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Some remarks on a theorem of Bergman

Quelques remarques sur un résultat de Bergman

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ABSTRACT

We extend a result of Bergman to show that any object in an arbitrary Grothendieck category may be expressed as an inverse limit of injectives. We also study inverse systems of κ-injective objects, where κ is an infinite regular cardinal.

RÉSUMÉ

Nous étendons un résultat de Bergman en montrant qu'on peut exprimer chaque objet dans une catégorie de Grothendieck comme la limite d'un système inverse d'objets injectifs. Nous étudions aussi les systèmes inverses d'objets κ-injectifs, où κ est un cardinal régulier infini.

1. Introduction

In a recent paper, Bergman [4] proved the following rather striking result: any left module M over a ring R can be expressed as the inverse limit of injective modules. It is natural to ask whether such a result holds in other similar categories, such as the category of sheaves of $\mathcal{O}_X$-modules over a scheme $(X, \mathcal{O}_X)$, and more generally in any Grothendieck category. We know that Grothendieck categories abound in nature: for example, if Y is a topological space and $\mathcal{O}$ is any sheaf of rings on Y, the categories $\mathcal{O}$-*Premod* and $\mathcal{O}$-*Mod* respectively of presheaves and sheaves of $\mathcal{O}$-modules are Grothendieck categories (see [3, I.3, II.4]). For a concentrated (i.e., quasi-compact and quasi-separated) scheme X, the category of quasi-coherent sheaves on X is a Grothendieck category. Further, given a Grothendieck category $\mathbf{C}$ and a small category $\mathcal{I}$, the category $[\mathcal{I}, \mathbf{C}]$ of functors from $\mathcal{I}$ to $\mathbf{C}$ is again a Grothendieck category. The purpose of this paper is to extend the result of Bergman [4] to arbitrary Grothendieck categories. Even in the case of quasi-coherent sheaves over a concentrated scheme, the fact that every object can be expressed as an inverse limit of injectives does not seem to have appeared before in the literature.

We now describe the structure of the paper in more detail: in Section 2, we adapt the result of Bergman [4, Theorem 2] to show that any object $\mathcal{A}$ in a Grothendieck category $\mathbf{C}$ can be expressed as the inverse limit of a system of injectives

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connected by monomorphisms. We further extend our result to the category $Fil^f(\mathbf{C})$ of finitely filtered objects of $\mathbf{C}$. Note that in general, $Fil^f(\mathbf{C})$ is not even an Abelian category, let alone a Grothendieck category. Further, for any given $\mathcal{A} \in \mathbf{C}$, one would also like to think about the cardinality of the inverse system whose limit is the object $\mathcal{A}$. Therefore, we consider, more generally, objects $\mathcal{G} \in \mathbf{C}$ that are "κ-injective" for some infinite regular cardinal κ, i.e., any morphism $h : \mathcal{K} \longrightarrow \mathcal{G}$ from a κ-generated subobject $\mathcal{K}$ of any $\mathcal{L} \in \mathbf{C}$ extends to all of $\mathcal{L}$. We recall that an object $\mathcal{K} \in \mathbf{C}$ is said to be "κ-generated" if the functor $Hom_{\mathbf{C}}(\mathcal{K}, \_\_) : \mathbf{C} \longrightarrow Sets$ preserves κ-filtered colimits of monomorphisms. Then, we show that in a Grothendieck category $\mathbf{C}$, every object may be expressed as the limit of an inverse system of cardinality κ consisting of κ-injective objects connected by monomorphisms. Finally, in Section 2, we also show that when the Grothendieck category $\mathbf{C}$ is locally κ-generated, the κ-injective objects are exactly those that can be expressed as the colimit of a κ-filtered system of injectives connected by monomorphisms.

In particular, when $\kappa = \aleph_0$, the κ-injective objects are the "finitely injective objects", a notion that was introduced for modules in [12]. In Section 3, we show that if $\mathcal{C}$ is a Grothendieck category with a finitely generated generator, every object may be expressed as the inverse limit of a system of finitely injective objects connected by epimorphisms. It is clear that the direct sum of a family of injectives is finitely injective. This raises the question of whether all finitely injective objects arise in this manner. However, we show that if $\mathcal{H}$ is an injective and S is any set such that the direct sum $\mathcal{H}(S)$ of S-copies of $\mathcal{H}$ is not injective, the injective hull $E(\mathcal{H}(S))$ must contain a finitely injective subobject that cannot be expressed as the direct sum of a family of injectives.

2. Inverse limits of systems of monomorphisms

Throughout this section, we let $\mathbf{C}$ be a Grothendieck Abelian category, that is an (AB5) category with a generator. For general properties of Grothendieck categories, we refer the reader, for instance, to Stenström [16] (or to Grothendieck's Tôhoku paper [6]). For the sake of definiteness, we choose a generator $\mathcal{U}$ for the Grothendieck category $\mathbf{C}$.

We now make the following construction. Let $\{\mathcal{M}_i\}_{i\in I}$ be a collection of objects of $\mathbf{C}$ indexed by a set I and let κ be an infinite regular cardinal. Let $P_\kappa(I)$ denote the collection of all subsets of I of cardinality $< \kappa$. Then, for any subset $J \in P_\kappa(I)$, consider the product $\prod_{j\in J} \mathcal{M}_j \in \mathbf{C}$ (note that a Grothendieck category is complete and contains all small products). Further, for any J, $J' \in P_\kappa(I)$ with $J \subseteq J'$, by considering zero maps $\prod_{j\in J} M_j \longrightarrow M_{j'}$ for each $j' \in J' \backslash J$, we obtain connecting morphisms $\prod_{j\in J} \mathcal{M}_j \longrightarrow \prod_{j''\in J'} \mathcal{M}_{j''}$. We now set:

$$\prod^{\kappa}_{i\in I} \mathcal{M}_i := \varinjlim_{J\in P_\kappa(I)} \prod_{j\in J} \mathcal{M}_j \tag{2.1}$$

and the colimit exists because $\mathbf{C}$ is a Grothendieck category (hence cocomplete). By a subobject $\mathcal{N}$ of some $\mathcal{M} \in \mathbf{C}$, we will mean a monomorphism $\mathcal{N} \longrightarrow \mathcal{M}$ and, by abuse of notation, we shall write $\mathcal{N} \subseteq \mathcal{M}$.

Lemma 2.1. *For any object $\mathcal{M} \in \mathbf{C}$, there exists an infinite regular cardinal κ such that every subobject $\mathcal{N} \subseteq \mathcal{M}$ is κ-compact, i.e., the corepresentable functor $Hom_{\mathbf{C}}(\mathcal{N}, \_\_) : \mathbf{C} \longrightarrow Sets$ preserves κ-filtered colimits.*

Proof. Since $\mathbf{C}$ is a Grothendieck category, it is also locally presentable and hence every object $\mathcal{P} \in \mathbf{C}$ is α-compact for some infinite regular cardinal α (see, for instance, [9, Proposition A.2]). Further, in a Grothendieck category, the class of subobjects of any given object is a set (see, for instance, [16, Proposition IV.6.6]). Accordingly, for each subobject $\mathcal{N} \subseteq \mathcal{M}$, we may choose an infinite regular cardinal $\alpha_{\mathcal{N}}$ such that $\mathcal{N}$ is $\alpha_{\mathcal{N}}$-compact. Since the subobjects of $\mathcal{M}$ form a set, we may consider the supremum of $\alpha_{\mathcal{N}}$ for all $\mathcal{N} \subseteq \mathcal{M}$ and choose an infinite regular cardinal κ such that every subobject of $\mathcal{M}$ is κ-compact. □

Following Lemma 2.1, for any object $\mathcal{M} \in \mathbf{C}$, we denote by $\kappa(\mathcal{M})$ the smallest infinite regular cardinal such that every subobject of $\mathcal{M}$ is $\kappa(\mathcal{M})$-compact.

Proposition 2.2. *Let $\{\mathcal{G}_i\}_{i\in I}$ be a collection of injective objects of $\mathbf{C}$ indexed by a set I. Let $\mathcal{U}$ be a generator for $\mathbf{C}$ and let $\kappa(\mathcal{U})$ be the infinite regular cardinal as defined above. Then, $\prod^{\kappa(\mathcal{U})}_{i\in I} \mathcal{G}_i$ is injective.*

Proof. We know that $\mathcal{U}$ is a generator for the Grothendieck category $\mathbf{C}$. Hence, in order to show that $\prod^{\kappa(U)}_{i\in I} \mathcal{G}_i$ is injective, it suffices to show (see, for instance, [16, V.2.9]) that any morphism to $\prod^{\kappa(U)}_{i\in I} \mathcal{G}_i$ from a subobject of $\mathcal{U}$ extends to all of $\mathcal{U}$. We consider some subobject $\mathcal{V} \subseteq \mathcal{U}$ and a morphism $f : \mathcal{V} \longrightarrow \prod^{\kappa(U)}_{i\in I} \mathcal{G}_i$. Then, since $\kappa(\mathcal{U})$ is a regular cardinal, the union of fewer than $\kappa(\mathcal{U})$ subsets from $P_{\kappa(\mathcal{U})}(I)$ must have cardinality less than $\kappa(\mathcal{U})$. Hence, $P_{\kappa(\mathcal{U})}(I)$ is $\kappa(\mathcal{U})$-filtered. By definition of $\kappa(\mathcal{U})$, the subobject $\mathcal{V} \subseteq \mathcal{U}$ is $\kappa(\mathcal{U})$-compact. Then, $Hom_{\mathbf{C}}(\mathcal{V}, \prod^{\kappa(\mathcal{U})}_{i\in I} \mathcal{G}_i) = Hom_{\mathbf{C}}(\mathcal{V}, \varinjlim_{J\in P_{\kappa(\mathcal{U})}(I)} \prod_{j\in J} \mathcal{G}_j) \cong \varinjlim_{J\in P_{\kappa(\mathcal{U})}(I)} Hom_{\mathbf{C}}(\mathcal{V}, \prod_{j\in J} \mathcal{G}_j)$. Consequently, given $f : \mathcal{V} \longrightarrow \prod^{\kappa(\mathcal{U})}_{i\in I} \mathcal{G}_i$, there exists $J_0 \in P_{\kappa(\mathcal{U})}(I)$ such that f factors through

$\prod_{j\in J_0}\mathcal{G}_j$. Each $\mathcal{G}_j$ being injective, so is the product $\prod_{j\in J_0}\mathcal{G}_j$ and hence the morphism $\mathcal{V}\longrightarrow\prod_{j\in J_0}\mathcal{G}_j$ extends to a morphism $\mathcal{U}\longrightarrow\prod_{j\in J_0}\mathcal{G}_j$. Composing with the canonical morphism $\prod_{j\in J_0}\mathcal{G}_j\longrightarrow\prod_{i\in I}^{\kappa(\mathcal{U})}\mathcal{G}_i$, we obtain an extension of f to a morphism $\mathcal{U}\longrightarrow\prod_{i\in I}^{\kappa(\mathcal{U})}\mathcal{G}_i$. This proves the result. □

We are now ready to show that any object in **C** may be expressed as an inverse limit of injectives.

Proposition 2.3. *Let* **C** *be a Grothendieck category. Then, every object* $\mathcal{A}\in$ **C** *may be expressed as an inverse limit of a system of injective objects connected by monomorphisms.*

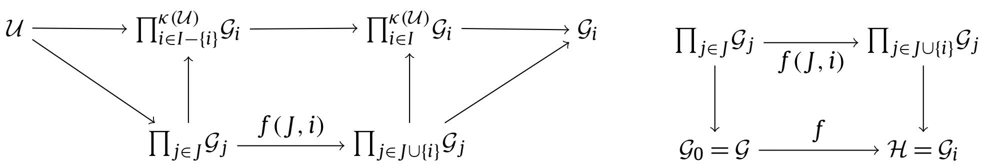
Proof. Since **C** has enough injectives, we can choose a morphism $f:\mathcal{G}\longrightarrow\mathcal{H}$ of injectives such that $\mathcal{A}=\mathrm{Ker}(f)$. We now consider the regular cardinal $\kappa(\mathcal{U})$ as in Proposition 2.2 and choose any set I having cardinality $\geq\kappa(\mathcal{U})$. We choose an arbitrary element of I which we denote by 0. We then define a family $\{\mathcal{G}_i\}_{i\in I}$ of injectives by setting $\mathcal{G}_0:=\mathcal{G}$ and $\mathcal{G}_i:=\mathcal{H}$ for all $i\in I-\{0\}$. Further, we set $\mathcal{P}:=\prod_{i\in I}^{\kappa(\mathcal{U})}\mathcal{G}_i$.

Let $F(I-\{0\})$ be the collection of finite (possibly empty) subsets of $I-\{0\}$. Now, let $S\in F(I-\{0\})$ and consider some $J\in P_{\kappa(\mathcal{U})}(I-S)$ with $0\in J$. Then, for each $s\in S$, we have a morphism:

$$\prod_{j\in J}\mathcal{G}_j\longrightarrow\mathcal{G}_0=\mathcal{G}\xrightarrow{f}\mathcal{H}=\mathcal{G}_s \tag{2.2}$$

where the first map in (2.2) is just the projection onto $\mathcal{G}_0$. Combining the morphisms in (2.2) for all $s\in S$ with the identity $1:\prod_{j\in J}\mathcal{G}_j\longrightarrow\prod_{j\in J}\mathcal{G}_j$, we obtain a monomorphism $f(J,S):\prod_{j\in J}\mathcal{G}_j\longrightarrow\prod_{j\in J\cup S}\mathcal{G}_j$ which we compose with the canonical inclusion $\prod_{j\in J\cup S}\mathcal{G}_j\longrightarrow\prod_{i\in I}^{\kappa(U)}\mathcal{G}_i$. Further, since the sets $J\in P_{\kappa(\mathcal{U})}(I-S)$ with $0\in J$ are cofinal in $P_{\kappa(\mathcal{U})}(I-S)$, the monomorphisms $\prod_{j\in J}\mathcal{G}_j\longrightarrow\prod_{i\in I}^{\kappa(U)}\mathcal{G}_i$ together give us a monomorphism $\prod_{i\in I-S}^{\kappa(U)}\mathcal{G}_i\longrightarrow\prod_{i\in I}^{\kappa(U)}\mathcal{G}_i$. From Proposition 2.2, it follows that each $\mathcal{P}_S:=\prod_{i\in I-S}^{\kappa(\mathcal{U})}\mathcal{G}_i$ (and in particular, $\mathcal{P}=\mathcal{P}_\phi=\prod_{i\in I}^{\kappa(\mathcal{U})}\mathcal{G}_i$) is injective. More generally, for any two subsets $S,T\in F(I-\{0\})$ with $S\subseteq T$, we similarly obtain an inclusion $\mathcal{P}_T\hookrightarrow\mathcal{P}_S$. This system is filtered, because we have inclusions $\mathcal{P}_{S_1\cup S_2}\hookrightarrow\mathcal{P}_{S_1}$ and $\mathcal{P}_{S_1\cup S_2}\hookrightarrow\mathcal{P}_{S_2}$ for any $S_1,S_2\in F(I-\{0\})$. Let $\mathcal{Q}$ denote the subobject of $\mathcal{P}$ that is the inverse limit of this system of injective objects $\mathcal{P}_S$ with $S\in F(I-\{0\})$. We will show that $\mathcal{A}=\mathrm{Ker}(f:\mathcal{G}\longrightarrow\mathcal{H})\cong\mathcal{Q}$.

For each $i\in I$, we have a canonical morphism $q_i:\mathcal{Q}\hookrightarrow\mathcal{P}\longrightarrow\mathcal{G}_i$. For $i\neq 0$, q_i factors through $\prod_{i\in I-\{i\}}^{\kappa(\mathcal{U})}\mathcal{G}_i$ and hence $q_i=0$. We claim that $f\circ q_0=0$. For this, we choose any morphism $h:\mathcal{U}\longrightarrow\mathcal{Q}$ from the generator $\mathcal{U}$. Then, for any $0\neq i\in I$, we can find commutative diagrams:



where $J\subseteq I-\{i\}$ is a subset containing 0 and of cardinality $<\kappa(\mathcal{U})$ such that $\mathcal{U}\longrightarrow\prod_{i\in I-\{i\}}^{\kappa(\mathcal{U})}\mathcal{G}_i$ factors through $\prod_{j\in J}\mathcal{G}_j$ (since $\mathcal{U}$ is $\kappa(\mathcal{U})$-compact). From these diagrams, it follows that $0=(f\circ q_0)\circ h$. Since $\mathcal{U}$ is a generator, we conclude that $f\circ q_0=0$ and hence $q_0:\mathcal{Q}\longrightarrow\mathcal{G}_0$ must factor via some $q':\mathcal{Q}\longrightarrow\mathrm{Ker}(f)$. On the other hand, we note that $\mathrm{Ker}(f)$ itself admits monomorphisms $k_S:\mathrm{Ker}(f)\hookrightarrow\mathcal{P}_S$ for $S\in F(I-\{0\})$ compatible with the inclusions $\mathcal{P}_T\hookrightarrow\mathcal{P}_S$ for $S\subseteq T\in F(I-\{0\})$. If we let $p_i:\mathcal{P}_S\longrightarrow\mathcal{G}_i$ be the projection of $\mathcal{P}_S\subseteq\prod_{i\in I}\mathcal{G}_i$ upon each $\mathcal{G}_i$, we get $p_i\circ k_S\circ q'=0=q_i:\mathcal{Q}\longrightarrow\mathcal{G}_i$ for $i\neq 0$ and $p_0\circ k_S\circ q'=q_0:\mathcal{Q}\longrightarrow\mathcal{G}_0$. Accordingly, the canonical morphisms $\mathcal{Q}\longrightarrow\mathcal{P}_S$ factor via $q':\mathcal{Q}\longrightarrow\mathrm{Ker}(f)$. Finally, we notice that the monomorphisms $k_S:\mathrm{Ker}(f)\hookrightarrow\mathcal{P}_S$ together induce a monomorphism $\mathrm{Ker}(f)\hookrightarrow\mathcal{Q}$. Hence, any family of morphisms $\mathcal{M}\longrightarrow\mathcal{P}_S$, $S\in F(I-\{0\})$ from some given $\mathcal{M}\in$ **C** factors uniquely through $\mathrm{Ker}(f)$. Hence, $\mathcal{Q}\cong\mathrm{Ker}(f)\cong\mathcal{A}$. □

We will now extend the result of Proposition 2.3 to the category $Fil^f(\mathbf{C})$ of objects of **C** equipped with a finite filtration. An object $(\mathcal{A},F)$ of $Fil^f(\mathbf{C})$ will consist of an object $A\in$ **C** along with a family of subobjects $\{F^p\mathcal{A}\}_{p\in\mathbb{Z}}$ such that each $F^{p+1}\mathcal{A}\subseteq F^p(\mathcal{A})$. Further, the filtration being finite, there exist integers $a\leq b$ such that $F^a\mathcal{A}=A$ and $F^b\mathcal{A}=0$. In general, the category $Fil^f(\mathbf{C})$ is not even an Abelian category, let alone a Grothendieck category. For more on the category $Fil^f(\mathbf{C})$ and on filtered derived categories, the reader may see, for example, Deligne [5], Illusie [10] and Schneiders [15]. A filtered injective object $(\mathcal{G},F)\in Fil^f(\mathbf{C})$ is given by a finite direct sum $\mathcal{G}=\oplus_{a\leq i\leq b}\mathcal{G}_n$ of injectives $\mathcal{G}_n$ with $F^p\mathcal{G}:=\oplus_{i\geq p}\mathcal{G}_n$ (see [14, Tag 05TP]). Hence, each $F^p\mathcal{G}$ is injective.

Proposition 2.4. *Let* **C** *be a Grothendieck category. Then, every object of* $Fil^f(\mathbf{C})$ *may be expressed as the inverse limit of a system of filtered injective objects in* $Fil^f(\mathbf{C})$.

Proof. Given some $(\mathcal{A},F)\in Fil^f(\mathbf{C})$, we can express $(\mathcal{A},F)=\mathrm{Ker}(f:(\mathcal{G},F)\longrightarrow(\mathcal{H},F))$ for some filtered injectives $(\mathcal{G},F)$ and $(\mathcal{H},F)$ (see [14, Tags 05TS, 05TT]). Let $\mathcal{U}$ be a generator for **C** and let $\kappa(\mathcal{U})$ be as before. We choose a set I of cardinality

$\geq \kappa(\mathcal{U})$ and fix an element $0 \in I$. We also set $\mathcal{G}_0 := \mathcal{G}$ and $\mathcal{G}_i := \mathcal{H}$ for all $i \in I - \{0\}$. Then, for any finite (possibly empty) subset S of $I - \{0\}$, we set $F^p\mathcal{P}_S := \prod_{I-S}^{\kappa(\mathcal{U})} F^p\mathcal{G}_i$, $p \in \mathbb{Z}$. Since this construction as defined in (2.1) involves only products and filtered colimits, we see that $\{F^p\mathcal{P}_S\}_{p\in\mathbb{Z}}$ is indeed a filtration on $\mathcal{P}_S := \prod_{I-S}^{\kappa(\mathcal{U})} \mathcal{G}_i$. Further, since each $\mathcal{G}_i$ is equal to either $\mathcal{G}$ or $\mathcal{H}$, $\{F^p\mathcal{P}_S\}_{p\in\mathbb{Z}}$ is a finite filtration. From the definition, it is clear that each $F^p\mathcal{P}_S$ is injective. As in Proposition 2.3, it now follows that each $F^p\mathcal{A}$ is the inverse limit of the objects $F^p\mathcal{P}_S$ and hence $(\mathcal{A}, F)$ is the inverse limit of the system $(\mathcal{P}_S, F)$ as S varies over all finite subsets of $I - \{0\}$. □

We observe that while Proposition 2.3 allows us to express any object in **C** as an inverse limit of injectives, it does not give us much control over the size of the inverse system, an issue that Bergman has also pondered over in [4]. In this regard, we recall the following definition from [12]: an object $\mathcal{G} \in \mathbf{C}$ is said to be finitely injective if any morphism $h : \mathcal{K} \longrightarrow \mathcal{G}$ from a finitely generated subobject $\mathcal{K}$ of any $\mathcal{L} \in \mathbf{C}$ extends to a morphism $h' : \mathcal{L} \longrightarrow \mathcal{G}$. Recall that an object $\mathcal{K} \in \mathbf{C}$ is said to be finitely generated if the functor $Hom_{\mathbf{C}}(\mathcal{K}, \_\_) : \mathbf{C} \longrightarrow Sets$ preserves filtered colimits of monomorphisms. More generally, for any infinite regular cardinal κ, an object $\mathcal{K} \in \mathbf{C}$ is κ-generated if $Hom_{\mathbf{C}}(\mathcal{K}, \_\_)$ preserves κ-filtered colimits of monomorphisms. Analogously, we will say that $\mathcal{G} \in \mathbf{C}$ is κ-injective if any morphism $h : \mathcal{K} \longrightarrow \mathcal{G}$ from a κ-generated subobject $\mathcal{K}$ of any $\mathcal{L} \in \mathbf{C}$ extends to a morphism $h' : \mathcal{L} \longrightarrow \mathcal{G}$.

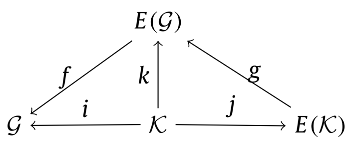
Proposition 2.5. *(a) The colimit of any κ-filtered system $\{\mathcal{G}_j\}_{j\in J}$ of monomorphisms of injective objects $\mathcal{G}_j \in \mathbf{C}$ is κ-injective.*
*(b) In a Grothendieck category **C**, every object $\mathcal{A} \in \mathbf{C}$ may be expressed as the inverse limit of a system of cardinality κ consisting of κ-injective objects connected by monomorphisms.*

Proof. For part (a), let $\mathcal{G} := colim_{j\in J}\mathcal{G}_j$. If $\mathcal{K} \subseteq \mathcal{L}$ is a κ-generated subobject, any morphism $h : \mathcal{K} \longrightarrow \mathcal{G}$ factors through $\mathcal{G}_{j_0} \longrightarrow \mathcal{G}$ for some $j_0 \in J$. Then, $\mathcal{G}_{j_0}$ being injective, h extends to a morphism $h' : \mathcal{L} \longrightarrow \mathcal{G}_{j_0} \longrightarrow \mathcal{G}$. For part (b), we choose a set I of cardinality κ. We use the same notation as in the proof of Proposition 2.3. Since $\prod_{i\in I-S}^{\kappa} \mathcal{G}_i$ can be expressed as a filtered colimit of monomorphisms as defined in (2.1), it follows from part (a) that $\mathcal{P}_S = \prod_{i\in I-S}^{\kappa} \mathcal{G}_i$ is κ-injective for each $S \in F(I - \{0\})$. Then, the same proof applies to show that any $\mathcal{A} \in \mathbf{C}$ may be expressed as the inverse limit of $\mathcal{P}_S$ as $S \in F(I - \{0\})$. Finally, it is clear that the cardinality of the collection $F(I - \{0\})$ is equal to $|I| = \kappa$. □

Following Proposition 2.5(a), it is natural to ask whether all κ-injectives in **C** may be recovered as colimits of κ-filtered systems of injectives connected by monomorphisms. We will now show that this is actually true when **C** is locally κ-generated, i.e., every object in **C** can be expressed as a κ-filtered union of its κ-generated subobjects. In fact, for $\kappa = \aleph_0$, the first infinite cardinal, several important Grothendieck categories are locally finitely generated: for example, if Y is a locally Noetherian space and $\mathcal{O}$ is a sheaf of rings on Y, the Grothendieck category $\mathcal{O} - Mod$ of sheaves of $\mathcal{O}$-modules is locally finitely generated (see [11, Theorem 3.5]). On the other hand, it is also possible that a Grothendieck category may not contain any non-zero finitely generated objects (see [16, pp. 122]).

Proposition 2.6. *Let κ be an infinite regular cardinal and let **C** be a locally κ-generated Grothendieck category. Then, every κ-injective object can be expressed as the colimit of a κ-filtered system of injectives connected by monomorphisms.*

Proof. Let $\mathcal{G}$ be a κ-injective object and let $\mathcal{K} \subseteq \mathcal{G}$ be a κ-generated subobject. Let $E(\mathcal{G})$ (resp. $E(\mathcal{K})$) denote the injective hull of $\mathcal{G}$ (resp. $\mathcal{K}$). We now have a commutative diagram:



In this diagram, the morphism g exists because $E(\mathcal{G})$ is an injective containing $\mathcal{K}$ and the morphism f exists because $\mathcal{G}$ is κ-injective and $\mathcal{K}$ is a κ-generated subobject of $E(\mathcal{G})$. It is clear that $f \circ g \circ j = i$ is a monomorphism. We set $\mathcal{L} := \mathrm{Ker}(f \circ g) \subseteq E(\mathcal{K})$ and suppose that $\mathcal{L} \neq 0$. Then since $\mathcal{K} \subseteq E(\mathcal{K})$ is an essential extension, the subobject $\mathcal{L}' := lim(\mathcal{K} \longrightarrow E(\mathcal{K}) \longleftarrow \mathcal{L})$ of $E(\mathcal{K})$ is non-zero. But then the composition $\mathcal{L}' \longrightarrow \mathcal{K} \overset{j}{\longrightarrow} E(\mathcal{K}) \overset{f\circ g}{\longrightarrow} \mathcal{G}$ which factors through $\mathcal{L}$ is zero, thus contradicting the fact that $i = f \circ g \circ j$ is a monomorphism. Hence, $f \circ g : E(\mathcal{K}) \longrightarrow \mathcal{G}$ is a monomorphism and $\mathcal{G}$ contains the injective hull $E(\mathcal{K})$ of each κ-generated subobject $\mathcal{K}$. Finally, since **C** is locally κ-generated, $\mathcal{G}$ itself may be expressed as the κ-filtered union of its κ-generated subobjects and hence as the κ-filtered union of the injectives $E(\mathcal{K})$, where $\mathcal{K}$ varies over all the κ-generated subobjects of $\mathcal{G}$. □

We remark here that since Grothendieck categories are locally presentable, by the Local Generation Theorem (see [1, pp. 54]) it follows that for every Grothendieck category **C** there exists an infinite regular cardinal $\kappa_{\mathbf{C}}$ such that **C** is locally $\kappa_{\mathbf{C}}$-generated.

3. Inverse limits of systems of epimorphisms

In this section, we will suppose that the Grothendieck category **C** has a generator $\mathcal{U}$ that is also finitely generated, i.e., $Hom_{\mathbf{C}}(\mathcal{U}, \_\_) : \mathbf{C} \longrightarrow Sets$ preserves filtered colimits of monomorphisms. For a Gabriel-Popescu type characterization of such Grothendieck categories as quotients of module categories (see Albu [2, Proposition 5.4.7]). In this section, we will show that any object in **C** may be expressed as the inverse limit of a system of finitely injective objects connected by epimorphisms. First of all, we recall the following fact from [8, § 2]: there exists an inverse system of nonempty sets $\{S_\alpha, g_{\alpha\beta}\}_{\alpha\leq\beta\in\omega_1}$ indexed by the first uncountable ordinal ω_1 connected by surjective maps $g_{\alpha\beta} : S_\beta \longrightarrow S_\alpha$ such that the inverse limit is empty. We will now use this fact and adapt the argument of [4, Lemma 3] to show the following result.

Lemma 3.1. *Let $\mathcal{H}$ be an object of* **C** *and for any set S, let $\mathcal{H}(S)$ denote the direct sum of copies of $\mathcal{H}$ indexed by S. Then, $\{\mathcal{H}(S_\alpha), \mathcal{H}(g_{\alpha\beta})\}_{\alpha\leq\beta\in\omega_1}$ is an inverse system of objects connected by epimorphisms and whose limit is* 0*.*

Proof. For any $\alpha \leq \beta \in \omega_1$, it is clear that $\mathcal{H}(g_{\alpha\beta}) : \mathcal{H}(S_\beta) \longrightarrow \mathcal{H}(S_\alpha)$ is an epimorphism. We set $\mathcal{L}$ to be the inverse limit of the system and consider a morphism $h : \mathcal{U} \longrightarrow \mathcal{L}$. Since $\mathcal{U}$ is finitely generated, for each $\alpha \in \omega_1$, we can choose a finite subset $T'_\alpha \subseteq S_\alpha$ such that the induced map $\mathcal{U} \longrightarrow \mathcal{H}(S_\alpha)$ factors through $\mathcal{H}(T'_\alpha)$. Suppose that the map $\mathcal{U} \longrightarrow \mathcal{H}(S_\alpha)$ factors through $\mathcal{H}(T''_\alpha)$ for some other finite $T''_\alpha \subseteq S_\alpha$. Then, if we take any $x \in T'_\alpha \backslash T''_\alpha$, we see that the projection $\mathcal{U} \longrightarrow \mathcal{H}(T'_\alpha) \longrightarrow \mathcal{H}$ to the x-th copy of $\mathcal{H}$ in the direct sum $\mathcal{H}(T'_\alpha)$ is zero. Hence, $\mathcal{U} \longrightarrow \mathcal{H}(S_\alpha)$ factors through $\mathcal{H}(T'_\alpha - \{x\})$. Repeating this argument with all elements in $(T'_\alpha\backslash T''_\alpha) \cup (T''_\alpha\backslash T'_\alpha)$, we see that $\mathcal{U} \longrightarrow \mathcal{H}(S_\alpha)$ factors through $\mathcal{H}(T'_\alpha \cap T''_\alpha)$. Hence, for each $\alpha \in \omega_1$, we can choose a unique smallest finite set $T_\alpha \subseteq S_\alpha$ such that $\mathcal{U} \longrightarrow \mathcal{H}(S_\alpha)$ factors through $\mathcal{H}(T_\alpha)$. Further, for any $\alpha \leq \beta$, it follows that $T_\alpha \subseteq g_{\alpha\beta}(T_\beta)$ and hence $|T_\alpha| \leq |T_\beta|$.

Further, for any $n \geq 0$, choose $\alpha_n \in \omega_1$ such that $|T_{\alpha_n}| = n$ (whenever possible) and let $\gamma \in \omega_1$ be the supremum of these countably many $\{\alpha_n\}_{n\geq 0}$. We set $M := |T_\gamma|$ and consider some $\beta \geq \gamma$. Then, if $|T_\beta| = M'$, we know that $\beta \geq \gamma$ implies $M' \geq M$. But, on the other hand, the supremum $\gamma \geq \alpha_{M'}$ gives $M \geq M'$. Hence, $M = M'$. Then, for any $\beta' \geq \beta \geq \gamma$, $T_\beta \subseteq g_{\beta\beta'}(T_{\beta'})$ gives us a bijection $T_{\beta'} \longrightarrow T_\beta$. Unless $M = 0$, these bijections would give us elements in the empty inverse limit of $\{S_\alpha\}_{\alpha\in\omega_1}$. Hence, $M = 0$ and $h : \mathcal{U} \longrightarrow \mathcal{L}$ must be 0, i.e., $\mathcal{L} = 0$. □

Proposition 3.2. *Let* **C** *be a Grothendieck category having a finitely generated generator. Then, every object in* **C** *may be expressed as the limit of an inverse system of finitely injective objects connected by epimorphisms.*

Proof. We choose some $\mathcal{A} \in \mathbf{C}$ and express it as $\mathcal{A} = \mathrm{Ker}(f : \mathcal{G} \longrightarrow \mathcal{H})$ with both $\mathcal{G}$, $\mathcal{H}$ injective. We consider the inverse system $\{\mathcal{G} \oplus \mathcal{H}(S_\alpha)\}_{\alpha\in\omega_1}$. For each $\alpha \in \omega_1$, the identity maps $\mathcal{H} \longrightarrow \mathcal{H}$ induce a "sum map" $\Sigma_\alpha : \mathcal{H}(S_\alpha) \longrightarrow \mathcal{H}$. Together with $f : \mathcal{G} \longrightarrow \mathcal{H}$, we obtain a morphism $(f - \Sigma_\alpha) : \mathcal{G} \oplus \mathcal{H}(S_\alpha) \longrightarrow \mathcal{H}$. However, from Lemma 3.1, we know that the limit of $\mathcal{H}(S_\alpha)$ is zero and hence the limit of the morphisms $(f - \Sigma_\alpha)$ over all $\alpha \in \omega_1$ is $f : \mathcal{G} \longrightarrow \mathcal{H}$. Accordingly, we obtain $\mathcal{A} = \mathrm{Ker}(f) = \varprojlim_{\alpha\in\omega_1} \mathrm{Ker}(f - \Sigma_\alpha)$. Since the $\mathcal{G} \oplus \mathcal{H}(S_\alpha)$ are connected by epimorphisms and epimorphisms are stable under pullbacks in an Abelian category, it follows that the kernels $\mathrm{Ker}(f - \Sigma_\alpha)$, $\alpha \in \omega_1$ are also connected by epimorphisms.

It remains to show that each $\mathrm{Ker}(f - \Sigma_\alpha)$ is finitely injective. For this, we choose some element $i_\alpha \in S_\alpha$ and put $S'_\alpha = S_\alpha - \{i_\alpha\}$. Let Σ'_α denote the restriction of $\Sigma_\alpha : \mathcal{H}(S_\alpha) \longrightarrow \mathcal{H}$ to the subobject $\mathcal{H}(S'_\alpha) \subseteq \mathcal{H}(S_\alpha)$. We notice that the morphism $(f - \Sigma_\alpha)$ may be expressed componentwise as $(f - \Sigma_\alpha) = ((f - \Sigma'_\alpha), (-1)) : (\mathcal{G} \oplus \mathcal{H}(S_\alpha)) = (\mathcal{G} \oplus \mathcal{H}(S'_\alpha)) \oplus \mathcal{H} \longrightarrow \mathcal{H}$. Accordingly, we see that:

$$\mathrm{Ker}(f - \Sigma_\alpha) = \lim(\mathcal{G} \oplus \mathcal{H}(S'_\alpha) \xrightarrow{f-\Sigma'_\alpha} \mathcal{H} \xleftarrow{\;1\;} \mathcal{H}) \tag{3.1}$$

which gives $\mathrm{Ker}(f - \Sigma_\alpha) \cong \mathcal{G} \oplus \mathcal{H}(S'_\alpha)$. Since a direct sum of injectives is finitely injective, this proves the result. □

In Proposition 3.2, we see that if the direct sum of injectives in **C** were injective (for example, the Grothendieck category **C** were also locally Noetherian), every object could be expressed as an inverse limit of injectives connected by epimorphisms. Otherwise, it is clear that any direct sum of injectives is finitely injective and it is natural to ask whether all finitely injective objects arise in this manner. However, we show the following result, for which we adapt the proof of [7, Theorem 2.1] (see also [13] for a study of related questions in the case of modules).

Proposition 3.3. *Let* **C** *be a Grothendieck category with a finitely generated generator $\mathcal{U}$. Let $\mathcal{H}$ be an injective object and S be a set such that the direct sum $\mathcal{H}(S)$ of copies of $\mathcal{H}$ indexed by S is not injective. Then, the injective hull $E(\mathcal{H}(S))$ contains a finitely injective subobject that cannot be expressed as a direct sum of a family of injectives.*

Proof. We set $\mathcal{G} := \mathcal{H}(S)$. Since $\mathcal{G}$ is not injective, there exists a morphism $f : \mathcal{V} \longrightarrow \mathcal{G}$ from some subobject $\mathcal{V} \subseteq \mathcal{U}$ that cannot be extended to $\mathcal{U}$. We now let Ω be the collection of all subobjects $\mathcal{G} \subseteq \mathcal{M} \subseteq E(\mathcal{G})$ such that $\mathcal{M}$ is finitely injective and the composed morphism $f_{\mathcal{M}} : \mathcal{V} \xrightarrow{f} \mathcal{G} \hookrightarrow \mathcal{M}$ does not extend to a morphism $\mathcal{U} \longrightarrow \mathcal{M}$. Clearly, $\mathcal{G} \in \Omega$. Further, given a totally ordered collection $\{\mathcal{M}_i\}_{i\in I}$ of objects in Ω, any morphism to the colimit $\mathcal{M}(I) := \varinjlim_{i\in I} \mathcal{M}_i$ from a finitely generated object factors through $\mathcal{M}_{i_0}$ for some $i_0 \in I$. Hence, $\mathcal{M}(I)$ is finitely injective. Further, since the generator $\mathcal{U}$ is also finitely generated, any extension of $f_{\mathcal{M}(I)} : \mathcal{V} \xrightarrow{f} \mathcal{G} \hookrightarrow \mathcal{M}(I)$ to a morphism $\mathcal{U} \longrightarrow \mathcal{M}(I)$ would factor through $\mathcal{M}_{i_0'}$ for some $i_0' \in I$, giving us an extension of the morphism $f_{\mathcal{M}_{i_0'}} : \mathcal{V} \longrightarrow \mathcal{M}_{i_0'}$ to $\mathcal{U}$. This would contradict $\mathcal{M}_{i_0'} \in \Omega$ and hence the upper bound $\mathcal{M}(I) \in \Omega$. By Zorn's Lemma, it follows that we can find some maximal object $\mathcal{N}$ in Ω and we will show that $\mathcal{N}$ cannot be expressed as a direct sum of injectives.

If not, suppose that $\{\mathcal{G}_j\}_{j\in J}$ is a family of injectives such that $\mathcal{N} = \bigoplus_{j\in J} \mathcal{G}_j$ and let $p_j : \mathcal{N} \longrightarrow \mathcal{G}_j$ be the canonical projections. Let $J' \subseteq J$ be the set of all $j \in J$ such that $p_j \circ f_{\mathcal{N}} \neq 0$. If J' is finite, the morphism $f_{\mathcal{N}} : \mathcal{V} \longrightarrow \mathcal{N}$ factors through the injective $\bigoplus_{j\in J'} \mathcal{G}_j$ and hence extends to a morphism $\mathcal{U} \longrightarrow \bigoplus_{j\in J'} \mathcal{G}_j \longrightarrow \mathcal{N}$ which is a contradiction. Hence, J' is infinite and we choose two disjoint infinite subsets J_1 and J_2 such that $J' = J_1 \cup J_2$. We let $p_{J_1} : \mathcal{N} \longrightarrow \bigoplus_{j\in J_1} \mathcal{G}_j$ and $p_{J_2} : \mathcal{N} \longrightarrow \bigoplus_{j\in J_2} \mathcal{G}_j$ denote the canonical projections.

Suppose now that $p_{J_1} \circ f_{\mathcal{N}} : \mathcal{V} \longrightarrow \bigoplus_{j\in J_1} \mathcal{G}_j$ extends to a morphism $h : \mathcal{U} \longrightarrow \bigoplus_{j\in J_1} \mathcal{G}_j$. Since $\mathcal{U}$ is finitely generated, there must be a finite subset $J_1' \subseteq J_1$ such that h factors through $\bigoplus_{j\in J_1'} \mathcal{G}_j$. But then, for any $j \in J_1 \backslash J_1'$ (which is nonempty because J_1 is infinite), we must have $p_j \circ f_{\mathcal{N}} = 0$ which is a contradiction since $j \in J_1 \subseteq J'$. Hence, $p_{J_1} \circ f_{\mathcal{N}}$ (and similarly $p_{J_2} \circ f_{\mathcal{N}}$) does not extend to $\mathcal{U}$. This also implies that neither $\bigoplus_{j\in J_1} \mathcal{G}_j$ nor $\bigoplus_{j\in J_2} \mathcal{G}_j$ are injective. Then, $E(\bigoplus_{j\in J_1} \mathcal{G}_j) \supsetneq \bigoplus_{j\in J_1} \mathcal{G}_j$ and hence

$$\mathcal{N}' := E(\textstyle\bigoplus_{j\in J_1} \mathcal{G}_j) \bigoplus (\bigoplus_{j\in J\backslash J_1} \mathcal{G}_j) \supsetneq (\bigoplus_{j\in J_1} \mathcal{G}_j) \bigoplus (\bigoplus_{j\in J\backslash J_1} \mathcal{G}_j) = \mathcal{N} \tag{3.2}$$

Finally, since $J_2 \subseteq J\backslash J_1$, any extension of $f_{\mathcal{N}'} : \mathcal{V} \longrightarrow \mathcal{N}'$ to $\mathcal{U}$ would also extend the morphism $p_{J_2} \circ f_{\mathcal{N}} : \mathcal{V} \longrightarrow \bigoplus_{j\in J_2} \mathcal{G}_j$ to $\mathcal{U}$, which would be a contradiction. Hence, $\mathcal{N} \subsetneq \mathcal{N}' \in \Omega$, which contradicts maximality of $\mathcal{N}$ in Ω. □

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