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Functional analysis/Geometry

On norms taking integer values on the integer lattice

Sur les normes prenant des valeurs entières sur un réseau

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ABSTRACT

We present a new proof of Thurston's theorem that the unit ball of a seminorm on $\mathbb{R}^d$ taking integer values on $\mathbb{Z}^d$ is a polyhedron defined by finitely many inequalities with integer coefficients.

RÉSUMÉ

On présente une nouvelle preuve du théorème de Thurston selon lequel la boule unité d'une seminorme sur $\mathbb{R}^d$ prenant des valeurs entières sur $\mathbb{Z}^d$ est un polyèdre défini par un nombre fini d'inégalités à coefficients entiers.

Version française abrégée

Dans l'étude de ce qu'on appelle maintenant la *norme de Thurston* [2], Thurston démontre que la boule unité d'une seminorme N sur $\mathbb{R}^d$ prenant des valeurs entières sur $\mathbb{Z}^d$ est un polyèdre défini par un nombre fini d'inégalités à coefficients entiers. Le but de cette note est de donner une preuve différente et à mon avis plus directe de ce théorème. On travaille avec la boule unité duale B^*, et on démontre que ses points exposés sont contenus dans $\mathbb{Z}^d$, ce qui suffit puisque B^* est l'enveloppe convexe fermée de ses points exposés par le théorème classique de Straszewicz [1]. Soit donc $y_0 \in B^*$ un point exposé. Cela signifie qu'il est l'unique point d'intersection de B^* avec un hyperplan d'appui de B^*, dont on appellera x_0 un vecteur normal, normalisé pour que $\langle x_0, y_0\rangle = 1$. Un petit raisonnement géométrique (Fig. 1 et formule (2)) permet de se convaincre que, pour un point x de $\mathbb{R}^d$, $N(x)$ diffère de $\langle x, y_0\rangle$ d'une quantité tendant vers 0 lorsque x s'éloigne de l'origine en restant à distance bornée de la demi-droite dirigée par x_0. En appliquant ceci à une suite de tels points $x_n \in \mathbb{Z}^d$ et à $x_n + e_j$ avec e_j le j^e vecteur de la base canonique, on en déduit que $\langle e_j, y_0\rangle = \langle x_n + e_j, y_0\rangle - \langle x_n, y_0\rangle$ est arbitrairement proche de l'entier $N(x + e_j) - N(x)$, donc est entier, ce qu'il fallait démontrer.

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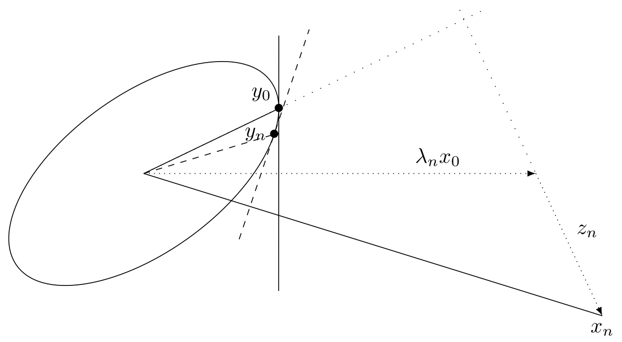

Fig. 1. All lines which look orthogonal are orthogonal.

English version

Consider $\mathbb{R}^d$ with its scalar product $\langle x, y\rangle = \sum_{i=1}^n x_i y_i$ and its associated norm $\|x\|$. The purpose of this note is to give an alternate proof of the following theorem of Thurston [2, Theorem 2].

Theorem 1 *(Thurston). If N is a seminorm on $\mathbb{R}^d$ taking integer values on $\mathbb{Z}^d$, then there is a finite subset $F \subset \mathbb{Z}^d$ such that $N(x) = \max_{y\in F}\langle x, y\rangle$ for all $x \in \mathbb{R}^d$.*

This theorem is important in connection with the Thurston norm [2], which is a seminorm on the second homology group $H_2(M;\mathbb{R})$ of an oriented 3-manifold M taking, by construction, integer values on $H_2(M;\mathbb{Z})$. Theorem 1 allows to deduce the important feature of this seminorm, i.e. that its unit ball is a polyhedron defined by linear inequalities with integer coefficients.

Thurston's proof of Theorem 1 is short and quite easy, but not very enlightening (at least to me). The proof presented here is direct, by showing that the set of exposed points of the dual unit ball $B^* = \{y \in \mathbb{R}^d, \langle x, y\rangle \le N(x) \forall x \in \mathbb{R}^d\}$ is contained in $\mathbb{Z}^d$. The set B^* is a convex compact subset of $\mathbb{R}^d$ symmetric around 0, with non-empty interior if and only if N is a norm. By Hahn–Banach, B^* allows us to recover N by

$$N(x) = \max_{y\in B^*}\langle x, y\rangle. \tag{1}$$

A point $y_0 \in B^*$ is called exposed if there is a supporting hyperplane that intersects B^* at y_0 only, or equivalently if there is $x_0 \in \mathbb{R}^d$ that *exposes* y_0, *i.e.* such that $\langle x_0, y\rangle < \langle x_0, y_0\rangle$ for every $y \in B^* \setminus \{y_0\}$. We shall use the following characterization of exposed points, which asserts that the N-seminorm of a point far in the direction of x_0 is almost attained at y_0.

Lemma 0.1. *Let C be a compact convex set and $y_0 \in C$ be a point exposed by $x_0 \in \mathbb{R}^d$. If $x_n \in \mathbb{R}^d$ is such that $\sup_n \|x_n - nx_0\| < \infty$, then $\lim_n N(x_n) - \langle x_n, y_0\rangle = 0$.*

Before we prove this lemma, let us explain how it implies Thurston's theorem. Let $y_0 \in B^*$ be a point exposed by x_0. For $1 \le j \le d$, let $e_j \in \mathbb{Z}^d$ be the j-th coordinate vector. Pick $x_n \in \mathbb{Z}^d$ the closest point (in Euclidean distance) to nx_0. By the lemma and the fact that $N(x_n) \in \mathbb{Z}$ we have $\lim_n d(\langle x_n, y_0\rangle, \mathbb{Z}) = 0$. Similarly, by the lemma applied to $x_n + e_j$, we have $\lim_n d(\langle x_n + e_j, y_0\rangle, \mathbb{Z}) = 0$, and hence

$$d(\langle e_j, y_0\rangle, \mathbb{Z}) \le \lim_n d(\langle x_n, y_0\rangle, \mathbb{Z}) + d(\langle x_n + e_j, y_0\rangle, \mathbb{Z}) = 0.$$

This proves that $\langle e_j, y_0\rangle \in \mathbb{Z}$.

Since the preceding is valid for every j and every exposed point y_0, we have proved that the set $\exp(B^*)$ of *exposed points* of B^* is contained in $\mathbb{Z}^d$; it is finite because it is bounded. We conclude the proof by the classical theorem by Straszewicz [1], which establishes that, as every convex compact subset of $\mathbb{R}^d$, B^* is the closed convex hull of $\exp(B^*)$, and by (1) $N(x) = \sup_{y\in\exp(B^*)}\langle x, y\rangle$.

Proof of Lemma 0.1. We can normalize x_0 so that $\langle x_0, y_0\rangle = 1$ (unless $\langle x_0, y_0\rangle = 0$, which implies that $C = \{0\}$). In particular $N(x_0) = 1$.

Decompose $x_n = \lambda_n x_0 + z_n$ with $z_n \perp y_0$, so $\lambda_n = \langle x_n, y_0\rangle \le N(x_n)$. Also, z_n is bounded because x_n stays at a bounded distance from nx_0 and $\lambda_n \sim n$ is positive for n large.

Let $y_n \in C$ such that $N(x_n) = \langle x_n, y_n\rangle$. See Fig. 1. Since x_0 exposes y_0 and C is compact, we have $\lim_n \|y_n - y_0\| = 0$ (if $y' \in C$ is an accumulation point of the sequence y_n, then we have $N(x_0) = \lim_n N(x_n/n) = \lim_n\langle x_n/n, y_n\rangle = \langle x_0, y'\rangle$, so that $y' = y_0$ because x_0 exposes y_0). Using that $\langle x_0, y_n\rangle \le 1$ and that $\langle z_n, y_n\rangle = \langle z_n, y_n - y_0\rangle$, we therefore get by the Cauchy–Schwarz inequality:

$$\lambda_n \le N(x_n) = \lambda_n\langle x_0, y_n\rangle + \langle z_n, y_n\rangle \le \lambda_n + \|z_n\|\|y_n - y_0\| = \lambda_n + o(1). \qquad \square \tag{2}$$

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