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Partial differential equations

On uniqueness for a rough transport–diffusion equation

Sur l'unicité pour une équation de transport–diffusion irrégulière

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A B S T R A C T

In this Note, we study a transport–diffusion equation with rough coefficients, and we prove that solutions are unique in a low-regularity class.

R É S U M É

Dans cette Note, nous étudions une équation de transport–diffusion à coefficients irréguliers, et nous prouvons l'unicité de sa solution dans une classe de fonctions peu régulières.

1. Introduction

In this note, we address the problem of uniqueness for a transport–diffusion equation with rough coefficients. Our primary interest and motivation is a uniqueness result for an equation obeyed by the vorticity of a Leray-type solution of the Navier–Stokes equation in the full, three-dimensional space [5]. The main theorem of this note is the following.

Theorem 1.1. *Let v be a divergence free vector field in $L^2(\mathbb{R}_+, \dot{H}^1(\mathbb{R}^3))$ and a be a function in $L^2(\mathbb{R}_+ \times \mathbb{R}^3)$. Assume that a is a distributional solution of the Cauchy problem*

$$(C) \begin{cases} \partial_t a + \nabla \cdot (av) - \Delta a = 0 \\ a(0) = 0, \end{cases} \tag{1}$$

where the initial condition is understood in the distributional sense. Then a is identically zero on $\mathbb{R}_+ \times \mathbb{R}^3$.

As a preliminary remark, the assumptions on both v and a entail that $\partial_t a$ belongs to $L^1_{\mathrm{loc}}(\mathbb{R}_+, H^{-2}(\mathbb{R}^3))$ and thus, in particular, a is also in $\mathcal{C}(\mathbb{R}_+, \mathcal{D}'(\mathbb{R}^3))$. In Theorem 1.1, a is to be thought of as a scalar component of the vorticity of v, which is in the original problem a Leray solution of the Navier–Stokes equation. In particular, we only know that a belongs to $L^2(\mathbb{R}_+ \times \mathbb{R}^3)$ and $L^\infty(\mathbb{R}_+, \dot{H}^{-1}(\mathbb{R}^3))$, though we will not use the second assumption. The reader accustomed to

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three-dimensional fluid mechanics will notice that, comparing the above equation with the actual vorticity equations in $3D$, a term of the type $a\partial_i v$ is missing. In the original problem, where this Theorem first appeared, we actually rely on a double application of Theorem 1.1. For some technical reasons, only the second application of Theorem 1.1 takes in account the above-mentioned term.

As opposed to the standard DiPerna–Lions theory, we cannot assume that a is in $L^\infty(\mathbb{R}_+, L^p(\mathbb{R}^3))$ for some $p \geq 1$. However, our proof does bear a resemblance to the work of DiPerna and Lions; our result may thus be viewed as a generalization of their techniques, see [1], [2], [3] and [4]. Because of the low regularity of both the vector field v and the scalar field a, the use of energy-type estimates seems difficult. This is the main reason why we rely instead on a duality argument, embodied by the following theorem.

Theorem 1.2. *Given v a divergence free vector field in $L^2(\mathbb{R}_+, \dot{H}^1(\mathbb{R}^3))$ and a smooth φ_0 in $\mathcal{D}(\mathbb{R}^3)$, there exists a distributional solution of the Cauchy problem*

$$(C')\begin{cases} \partial_t\varphi - v\cdot\nabla\varphi - \Delta\varphi = 0 \\ \varphi(0) = \varphi_0 \end{cases} \tag{2}$$

with the bounds

$$\|\varphi(t)\|_{L^\infty(\mathbb{R}^3)} \leq \|\varphi_0\|_{L^\infty(\mathbb{R}^3)} \tag{3}$$

and

$$\|\partial_j\varphi(t)\|^2_{L^2(\mathbb{R}^3)} + \int_0^t \|\nabla\partial_j\varphi(s)\|^2_{L^2(\mathbb{R}^3)}\mathrm{d}s \leq \|\partial_j\varphi_0\|^2_{L^2(\mathbb{R}^3)} + \|\varphi_0\|^2_{L^\infty(\mathbb{R}^3)}\|\partial_j v\|^2_{L^2(\mathbb{R}_+\times\mathbb{R}^3)} \tag{4}$$

for $j = 1, 2, 3$ and any positive time t.

By reversing the arrow of time, this amounts to build, for any strictly positive T, a solution on $[0, T]\times\mathbb{R}^3$ of the Cauchy problem

$$(-C')\begin{cases} -\partial_t\varphi - v\cdot\nabla\varphi - \Delta\varphi = 0 \\ \varphi(T) = \varphi_T, \end{cases} \tag{5}$$

where we have set $\varphi_T := \varphi_0$ for the reader's convenience.

2. Proofs

We begin with the dual existence result.

Proof (of Theorem 1.2). Let us choose some mollifying kernel $\rho = \rho(t, x)$ and denote $v^\delta := \rho_\delta * v$, where $\rho_\delta(t, x) := \delta^{-4}\rho(\frac{t}{\delta}, \frac{x}{\delta})$. Let (C'_δ) be the Cauchy problem (C'), where we replaced v by v^δ. The existence of a (smooth) solution φ^δ to (C'_δ) is then easily obtained thanks to, for instance, a Friedrichs method combined with heat kernel estimates. We now turn to estimates uniform in the regularization parameter δ. The first one is a sequence of energy estimates done in L^p with $p \geq 2$, which yields the maximum principle in the limit. Multiplying the equation on φ^δ by $\varphi^\delta|\varphi^\delta|^{p-2}$ and integrating in space and time, we get

$$\frac{1}{p}\|\varphi^\delta(t)\|^p_{L^p(\mathbb{R}^3)} + (p-1)\int_0^t \|\nabla\varphi^\delta(s)|\varphi^\delta(s)|^{\frac{p-2}{2}}\|^2_{L^2(\mathbb{R}^3)}\mathrm{d}s = \frac{1}{p}\|\varphi_0\|^p_{L^p(\mathbb{R}^3)}. \tag{6}$$

Discarding the gradient term, taking the p-th root on both sides and letting p go to infinity gives

$$\|\varphi^\delta(t)\|_{L^\infty(\mathbb{R}^3)} \leq \|\varphi_0\|_{L^\infty(\mathbb{R}^3)}. \tag{7}$$

To obtain the last estimate, let us derive for $1 \leq j \leq 3$ the equation satisfied by $\partial_j\varphi^\delta$. We have

$$\partial_t\partial_j\varphi^\delta - v^\delta\cdot\nabla\partial_j\varphi^\delta - \Delta\partial_j\varphi^\delta = \partial_j v^\delta\cdot\nabla\varphi^\delta. \tag{8}$$

Multiplying this new equation by $\partial_j\varphi^\delta$ and integrating in space and time gives

$$\frac{1}{2}\|\partial_j\varphi^\delta(t)\|^2_{L^2(\mathbb{R}^3)} + \int_0^t \|\nabla\partial_j\varphi^\delta(s)\|^2_{L^2(\mathbb{R}^3)}\,\mathrm{d}s = \frac{1}{2}\|\partial_j\varphi_0\|^2_{L^2(\mathbb{R}^3)} + \int_0^t\int_{\mathbb{R}^3} \partial_j\varphi^\delta(s,x)\partial_j v^\delta(s,x)\cdot\nabla\varphi^\delta(s,x)\,\mathrm{d}x\,\mathrm{d}s. \tag{9}$$

Since v is divergence free, the gradient term in the left-hand side does not contribute to Equation (9). Denote by $I(t)$ the last integral written above. Integrating by parts and recalling that v is divergence free, we have

$$\begin{aligned} I(t) &= -\int_0^t \int_{\mathbb{R}^3} \varphi^\delta(s,x)\partial_j v^\delta(s,x)\cdot\nabla\partial_j\varphi^\delta(s,x)\,\mathrm{d}x\,\mathrm{d}s \\ &\le \|\varphi_0\|_{L^\infty(\mathbb{R}^3)}\int_0^t \|\partial_j v^\delta(s)\|_{L^2(\mathbb{R}^3)}\|\nabla\partial_j\varphi^\delta(s)\|_{L^2(\mathbb{R}^3)}\,\mathrm{d}s \\ &\le \frac{1}{2}\int_0^t \|\nabla\partial_j\varphi^\delta(s)\|^2_{L^2(\mathbb{R}^3)}\,\mathrm{d}s + \frac{1}{2}\|\varphi_0\|^2_{L^\infty(\mathbb{R}^3)}\int_0^t \|\partial_j v^\delta(s)\|^2_{L^2(\mathbb{R}^3)}\,\mathrm{d}s. \end{aligned}$$

And finally, the energy estimate on $\partial_j\varphi^\delta$ reads

$$\|\partial_j\varphi^\delta(t)\|^2_{L^2(\mathbb{R}^3)} + \int_0^t \|\nabla\partial_j\varphi^\delta(s)\|^2_{L^2(\mathbb{R}^3)}\,\mathrm{d}s \le \|\partial_j\varphi_0\|^2_{L^2(\mathbb{R}^3)} + \|\varphi_0\|^2_{L^\infty(\mathbb{R}^3)}\|\partial_j v\|^2_{L^2(\mathbb{R}_+,\times\mathbb{R}^3)}. \tag{10}$$

Thus, the family $(\varphi^\delta)_\delta$ is bounded in $L^\infty(\mathbb{R}_+, H^1(\mathbb{R}^3)) \cap L^2(\mathbb{R}_+, \dot{H}^2(\mathbb{R}^3)) \cap L^\infty(\mathbb{R}_+ \times \mathbb{R}^3)$. Up to some extraction, we have the weak convergence of $(\varphi^\delta)_\delta$ in $L^2(\mathbb{R}_+, \dot{H}^2(\mathbb{R}^3))$ and its weak-$*$ convergence in $L^\infty(\mathbb{R}_+, H^1(\mathbb{R}^3)) \cap L^\infty(\mathbb{R}_+ \times \mathbb{R}^3)$ to some function φ.

By interpolation, we also have $\nabla\varphi^\delta \rightharpoonup \nabla\varphi$ weakly in $L^4(\mathbb{R}_+, \dot{H}^{\frac{1}{2}}(\mathbb{R}^3))$ as $\delta \to 0$. As a consequence, because $v^\delta \to v$ strongly in $L^2(\mathbb{R}_+, \dot{H}^1(\mathbb{R}^3))$ as $\delta \to 0$, the following convergences hold:

$$\begin{aligned} \Delta\varphi^\delta &\rightharpoonup \Delta\varphi \text{ in } L^2(\mathbb{R}_+ \times \mathbb{R}^3); \\ v^\delta\cdot\nabla\varphi^\delta\,,\ \partial_t\varphi^\delta &\rightharpoonup v\cdot\nabla\varphi\,,\ \partial_t\varphi \text{ in } L^{\frac{4}{3}}(\mathbb{R}_+, L^2(\mathbb{R}^3)). \end{aligned}$$

In particular, such a φ is a distributional solution of (C') with the desired regularity. □

We now state a Lemma that will be useful in the final proof.

Lemma 2.1. *Let v be a fixed, divergence-free vector field in $L^2(\mathbb{R}_+, \dot{H}^1(\mathbb{R}^3))$. Let $(\varphi^\delta)_\delta$ be a bounded family in $L^\infty(\mathbb{R}_+ \times \mathbb{R}^3)$. Let $\rho = \rho(x)$ be some smooth function supported inside the unit ball of $\mathbb{R}^3$ and define $\rho_\varepsilon := \varepsilon^{-3}\rho\left(\frac{\cdot}{\varepsilon}\right)$. Define the commutator $C^{\varepsilon,\delta}$ by*

$$C^{\varepsilon,\delta}(s,x) := v(s,x)\cdot(\nabla\rho_\varepsilon * \varphi^\delta(s))(x) - (\nabla\rho_\varepsilon * (v(s)\varphi^\delta(s)))(x).$$

Then

$$\|C^{\varepsilon,\delta}\|_{L^2(\mathbb{R}_+\times\mathbb{R}^3)} \le \|\nabla\rho\|_{L^1(\mathbb{R}^3)}\|\nabla v\|_{L^2(\mathbb{R}_+,\dot{H}^1(\mathbb{R}^3))}\|\varphi^\delta\|_{L^\infty(\mathbb{R}_+,H^1(\mathbb{R}^3))}. \tag{11}$$

This type of lemma is absolutely not new. Actually, it is strongly reminiscent of Lemma II.1 in [2] and serves the same purpose. We are now in position to prove the main theorem of this note.

Proof (of Theorem 1.1). Let $\rho = \rho(x)$ be a radial mollifying kernel and define $\rho_\varepsilon(x) := \varepsilon^{-3}\rho(\frac{x}{\varepsilon})$. Convolving the equation on a by ρ_ε gives, denoting $a_\varepsilon := \rho_\varepsilon * a$,

$$(C_\varepsilon)\ \ \partial_t a_\varepsilon + \nabla\cdot(a_\varepsilon v) - \Delta a_\varepsilon = \nabla\cdot(a_\varepsilon v) - \rho_\varepsilon * \nabla\cdot(av). \tag{12}$$

Notice that even without any smoothing in time, a_ε, $\partial_t a_\varepsilon$ are in $L^\infty(\mathbb{R}_+, \mathcal{C}^\infty(\mathbb{R}^3))$ and $L^1(\mathbb{R}_+, \mathcal{C}^\infty(\mathbb{R}^3))$ respectively, which is enough to make the upcoming computations rigorous. In what follows, we let φ^δ be a solution of the Cauchy problem $(-C'_\delta)$, with $(-C'_\delta)$ being $(-C')$ with v replaced by v^δ. Let us now multiply, for $\delta, \varepsilon > 0$ the equation (C_ε) by φ^δ and integrate in space and time. After integrating by parts (which is justified by the high regularity of the terms we have written), we get

$$\int_0^T \int_{\mathbb{R}^3} \partial_t a_\varepsilon(s,x)\varphi^\delta(s,x)\,\mathrm{d}x\,\mathrm{d}s = \langle a_\varepsilon(T), \varphi_T\rangle_{\mathcal{D}'(\mathbb{R}^3),\mathcal{D}(\mathbb{R}^3)} - \int_0^T \int_{\mathbb{R}^3} a_\varepsilon(s,x)\partial_t\varphi^\delta(s,x)\,\mathrm{d}x\,\mathrm{d}s$$

and

$$\int_0^T \int_{\mathbb{R}^3} [\nabla \cdot (v(s,x)a_\varepsilon(s,x)) - \rho_\varepsilon(x) * \nabla \cdot (v(s,x)a(s,x))]\, \varphi^\delta(s,x)\, \mathrm{d}x\, \mathrm{d}s = \int_0^T \int_{\mathbb{R}^3} a(s,x) C^{\varepsilon,\delta}(s,x)\, \mathrm{d}x\, \mathrm{d}s,$$

where the commutator $C^{\varepsilon,\delta}$ has been defined in the Lemma. From these two identities, it follows that

$$\langle a_\varepsilon(T), \varphi_T \rangle_{\mathcal{D}'(\mathbb{R}^3),\mathcal{D}(\mathbb{R}^3)} = \int_0^T \int_{\mathbb{R}^3} a(s,x) C^{\varepsilon,\delta}(s,x)\, \mathrm{d}x\, \mathrm{d}s \\ - \int_0^T \int_{\mathbb{R}^3} a_\varepsilon(s,x) \left(-\partial_t \varphi^\delta(s,x) - v(s,x) \cdot \nabla \varphi^\delta(s,x) - \Delta \varphi^\delta(s,x)\right) \mathrm{d}x\, \mathrm{d}s.$$

From the Lemma, we know that $(C^{\varepsilon,\delta})_{\varepsilon,\delta}$ is bounded in $L^2(\mathbb{R}^+ \times \mathbb{R}^3)$. Because $v \cdot \nabla \varphi^\delta \to v \cdot \nabla \varphi$ in $L^{\frac{4}{3}}(\mathbb{R}^+, L^2(\mathbb{R}^3))$ as $\delta \to 0$, the only weak limit point in $L^2(\mathbb{R}_+ \times \mathbb{R}^3)$ of the family $(C^{\varepsilon,\delta})_{\varepsilon,\delta}$ as $\delta \to 0$ is $C^{\varepsilon,0}$. Thanks to the smoothness of a_ε for each fixed ε, we can take the limit $\delta \to 0$ in the last equation, which leads to

$$\langle a_\varepsilon(T), \varphi_T \rangle_{\mathcal{D}'(\mathbb{R}^3),\mathcal{D}(\mathbb{R}^3)} = \int_0^T \int_{\mathbb{R}^3} a(s,x) C^{\varepsilon,0}(s,x)\, \mathrm{d}x\, \mathrm{d}s. \tag{13}$$

Again, the family $(C^{\varepsilon,0})_\varepsilon$ is bounded in $L^2(\mathbb{R}_+ \times \mathbb{R}^3)$ and its only limit point as $\varepsilon \to 0$ is 0, simply because $v \cdot \nabla \varphi_\varepsilon - \rho_\varepsilon * (v \cdot \nabla \varphi) \to 0$ in $L^{\frac{4}{3}}(\mathbb{R}^+, L^2(\mathbb{R}^3))$. Taking the limit $\varepsilon \to 0$, we finally obtain

$$\langle a(T), \varphi_T \rangle_{\mathcal{D}'(\mathbb{R}^3),\mathcal{D}(\mathbb{R}^3)} = 0. \tag{14}$$

This being true for any test function φ_T, $a(T)$ is the zero distribution and finally $a \equiv 0$. $\square$

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