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Partial differential equations

Solenoidal extensions of vector fields in two-dimensional unbounded domains

Extensions de champs de vecteurs dans des ouverts non bornés

Michel Chipot

Institute of Mathematics, University of Zürich, Winterthurerstrasse 190, CH-8057 Zürich, Switzerland

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ABSTRACT

The goal of this note is to construct solenoidal extensions of vector fields defined on the boundary of simply connected domains having outlets to infinity and which satisfy the Leray–Hopf condition. The case of non-simply connected domains is also mentioned, in particular in the case when the domain admits a symmetry axis. This kind of extensions allows us to solve the stationary Navier–Stokes problem with nonhomogeneous boundary conditions in such domains.

RÉSUMÉ

Le but de cette note est la construction d'extensions de champs de vecteur definis sur la frontière de domaines simplement connexes ayant des canalisations allant à l'infini par des champs de vecteurs à divergence nulle satisfaisant la condition de Leray–Hopf. Le cas des ouverts non simplement connexes est évoqué, en particulier lorsque le domaine possède un axe de symétrie. Ces extensions permettent de résoudre les équations de Navier–Stokes avec des données au bord non homogènes dans ce type d'ouverts.

Version française abrégée

Le problème de la résolution du système de Navier–Stokes stationnaire introduit par J. Leray dans [11] a été résolu dans le cas des ouverts bornés dans [7]. On considère ici un domaine Ω simplement connexe ayant $N \geq 2$ conduits extérieurs allant à l'infini, $\mathcal{O}_1, \ldots, \mathcal{O}_N$, que l'on suppose numérotés dans le sens des aiguilles d'une montre, comme sur la Fig. 2. Le conduit $\mathcal{O}_i$ a pour frontière Γ_i et Γ_{i+1} avec la convention que $\Gamma_{N+1} = \Gamma_1$. On se propose alors de montrer le théorème suivant.

E-mail address: m.m.chipot@math.uzh.ch.

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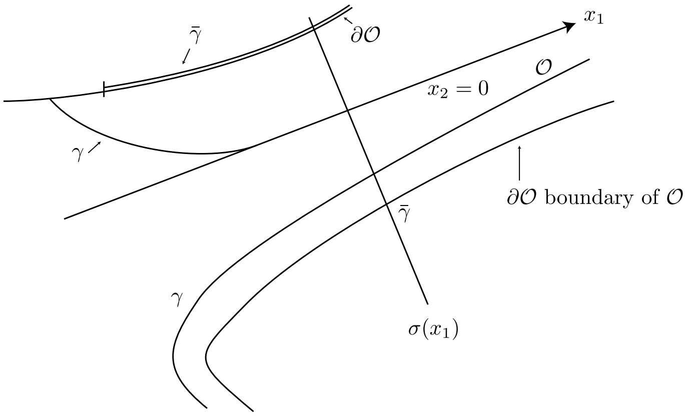

Fig. 1. An outlet at infinity.

Théorème. *Soit* $\mathbf{a} \in W^{\frac{1}{2},2}(\partial\Omega)$ *à support compact tel que*

$$\int_{\Gamma_i} \mathbf{a}\cdot\mathbf{n}\,\mathrm{d}\sigma = F_i,\ i = 1,\ldots,N.$$

Pour $i = 1,\ldots,N$ *soient* $\mathcal{F}_i \in \mathbb{R}$ *tels que*

$$\sum_{i=1}^{N}(F_i + \mathcal{F}_i) = 0.$$

Alors il existe une extension **A** *de* **a** *de divergence nulle satisfaisant l'inégalité de Leray–Hopf*

$$\Big|\int_{\Omega'} (\mathbf{w}\cdot\nabla)\mathbf{w}\cdot\mathbf{A}\,\mathrm{d}x\Big| \le C\epsilon \int_{\Omega'} |\nabla\mathbf{w}|^2\,\mathrm{d}x$$

(pour tout $\mathbf{w} \in (W^{1,2}(\Omega'))^2$ *de divergence nulle et de trace nulle sur* $\partial\Omega$*), mais également la condition de flux*

$$\int_{\sigma_i} \mathbf{A}\cdot\mathbf{n}\,\mathrm{d}\sigma = \mathcal{F}_i,\ i = 1,\ldots,N.$$

(Ω' *est un ouvert borné arbitraire contenu dans* Ω*,* C *est independante de* Ω'*,* **n** *est la normale unitaire à* $\partial\Omega$ *ou à* σ_i *la section du conduit* $\mathcal{O}_i$*.)*

Un théorème identique est vrai dans le cas où Ω n'est pas simplement connexe, pourvu par exemple que l'ouvert Ω possède un axe de symétrie coupant les differentes composantes connexes bornées de la frontière Ω. Ces résultats permettent de montrer l'existence d'une solution pour le problème de Navier–Stokes avec conditions au bord non homogènes et flux prescrit dans les differentes conduites (cf. [6,2,1]). Pour plus de détails sur différents théorèmes d'existence, on renvoie le lecteur à [9,10,5,4,8,12–15,17].

1. Solenoidal vector fields carrying a constant flux to infinity

An outlet at infinity in $\mathbb{R}^2$ is a domain $\mathcal{O}$ contained in a half space and containing a half straight line that we can suppose, in local coordinates, to be on $x_2 = 0$ (see Fig. 1). The boundary of the outlet will be described by a graph g – possibly two if the outlet is nonsymmetric – such that $x_2 = \pm g(x_1)$ for $x = (x_1, x_2) \in \partial\mathcal{O}$. The vector fields that we are going to construct will be defined using a curve γ and a part of the boundary of the outlet that we will denote by $\overline{\gamma}$.

- If γ cuts the boundary of the outlet, $\overline{\gamma}$ is then an infinite connected part of $\partial\mathcal{O}$ such that $\mathrm{dist}(\gamma, \overline{\gamma}) > 0$.
- If γ does not cut the boundary of the outlet $\overline{\gamma}$ is then an infinite connected component of $\partial\mathcal{O}$ (see both cases in Fig. 1).

Denote by ψ a smooth function such that

$$\psi(t) = \begin{cases} 0 \text{ if } t \le 0, \\ 1 \text{ if } t \ge 1, \end{cases}$$

and by ΔA the general distance to a set A introduced in [18]. For x between the curves γ, $\overline{\gamma}$ set for $\epsilon > 0$

$$\xi(x) = \psi(\epsilon \ln(\frac{\Delta\gamma}{\Delta\overline{\gamma}}(x))),\ \boldsymbol{\zeta}(x) = (\partial_{x_2}\xi(x), -\partial_{x_1}\xi(x)).$$

Since the generalized distance is a smooth function, so are ξ and $\boldsymbol{\zeta}$. Moreover $\boldsymbol{\zeta}$ is solenoidal and since $\Delta A(x) \sim \mathrm{dist}(x, A)$ (we write $f \sim g$ if for some constants $\alpha, \beta > 0$ it holds $\alpha f \le g \le \beta f$), one has $\xi(x) = 0$ for $\Delta\gamma \le \Delta\overline{\gamma}$, i.e. in a neighbourhood of γ, $\xi(x) = 1$ for $\Delta\gamma \ge e^{\frac{1}{\epsilon}}\Delta\overline{\gamma}$, i.e. in a neighbourhood of $\overline{\gamma}$, and the support of $\boldsymbol{\zeta}$ is contained in the set

$$\{x \mid e^{-\frac{1}{\epsilon}}\Delta\gamma(x) \le \Delta\overline{\gamma}(x) \le \Delta\gamma(x)\}.$$

Note also that if $\sigma(x_1)$ denotes a section of $\mathcal{O}$ – i.e. $\sigma(x_1) = \{x_2 \mid (x_1, x_2) \in \mathcal{O}\}$ – and if $\mathbf{e}_1$ is the unit normal to $\{x_1\} \times \sigma(x_1)$, one has

$$\int_{\sigma(x_1)} \boldsymbol{\zeta} \cdot \mathbf{e}_1 \,\mathrm{d}x_2 = \int_{\sigma(x_1)} \partial_{x_2}\xi(x)\,\mathrm{d}x_2 = \begin{cases} 1 \text{ when } \gamma \text{ cuts } \partial\mathcal{O}, \\ -1 \text{ when } \gamma \text{ does not cut } \partial\mathcal{O}, \end{cases} \tag{1}$$

assuming that we are in the case of the Fig. 1. Then one defines two “drains” $\mathbf{d}, \mathbf{D}$ by

$$\mathbf{d} = \boldsymbol{\zeta} \text{ when } \gamma \text{ cuts } \partial\mathcal{O},\ \mathbf{D} = -\boldsymbol{\zeta} \text{ when } \gamma \text{ does not cut } \partial\mathcal{O},$$

in such a way that these two drains satisfy

$$\int_{\sigma(x_1)} \mathbf{d} \cdot \mathbf{e}_1 \,\mathrm{d}x_2 = \int_{\sigma(x_1)} \mathbf{D} \cdot \mathbf{e}_1 \,\mathrm{d}x_2 = 1.$$

Using now the fact that, for any multiindex α, one has

$$|D^\alpha \Delta A| \le C_\alpha \mathrm{dist}(x, A)^{1-|\alpha|}$$

one can verify easily that (compare to [16,6])

$$|\partial_{x_i}\xi(x)| \le \frac{C\epsilon}{\mathrm{dist}(x, \overline{\gamma})}\ \forall\, i = 1, 2$$

where C is independent of ϵ. This implies via the Hardy inequality that the vector fields $\mathbf{d}$, $\mathbf{D}$ satisfy the Leray–Hopf inequality. Note that in addition – assuming g Lipschitz continuous – one can show that

$$|\partial_{x_i}\xi(x)| \le \frac{C(\epsilon)}{g(x_1)},\ |\partial^2_{x_i x_j}\xi(x)| \le \frac{C(\epsilon)}{g(x_1)^2}\ \forall\, i, j = 1, 2$$

where this time $C(\epsilon)$ is a constant depending on ϵ. These last estimates are useful for deducing an existence result (see [1,2]).

2. The case of a simply connected domain with N outlets

We consider here a domain Ω simply connected with $N \ge 2$ outlets $\mathcal{O}_1, \dots, \mathcal{O}_N$, which we suppose to be numbered clockwise according to Fig. 2.

The outlet $\mathcal{O}_i$ is having as boundaries Γ_i and Γ_{i+1} with the convention that $\Gamma_{N+1} = \Gamma_1$. Then we have

Theorem 2.1. *Let $\mathbf{a} \in W^{\frac{1}{2},2}(\partial\Omega)$ with compact support such that*

$$\int_{\Gamma_i} \mathbf{a} \cdot \mathbf{n} \,\mathrm{d}\sigma = F_i,\ i = 1, \dots, N.$$

For $i = 1, \dots, N$ let $\mathcal{F}_i \in \mathbb{R}$ be such that

$$\sum_{i=1}^{N} (F_i + \mathcal{F}_i) = 0 \tag{2}$$

then there exists a solenoidal extension $\mathbf{A}$ of $\mathbf{a}$ satisfying the Leray–Hopf inequality

$$|\int_{\Omega'} (\mathbf{w} \cdot \nabla)\mathbf{w} \cdot \mathbf{A} \,\mathrm{d}x| \le C\epsilon \int_{\Omega'} |\nabla \mathbf{w}|^2 \,\mathrm{d}x$$

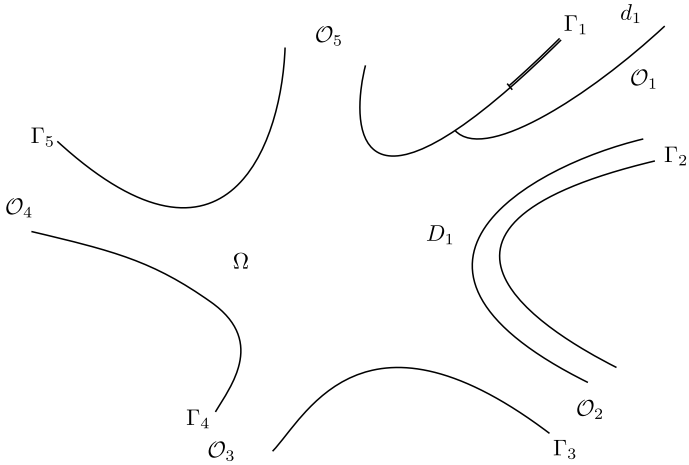

Fig. 2. A case with five outlets.

(for all $\mathbf{w} \in (W^{1,2}(\Omega'))^2$ *solenoidal vanishing on* $\partial\Omega$*) and such that*

$$\int_{\sigma_i} \mathbf{A} \cdot \mathbf{n} \, \mathrm{d}\sigma = \mathcal{F}_i, \; i = 1, \dots, N.$$

(Ω' *is a arbitrary bounded domain contained in* Ω*, C is independent of* Ω'*,* $\mathbf{n}$ *denotes the outward unit normal to* $\partial\Omega$ *or* σ_i *the section of the outlet* $\mathcal{O}_i$*.)*

Proof. For each outlet $\mathcal{O}_i$, we define draining vector fields $\mathbf{d}_i, \mathbf{D}_i$ as introduced above. (See Fig. 2 where we have shown the position of the support of these vector fields in the outlet $\mathcal{O}_1$.) Since they are solenoidal and vanish on a large part of the boundary, one verifies easily that one has (cf. (1))

$$\int_{\Gamma_i} \mathbf{d}_i \cdot \mathbf{n} \, \mathrm{d}\sigma = -\int_{\sigma_i} \mathbf{d}_i \cdot \mathbf{n} \, \mathrm{d}\sigma = -1, \; \int_{\sigma_i} \mathbf{D}_i \cdot \mathbf{n} \, \mathrm{d}\sigma = -\int_{\sigma_{i+1}} \mathbf{D}_i \cdot \mathbf{n} \, \mathrm{d}\sigma = 1. \tag{3}$$

Therefore, for each Γ_i, one has

$$\int_{\Gamma_i} (\mathbf{a} + F_i \mathbf{d}_i) \cdot \mathbf{n} \, \mathrm{d}\sigma = 0$$

and there exists a solenoidal extension of $\mathbf{a} + F_i \mathbf{d}_i$ that we will denote by $\mathbf{a}_{0,i}$ satisfying the Leray–Hopf inequality and having a support located near Γ_i (cf. [3,5,14]). Then one sets

$$\mathbf{A} = \sum_{i=1}^{N} \{\mathbf{a}_{0,i} - F_i \mathbf{d}_i + \sum_{j \le i-1} (F_j + \mathcal{F}_j) \mathbf{D}_{i-1} \}.$$

Since all the vector fields in the sum above satisfy the Leray–Hopf condition, so does $\mathbf{A}$. Note also that $\mathbf{D}_i$ vanishes on $\partial\Omega$ and $\mathbf{a}_{0,i} - F_i \mathbf{d}_i = 0$ on Γ_j for $j \neq i$. Thus on Γ_j one has

$$\mathbf{A} = \mathbf{a}_{0,j} - F_j \mathbf{d}_j = \mathbf{a} + F_j \mathbf{d}_j - F_j \mathbf{d}_j = \mathbf{a}.$$

Moreover, on $\mathcal{O}_j$ the only nonvanishing functions in the sum above are $\mathbf{a}_{0,j}, \; \mathbf{d}_j, \; \mathbf{D}_{j-1} \; \mathbf{D}_j$ and thus

$$\int_{\sigma_j} \mathbf{A} \cdot \mathbf{n} \, \mathrm{d}\sigma = \int_{\sigma_j} \Big\{ \mathbf{a}_{0,j} - F_j \mathbf{d}_j + \sum_{k \le j-1} (F_k + \mathcal{F}_k) \mathbf{D}_{j-1} + \sum_{k \le j} (F_k + \mathcal{F}_k) \mathbf{D}_j \Big\} \cdot \mathbf{n} \, \mathrm{d}\sigma.$$

By (3) it comes

$$\int_{\sigma_j} \mathbf{A} \cdot \mathbf{n} \, \mathrm{d}\sigma = -F_j - \sum_{k \le j-1} (F_k + \mathcal{F}_k) + \sum_{k \le j} (F_k + \mathcal{F}_k) = -F_j + F_j + \mathcal{F}_j = \mathcal{F}_j.$$

This completes the proof of the theorem. $\square$

Remark 1. In the case of a single outlet ($N = 1$), $A = \mathbf{a}_{0,1} - F_1\mathbf{d}_1$ and in this case, due to (2), $\mathcal{F}_1$ cannot be taken arbitrary. Else, when $N > 1$, since the fluxes can be chosen arbitrarily provided (2) holds, one can see that the corresponding Navier–Stokes problem can have infinitely many solutions.

Remark 2. In the case where Ω is not simply connected and possesses internal boundaries $\Gamma'_1, \dots, \Gamma'_n$ the same theorem holds provided

$$\int_{\Gamma'_i} \mathbf{a} \cdot \mathbf{n} \, \mathrm{d}\sigma = 0 \ \forall i = 1, \dots, n$$

(see also [5] IX.4).

Remark 3. In the case where $\mathbf{a}$, Ω are symmetric with respect to one direction (say $x_2 = 0$) if $\mathbf{A} = (A_1, A_2)$ is a solenoidal extension of $\mathbf{a}$, then $\tilde{\mathbf{A}} = \frac{1}{2}(A_1(x_1, x_2) + A_1(x_1, -x_2), A_2(x_1, x_2) - A_2(x_1, -x_2))$ provides a symmetric extension of $\mathbf{a}$ when the $\mathcal{F}_i$ are chosen equal on symmetric outlets. In the case of two symmetric outlets not self symmetric – i.e. not containing both the x_1-axis and exiting in opposite directions – since $\mathcal{F}_1 = \mathcal{F}_2$ and due to (2) this flux cannot be chosen arbitrary when one looks for a symmetric solution.

In the symmetric case and when Ω possesses internal boundaries $\Gamma'_1, \dots, \Gamma'_n$ symmetric and cut by the x_1-axis, there are two cases:

• the x_1-axis cuts Γ_i one boundary of the outlet, then – using the technique of [2] – one can drain the fluxes on this boundary and be reduced to the case treated above. In this case one will develop in [1] another technique;

• the x_1-axis does not cut a boundary of the outlet, then – using the technique of [2] – one can drain the flux at infinity and be reduced to the case above (compare to [6]).

Remark 4. The corresponding existence results for the Navier–Stokes problem with non-homogeneous boundary conditions will be developed in [1].

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