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Newton–Okounkov bodies and complexity functions

Corps de Newton–Okounkov et fonctions de complexité

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ABSTRACT

We show that quite universally the holonomicity of the complexity function of a divisor does not predict whether the Newton–Okounkov body is polyhedral for every choice of a flag.

RÉSUMÉ

Nous montrons que, pratiquement universellement, une condition de régularité de la cohomologie d'un diviseur grand sur une variété projective ne signifie pas que le corps de Newton–Okounkov correspondant est polyédrique.

Let X be a complex projective variety of dimension d with a complete flag of subvarieties $Y_\bullet : \ X = Y_0 \supset \ldots \supset Y_d$ such that Y_d is a smooth point for all Y_i. For any big divisor D on X, [9] and [4] construct a convex body $\Delta_{Y_\bullet}(X; D) \subseteq \mathbb{R}^d$. This Newton–Okounkov body encodes asymptotic information about the sections of multiples of D. For example, its Euclidean volume and the volume of D as a divisor on X (cf. [8, §2.2.C]) coincide up to a normalization factor.

Work of [9] and [2] suggests that the Newton–Okounkov bodies encode many important invariants of the divisor D. For example [2] shows that the set of bodies $\Delta_{Y_\bullet}(X; D)$ considering *all* flags $Y_\bullet$ as above recovers the numerical class of D. More recently, [5,6] recover the non-ample locus $\mathbb{B}_+(D)$ and non-nef locus $\mathbb{B}_-(D)$ of D from the bodies associated with certain flags. On the other hand, [7] shows that the particular shape of $\Delta_{Y_\bullet}(X; D)$ for many flags $Y_\bullet$ is not necessarily descriptive of the positivity properties of D.

In this note we further this investigation on algebro-geometric consequences of the shape of Newton–Okounkov bodies. We show that quite universally a certain regularity condition on the cohomology of D that has been considered by Katzarkov and Liu [3] among others does not determine the polyhedrality of $\Delta_{Y_\bullet}(X; D)$. The strategy is to lift a generalization of an example of [7] to some representative of each birational class of varieties of dimension $d \geqslant 4$.

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The complexity function of a functor F on a category $\mathcal{C}$ with respect to objects E_1, E_2 in $\mathcal{C}$ is defined in [1,3] by

$$E_{E_1,E_2,F}(x,q) = \sum_{n,i} \dim \operatorname{Ext}^i(E_1, F^n E_2) x^n q^i.$$

In particular, they ask whether $E_{E_1,E_2,F}$ is holonomic. This would imply that the function $E_{E_1,E_2,F}(x,q)$ is determined by a finite set of data. Concretely, a formal power series $f = \sum_{i,n} a_{i,n} x^n q^i$ is *holonomic*, or *D-finite* (see [11, Definition 15]), if there exist tuples of not all zero polynomials $p_0, \ldots, p_k$ and $g_0, \ldots, g_k$ in x and q such that f satisfies linear differential equations

$$p_0 \frac{\partial^k f}{\partial x^k} + p_1 \frac{\partial^{k-1} f}{\partial x^{k-1}} + \cdots + p_k f = 0$$
$$g_0 \frac{\partial^k f}{\partial q^k} + g_1 \frac{\partial^{k-1} f}{\partial q^{k-1}} + \cdots + g_k f = 0.$$

In general, $E_{E_1,E_2,F}(x,q)$ cannot be expected to be holonomic. [1] gives an example where X is a $\mathbb{P}^1$-bundle over a product of elliptic curves, $E_1 = E_2 = \mathcal{O}_X$, and F is the twist by an appropriately chosen big divisor. The authors suggest that this failure of being holonomic can be linked to the non-polyhedrality of the categorical Okounkov-body of D (cf. [3, Section 4]). We consider the following

Question. Let $X = Y_0 \supset Y_1 \supset \ldots \supset Y_d$ be an admissible flag on the projective variety X of dimension d, i.e., Y_k is irreducible of codimension k and smooth at the point Y_d for all k. Let D be a big divisor on X whose associated Newton–Okounkov body $\Delta_{Y_\bullet}(X; D)$ is nonpolyhedral. Is the complexity function

$$E_{X,D}(x,q) := \sum_{n,i} \dim H^i(X; \mathcal{O}_X(nD)) x^n q^i$$

necessarily non-holonomic?

Our first counterexample has D *ample*:

Example 1. (See [7].) Let $X = \mathbb{P}^2 \times \mathbb{P}^2$. Consider $E \subseteq \mathbb{P}^2$ an elliptic curve without complex multiplication. Construct the flag:

- $Y_0 = X$.
- $Y_1 = \mathbb{P}^2 \times E$.
- $Y_2 = E \times E$.
- Y_3 is general in the complete linear series $|f_1 + f_2 + \Delta_E|$ on Y_2. Here f_1, f_2 are the fibers of the projections to the factors in the product, and Δ is the diagonal.
- Y_4 is a general point on Y_3.

Let D be a divisor with associated line bundle $\mathcal{O}(3,1)$. [7, Example 3.4] shows that the associated Newton–Okounkov body $\Delta_{Y_\bullet}(X; D)$ is not polyhedral.

On the other hand, the complexity function of an ample divisor is holonomic by the following lemma. □

Lemma 2. *Let X be a projective variety of dimension d and let D be a semi-ample divisor on it. Then the complexity function $E_{X,D}(x,q)$ is holonomic.*

Proof. Assume first that $\mathcal{O}_X(D)$ is globally generated. Let $\pi : X \to \mathbb{P}^N$ be the morphism determined by the complete linear series $|D|$. Then $D = \pi^* \mathcal{O}_{\mathbb{P}^N}(1)$. The ampleness of the hyperplane class on $\mathbb{P}^N$, the Leray spectral sequence, and Serre vanishing imply

$$H^i(X; \mathcal{O}_X(nD)) = H^0\left(\mathbb{P}^N; R^i \pi_* \mathcal{O}_X \otimes \mathcal{O}_{\mathbb{P}^N}(n)\right)$$

for n sufficiently large. There exists polynomials P_i such that $P_i(n) = \dim H^0(\mathbb{P}^N; R^i \pi_* \mathcal{O}_X \otimes \mathcal{O}_{\mathbb{P}^N}(n))$ for any i and sufficiently large n. Then up to finitely many terms, which in any case do not influence holonomicity (cf. [10, Proposition 2.3.(ii)]),

$$E_{X,D}(x,q) = \sum_{i=0}^{d} q^i \sum_r P_i(r) x^r.$$

This is a rational function in x, polynomial in q. As such it is algebraic, and in particular holonomic (cf. [10, Proposition 2.3]).

When D is only semi-ample, let m be such that $|mD|$ is basepoint free. Let $\pi : X \to \mathbb{P}^N$ be the morphism determined by this linear series so that $\mathcal{O}_X(mD) = \pi^*\mathcal{O}_{\mathbb{P}^N}(1)$. Write $n = am + r$ with $0 \leqslant r < m$. Then

$$H^i(X; \mathcal{O}_X(nD)) = H^0(\mathbb{P}^N; R^i\pi_*\mathcal{O}_X(rD) \otimes \mathcal{O}_{\mathbb{P}^N}(a)).$$

For large n, as in the globally generated case, its dimension is $P_{i,r}(a)$ for some polynomial $P_{i,r}$. It is then easy to see that the complexity function is again algebraic. □

We have so far seen that the answer to the question above is negative by giving one example. In the remainder of this note we will see that in a sense it is universally negative. Concretely, we show that in each birational equivalence class of varieties of dimension $d \geqslant 4$ there exists a smooth model X carrying an admissible flag and a big and semi-ample divisor such that the corresponding Newton–Okounkov body is non-polyhedral, while by Lemma 2 the complexity function $E_{X,D}(x, q)$ is holonomic.

Theorem 3. *Let X be a normal projective variety of dimension $d \geqslant 4$. Then there exists a birational model $\widetilde{X} \to X$ containing an admissible flag $X_\bullet$ and a semi-ample divisor H such that the Newton–Okounkov body $\Delta_{X_\bullet}(\widetilde{X}; H)$ is non-polyhedral.*

Proof. Let us first assume that $d = 4$. Up to blowing-up, we may assume that there exists a generically finite morphism $\pi : X \to \mathbb{P}^2 \times \mathbb{P}^2$ with X smooth (e.g. choose an appropriate birational model of a Noether normalization $X \to \mathbb{P}^4$).

We may choose the flag elements in the example of [7] in the previous section such that $X_k := \pi^{-1}Y_k$ is irreducible and smooth for all $k < 4$. (If $Y_1 = \mathbb{P}^2 \times E$ is general in $|\mathcal{O}(0, 3)|$, then its inverse image X_1 is smooth irreducible. To construct Y_2, consider the first projection $\mathbb{P}^2 \times E \to \mathbb{P}^2$ and pullback a general $PGL(3)$ translate of $E \subseteq \mathbb{P}^2$. The pullback X_2 of Y_2 to X_1 is then smooth by Kleiman transversality, connected by the Fulton–Hansen theorem [8, Theorem 3.3.6], hence also irreducible. See also [8, Example 3.3.10]. The curve Y_3 can be chosen general in a very ample linear series, thus its pullback to X_2 is smooth irreducible.) Let X_4 be an arbitrary point in $\pi^{-1}Y_4$. Then $X_\bullet$ is an admissible flag on X. Put $H := \pi^*D$

We claim that for some $\varepsilon > 0$, the slice $\Delta_{X_\bullet}(X; H) \cap \big(\{0\} \times [0, \varepsilon] \times \mathbb{R}^2\big)$ is non-polyhedral. Pick $\varepsilon > 0$ such that $D|_{Y_1} - sY_2$ is ample for all $s \in [0, \varepsilon]$. It is not hard to check that for $s \in [0, \varepsilon]$ the identity

$$\Delta_{X_\bullet}(X; H) \cap \big(\{0\} \times \{s\} \times \mathbb{R}^2\big) = \Delta_{X_\bullet}(X_2; \pi^*(D|_{Y_1} - sY_2))$$

holds: [7, Proposition 3.1] handles the case of restricting ample divisors. The big and semiample case follows from this by the continuity of slices in the global Newton–Okounkov body of X by considering a collection of divisors E_s on X_1 such that $\pi^*(D|_{Y_1} - sY_2) - tE_s$ is ample on X_1 for all $t \in [0, 1]$ and $s \in (0, \varepsilon]$.

As in [7, Remark 3.3, Example 3.4], to conclude that $\Delta_{X_\bullet}(X; H)$ is not polyhedral, it is enough to show that the cone which is the translation by $[H|_{X_2}]$ of the convex span of $-[X_2|_{X_2}]$ and $-[X_3]$ meets the boundary of $\mathrm{Nef}(X_2)$ along a curve that is not piecewise linear.

For this, observe that $\pi^*\alpha \in \mathrm{Nef}(X_2)$ iff $\alpha \in \mathrm{Nef}(Y_2)$, and $\pi^*\alpha \in \overline{\mathrm{NE}}(X_2)$ iff $\alpha \in \overline{\mathrm{NE}}(Y_2)$. Similar equivalences hold for ample and for big classes respectively. Furthermore, $\mathrm{Nef}(Y_2) = \overline{\mathrm{NE}}(Y_2)$ by [7, Example 3.4] and $\pi^* : \mathrm{NS}(Y_2) \to \mathrm{NS}(X_2)$ is a linear injection, since π is dominant. Therefore the boundary curve (in the sense of the previous paragraph) on $\mathrm{Nef}(X_2)$ is identified via pullback with the boundary curve on $\mathrm{Nef}(Y_2)$. By [7, Example 3.4], the latter is conic, not piecewise linear.

For X of arbitrary dimension $d \geqslant 4$, as above we can assume that there is a generically finite dominant morphism $\pi : X \to \mathbb{P}^{d-2} \times \mathbb{P}^2$ with X smooth. Consider in the image the flag $Y_\bullet$ given as follows:

$$Y_k = \mathbb{P}^{d-2-k} \times \mathbb{P}^2$$

for $k \leqslant d-4$, and $Y_{d-3}, \ldots, Y_d$ is the flag from Example 1. We can again argue by Bertini type arguments that the pre-images $X_k := \pi^{-1}Y_k$ for $k < d$ are smooth and irreducible, thus any choice of the point X_d makes $X_\bullet$ into an admissible flag. Now the same arguments as above yield that for $D = \pi^*\mathcal{O}_{\mathbb{P}^{d-2}\times\mathbb{P}^2}(3, 1)$ and some $\varepsilon > 0$ the slice

$$\Delta_{X_\bullet}(X; D) \cap \big(\{0\}^{d-3} \times [0, \varepsilon] \times \mathbb{R}^2\big)$$

is non-polyhedral. □

Remark 4. Considering the original question, it is natural to ask whether at least the reverse implication is true, i.e., whether the polyhedrality of some Newton–Okounkov body implies holonomicity of the corresponding complexity function. However, the fact that Newton–Okounkov bodies do not carry information about all sections in each degree of a graded linear series suggests a negative answer to this question as well. A good candidate would be a linear series whose semi-group of valuation vectors is not finitely generated but whose Newton–Okounkov body is polyhedral nonetheless.

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