dilute granular flows. Although intermittency has been observed in several important studies of dense granular flows [9,4,3,6,2], dilute and dense granular flows are very different. The mechanisms and behavior that are observed in dense granular flows have little relevance to dilute flows [8]. In dense granular flows, particles are heavily constrained and experience long-lasting interactions with their neighbors. Moreover, friction between particles is often the principal mechanism for momentum transport.

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