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Harmonic analysis

L^p harmonic analysis for differential-reflection operators

Salem Ben Saïd [a], Asma Boussen [b], Mohamed Sifi [b]

[a] Institut Élie Cartan de Lorraine, Université de Lorraine, BP 239, 54506 Vandœuvre-Lès-Nancy, France
[b] Université de Tunis El Manar, Faculté des sciences de Tunis, LR11ES11 Laboratoire d'analyse mathématiques et applications, 2092 Tunis, Tunisia

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ABSTRACT

We introduce and study differential-reflection operators $\Lambda_{A,\varepsilon}$ acting on smooth functions defined on $\mathbb{R}$. Here A is a Sturm–Liouville function with additional hypotheses and $\varepsilon \in \mathbb{R}$. For special pairs (A, ε), we recover Dunkl's, Heckman's and Cherednik's operators (in one dimension).
As, by construction, the operators $\Lambda_{A,\varepsilon}$ are mixture of $\mathrm{d}/\mathrm{d}x$ and reflection operators, we prove the existence of an operator $V_{A,\varepsilon}$ so that $\Lambda_{A,\varepsilon} \circ V_{A,\varepsilon} = V_{A,\varepsilon} \circ \mathrm{d}/\mathrm{d}x$. The positivity of the intertwining operator $V_{A,\varepsilon}$ is also established.
Via the eigenfunctions of $\Lambda_{A,\varepsilon}$, we introduce a generalized Fourier transform $\mathscr{F}_{A,\varepsilon}$. For $-1 \leq \varepsilon \leq 1$ and $0 < p \leq \frac{2}{1+\sqrt{1-\varepsilon^2}}$, we develop an L^p-Fourier analysis for $\mathscr{F}_{A,\varepsilon}$, and then we prove an L^p-Schwartz space isomorphism theorem for $\mathscr{F}_{A,\varepsilon}$.
Details of this paper will be given in other articles [3] and [4].

RÉSUMÉ

Nous introduisons et étudions des opérateurs différentiels aux différences $\Lambda_{A,\varepsilon}$ agissant sur les fonctions régulières définies sur $\mathbb{R}$. Ici A est une fonction de Sturm–Liouville avec des hypothèses supplémentaires et $\varepsilon \in \mathbb{R}$. Pour des cas particuliers de paires (A, ε), nous obtenons les opérateurs de Dunkl, de Heckman et de Cherednik (unidimensionnels).
Comme, par construction, les opérateurs $\Lambda_{A,\varepsilon}$ entremêlent $\mathrm{d}/\mathrm{d}x$ et des opérateurs de réflexion, nous prouvons qu'il existe un opérateur $V_{A,\varepsilon}$ tel que $\Lambda_{A,\varepsilon} \circ V_{A,\varepsilon} = V_{A,\varepsilon} \circ \mathrm{d}/\mathrm{d}x$. La positivité de l'opérateur $V_{A,\varepsilon}$ a été établie.
À l'aide des fonctions propres de $\Lambda_{A,\varepsilon}$, nous introduisons une transformée de Fourier généralisée $\mathscr{F}_{A,\varepsilon}$. Nous développons de l'analyse de Fourier de type L^p pour $\mathscr{F}_{A,\varepsilon}$ quand $-1 \leq \varepsilon \leq 1$ et $0 < p \leq \frac{2}{1+\sqrt{1-\varepsilon^2}}$, et nous caractérisons l'image des p-espaces de Schwartz par $\mathscr{F}_{A,\varepsilon}$.
Les détails seront publiés dans d'autres articles [3] et [4].

E-mail addresses: salem.bensaid@univ-lorraine.fr (S. Ben Saïd), asma.boussen@live.fr (A. Boussen), mohamed.sifi@fst.rnu.tn (M. Sifi).

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1. A family of differential-reflection operators

It became apparent long ago that radial Fourier analysis on real-rank-one symmetric spaces is closely connected to certain classes of special functions in one variable:

- Bessel functions in connection with radial Fourier analysis on Euclidean spaces,
- Jacobi functions in connection with radial Fourier analysis on hyperbolic spaces.

We refer to [12] for a detailed exposition.

In the late 80's/early 90's Dunkl [10] found a remarkable family of commuting operators that now bear his name. In one dimension, this reads

$$D_\alpha f(x) = f'(x) + \frac{2\alpha+1}{x}\left(\frac{f(x)-f(-x)}{2}\right) \qquad \alpha \geq -1/2. \tag{1.1}$$

The eigenfunctions of Dunkl's operators, known as the Dunkl kernel, are the nonsymmetric version of Bessel functions.

Some years after [10], in [8] Cherednik wrote down a trigonometric variant of the Dunkl operator. In one dimension, this reads

$$T_{\alpha,\beta} f(x) = f'(x) + \Big\{(2\alpha+1)\coth x + (2\beta+1)\tanh x\Big\}\left(\frac{f(x)-f(-x)}{2}\right) - \varrho f(-x), \tag{1.2}$$

where $\alpha \geq \beta \geq -1/2$, $\alpha \neq -1/2$, and $\varrho = \alpha+\beta+1$. The eigenfunctions of Cherednik's operators, known as the Opdam functions [14], are the nonsymmetric version of the Jacobi functions. We mention that the trigonometric Dunkl operators were originally introduced by Heckman [11] in a different form. In one dimension, his operator reads:

$$S_{\alpha,\beta} f(x) = f'(x) + \Big\{(2\alpha+1)\coth x + (2\beta+1)\tanh x\Big\}\left(\frac{f(x)-f(-x)}{2}\right).$$

This paper gives some aspects of harmonic analysis associated with the following family of one dimensional (A,ε)-operators

$$\Lambda_{A,\varepsilon} f(x) = f'(x) + \frac{A'(x)}{A(x)}\left(\frac{f(x)-f(-x)}{2}\right) - \varepsilon\varrho f(-x),$$

where $\varepsilon \in \mathbb{R}$ and $A:\mathbb{R}\to\mathbb{R}^+$ satisfies the following conditions (cf. [5,6,16]):

(C1) $A(x) = |x|^{2\alpha+1}B(x)$, where $\alpha > -\frac{1}{2}$ and $B \in C^\infty(\mathbb{R})$ is even, positive, and $B(0)=1$.
(C2) On $\mathbb{R}^+ \setminus \{0\}$, A is increasing, whereas A'/A is decreasing. This condition implies that the limit $\varrho := \lim_{x\to+\infty} A'(x)/2A(x) \geq 0$ exists.
(C3) There exists a constant $\delta > 0$ such that for $x \gg 0$,

$$\frac{A'(x)}{A(x)} = \begin{cases} 2\varrho + e^{-\delta x}D(x) & \text{if } \varrho > 0,\\ \dfrac{2\alpha+1}{x} + e^{-\delta x}D(x) & \text{if } \varrho = 0,\end{cases} \tag{1.3}$$

with $|D^{(k)}(x)| \leq c_k$ for all $x \gg 0$ and $k \in \mathbb{N}$.

The function A and the real number ε are the deformation parameters giving back the above three operators (as special examples) when:

(1) $A(x) = A_\alpha(x) = |x|^{2\alpha+1}$ and ε arbitrary (Dunkl's operators D_α),
(2) $A(x) = A_{\alpha,\beta}(x) = |\sinh x|^{2\alpha+1}(\cosh x)^{2\beta+1}$ and $\varepsilon = 0$ (Heckman's operators $S_{\alpha,\beta}$),
(3) $A(x) = A_{\alpha,\beta}(x) = |\sinh x|^{2\alpha+1}(\cosh x)^{2\beta+1}$ and $\varepsilon = 1$ (Cherednik's operators $T_{\alpha,\beta}$).

Let $\lambda \in \mathbb{C}$ and consider the initial data problem

$$\Lambda_{A,\varepsilon} f(x) = i\lambda f(x), \qquad f(0) = 1, \tag{1.4}$$

where $f:\mathbb{R}\to\mathbb{C}$. We prove that:

Theorem 1.1.

I) *For $\lambda \in \mathbb{C}$, there exists a unique solution $\Psi_{A,\varepsilon}(\lambda,\cdot)$ to the problem (1.4). Further, for every $x \in \mathbb{R}$, the function $\lambda \mapsto \Psi_{A,\varepsilon}(\lambda, x)$ is analytic on $\mathbb{C}$.*

II) *Under the restriction* $-1 \le \varepsilon \le 1$, *for all* $x \in \mathbb{R}$ *we have:*
 1) *for* $\lambda \in \mathbb{R}$, *we have* $|\Psi_{A,\varepsilon}(\lambda, x)| \le \sqrt{2}$.
 2) *for* $\lambda \in \mathrm{i}\mathbb{R}$, *we have* $\Psi_{A,\varepsilon}(\lambda, x) > 0$.
 3) *Assume that* $\lambda \in \mathbb{C}$ *and* $|x| \ge x_0$ *with* $x_0 > 0$. *Then*

$$\left|\partial_x^N \Psi_{A,\varepsilon}(\lambda, x)\right| \le c(|\lambda|+1)^N(|x|+1)e^{(|\operatorname{Im}\lambda| - \varrho(1-\sqrt{1-\varepsilon^2}))\,|x|}.$$

 4) *Assume that* $\lambda \in \mathbb{C}$ *and* $x \in \mathbb{R}$. *Then*

$$\left|\partial_\lambda^M \Psi_{A,\varepsilon}(\lambda, x)\right| \le c|x|^M(|x|+1)e^{(|\operatorname{Im}\lambda| - \varrho(1-\sqrt{1-\varepsilon^2}))\,|x|}.$$

Sketch of Proof. I) The proof is based on the following facts:

Fact 1) Under the conditions (C1) and (C2), the Cauchy problem

$$\begin{cases} h''(x) + \frac{A'(x)}{A(x)}h'(x) = -(\mu^2 + \varrho^2)h(x) \\ h(0) = 1, \quad h'(0) = 0, \end{cases} \tag{1.5}$$

with $\mu \in \mathbb{C}$, admits a unique solution, which we denote by φ_μ (see [6,7]).

Fact 2) Define μ_ε so that $\mu_\varepsilon^2 = \lambda^2 + (\varepsilon^2 - 1)\varrho^2$. For $i\lambda \ne \varepsilon\varrho$, the function

$$\Psi_{A,\varepsilon}(\lambda, x) := \varphi_{\mu_\varepsilon}(x) + \frac{1}{i\lambda - \varepsilon\varrho}\varphi'_{\mu_\varepsilon}(x), \tag{1.6}$$

satisfies the problem (1.4).

Fact 3) We may rewrite (1.6) as

$$\Psi_{A,\varepsilon}(\lambda, x) = \varphi_{\mu_\varepsilon}(x) + (i\lambda + \varepsilon\varrho)\frac{\mathrm{sg}(x)}{A(x)}\int_0^{|x|} \varphi_{\mu_\varepsilon}(t)A(t)\,\mathrm{d}t, \tag{1.7}$$

which implies that $\lambda \mapsto \Psi_{A,\varepsilon}(\lambda, x)$ is analytic, and therefore the restriction on λ can be dropped. The uniqueness follows by standard arguments.

II.1) The proof is inspired by Opdam's proof of Proposition 6.1 in [14]. Using the fact that $\Psi_{A,\varepsilon}$ satisfies

$$\Psi'_{A,\varepsilon}(\lambda, x) = -\frac{A'(x)}{2A(x)}\Big(\Psi_{A,\varepsilon}(\lambda, x) - \Psi_{A,\varepsilon}(\lambda, -x)\Big) + \varepsilon\varrho\Psi_{A,\varepsilon}(\lambda, -x) + \mathrm{i}\lambda\Psi_{A,\varepsilon}(\lambda, x), \tag{1.8}$$

we prove that for all $x \in \mathbb{R}^+$, the derivative $\left\{|\Psi_{A,\varepsilon}(\lambda, -x)|^2 + |\Psi_{A,\varepsilon}(\lambda, x)|^2\right\}' \le 0$. This implies that for $x \in \mathbb{R}^+$, we have $|\Psi_{A,\varepsilon}(\lambda, -x)|^2 + |\Psi_{A,\varepsilon}(\lambda, x)|^2 \le |\Psi_{A,\varepsilon}(\lambda, 0)|^2 + |\Psi_{A,\varepsilon}(\lambda, 0)|^2 = 2$.

II.2) Assume that $\Psi_{A,\varepsilon}(\lambda, \cdot)$ is not strictly positive. Since $\Psi_{A,\varepsilon}(\lambda, 0) = 1 > 0$, it follows that $\Psi_{A,\varepsilon}(\lambda, \cdot)$ vanishes. Let x_0 be a zero of $\Psi_{A,\varepsilon}(\lambda, \cdot)$ so that $|x_0| = \inf\left\{|x| \,:\, \Psi_{A,\varepsilon}(\lambda, x) = 0\right\}$. We prove that $\Psi_{A,\varepsilon}(\lambda, \pm x_0) = 0$ and $\Psi'_{A,\varepsilon}(\lambda, \pm x_0) = 0$. Differentiating (1.8), we see that the second derivative of $\Psi_{A,\varepsilon}(\lambda, \cdot)$ vanishes at $\pm x_0$. Repeating the same argument over and over again to get $\Psi^{(k)}_{A,\varepsilon}(\lambda, \pm x_0) = 0$ for all $k \in \mathbb{N}$. Since $\Psi_{A,\varepsilon}(\lambda, \cdot)$ is a real analytic function, we deduce that $\Psi_{A,\varepsilon}(\lambda, x) = 0$ for all $x \in \mathbb{R}$. This contradicts $\Psi_{A,\varepsilon}(\lambda, 0) = 1$.

II.3) If $N = 0$ we show that for $\lambda \in \mathbb{C}$ we have

$$|\Psi_{A,\varepsilon}(\lambda, x)| \le \Psi_{A,\varepsilon}(0, x)\,\mathrm{e}^{|\operatorname{Im}\lambda|\,|x|}, \tag{1.9}$$

where $\Psi_{A,\varepsilon}(0, x) = 1$ for $\varepsilon = 0$, and $\Psi_{A,\varepsilon}(0, x) \le c_\varepsilon(|x|+1)\,\mathrm{e}^{-\varrho(1-\sqrt{1-\varepsilon^2})|x|}$ for $\varepsilon \ne 0$. So assume $N \ge 1$. The identity (1.8) allows us to express the derivatives of $\Psi_{A,\varepsilon}(\lambda, \cdot)$ in terms of lower-order derivatives. On the other hand, since $A'/(2A)$ satisfies the condition (C3), it follows that

$$\left|\left(\frac{A'(x)}{2A(x)}\right)^{(N)}\right| \le C, \qquad \forall\, |x| \ge x_0 \text{ with } x_0 > 0.$$

II.4) If $M = 0$ this is just (1.9). So assume $M \ge 1$. If $x = 0$, the statement follows from Liouville's theorem. If $x \ne 0$, apply Cauchy's integral formula for $\Psi_{A,\varepsilon}(\lambda, x)$ over a circle with radius proportional to $\frac{1}{|x|}$, centered at λ in the complex plane. □

2. The existence and the positivity of an intertwining operator

Recall from the (sketch of) proof of Theorem 1.1.I the function φ_μ which is the unique solution to the Cauchy problem (1.5). By [6] we have the following Laplace type representation

$$\varphi_\mu(x) = \int_0^{|x|} K(|x|, y)\cos(\mu y)\,\mathrm{d}y \qquad x \in \mathbb{R}^*, \tag{2.1}$$

where $K(|x|, \cdot)$ is a non-negative even continuous function supported in $[-|x|, |x|]$. Using a Delsarte type operator introduced in [15, Proposition 2.1] (see also Theorem 5.1 in [13]), we prove that the integral representation (2.1) can be rewritten as

$$\varphi_{\mu_\varepsilon}(x) = \int_0^{|x|} K_\varepsilon(|x|, y)\cos(\lambda y)\,\mathrm{d}y \qquad x \in \mathbb{R}^*, \tag{2.2}$$

where the relationship between μ_ε and λ is given by $\mu_\varepsilon^2 = \lambda^2 + (\varepsilon^2 - 1)\varrho^2$. Here $K_\varepsilon(|x|, \cdot)$ is even, continuous and supported in $[-|x|, |x|]$. Now, in view of the expression (1.7) of the eigenfunction $\Psi_{A,\varepsilon}(\lambda, x)$, we deduce that

$$\Psi_{A,\varepsilon}(\lambda, x) = \int_{|y|<|x|} \mathbb{K}_\varepsilon(x, y)\,\mathrm{e}^{\mathrm{i}\lambda y}\,\mathrm{d}y \qquad x \in \mathbb{R}^*, \tag{2.3}$$

where $\mathbb{K}_\varepsilon(x, \cdot)$ is a continuous function supported in $[-|x|, |x|]$. This integral representation of $\Psi_{A,\varepsilon}(\lambda, x)$ is the starting point for obtaining an intertwining operator between the operator $\Lambda_{A,\varepsilon}$ and the ordinary derivative $\mathrm{d}/\mathrm{d}x$. More precisely, for $f \in C^\infty(\mathbb{R})$, we define $V_{A,\varepsilon} f$ by

$$V_{A,\varepsilon} f(x) = \begin{cases} \displaystyle\int_{|y|<|x|} \mathbb{K}_\varepsilon(x, y) f(y)\,\mathrm{d}y & x \neq 0 \\ f(0) & x = 0, \end{cases} \tag{2.4}$$

where the kernel $\mathbb{K}_\varepsilon(x, y)$ is as in (2.3).

Theorem 2.1.

1) *The operator $V_{A,\varepsilon}$ is the unique automorphism of $C^\infty(\mathbb{R})$ such that*

$$\Lambda_{A,\varepsilon} \circ V_{A,\varepsilon} = V_{A,\varepsilon} \circ \frac{\mathrm{d}}{\mathrm{d}x}. \tag{2.5}$$

2) *For all $(x, y) \in \mathbb{R}^* \times \mathbb{R}$, the kernel $\mathbb{K}_\varepsilon(x, y)$ is positive.*

The positivity of $V_{A,\varepsilon}$ played a fundamental role in [2] in establishing an analogue of Beurling's theorem, and its relatives such as theorems of type Gelfand–Shilov, Morgan's, Hardy's, and Cowling–Price in the setting of this paper.

For $\varepsilon = 0$ and 1, the positivity of $\mathbb{K}_\varepsilon(x, y)$ can be found in [17] and [18].

Sketch of Proof of Theorem 2.1. 1) Write f as the superposition $f = f_\mathrm{e} + f_\mathrm{o}$ of an even function f_e and an odd function f_o. We prove that $V_{A,\varepsilon}$ can be expressed as

$$V_{A,\varepsilon} f(x) = \left(\mathrm{id} + \varepsilon\varrho\mathbb{M}\right) \circ \mathbb{A}_\varepsilon f_\mathrm{e}(x) + \mathbb{M} \circ \mathbb{A}_\varepsilon f_\mathrm{o}'(x), \tag{2.6}$$

where

$$\mathbb{M}h(x) := \frac{\mathrm{sg}(x)}{A(x)} \int_0^{|x|} h(t)A(t)\,\mathrm{d}t$$

and

$$\mathbb{A}_\varepsilon f(x) := \frac{1}{2} \int_{|y|<|x|} K_\varepsilon(|x|, y) f(y)\,\mathrm{d}y,$$

with $K_\varepsilon(|x|, y)$ is as in (2.2). The transform $\mathbb{M}$ is an isomorphism from $C_e^\infty(\mathbb{R})$ to $C_o^\infty(\mathbb{R})$ and its inverse is given by $\mathbb{M}^{-1} = \frac{d}{dx} + \frac{A'(x)}{A(x)}\,\mathrm{id}$, while $\mathbb{A}_\varepsilon$ is an automorphism of $C_e^\infty(\mathbb{R})$. Further, $(d^2/dx^2 + (A'/A)(x)d/dx) \circ \mathbb{A}_\varepsilon = \mathbb{A}_\varepsilon \circ (d^2/dx^2 - \varepsilon^2\varrho^2)$ and $\Lambda_{A,\varepsilon} \circ \mathbb{M} = \mathrm{id} + \varepsilon\varrho\mathbb{M}$. Now, the first statement follows from (2.6). The uniqueness of $V_{A,\varepsilon}$ is due to the fact that the unique solution $\Psi_{A,\varepsilon}$ to the problem (1.4) can be written as $\Psi_{A,\varepsilon}(\lambda, x) = V_{A,\varepsilon}(e^{i\lambda\cdot})(x)$ (see (2.3)).

2) For a linear operator L on $\mathscr{D}(\mathbb{R})$ we denote by ${}^tL$ its dual operator in the sense that $\int_\mathbb{R} Lf(x)g(x)A(x)dx = \int_\mathbb{R} f(y)\,{}^tLg(y)dy$.

It is more convenient to deal with the dual operator ${}^tV_{A,\varepsilon}$ than with $V_{A,\varepsilon}$. For $g \in \mathscr{D}(\mathbb{R})$, we have ${}^tV_{A,\varepsilon}g(y) = \int_{|x|>|y|} \mathbb{K}_\varepsilon(x, y)g(x)A(x)\,dx$. We shall prove that if $g \geq 0$ then ${}^tV_{A,\varepsilon}g \geq 0$. For $s > 0$ and $u, v \in \mathbb{R}$, let $p_s(u, v) := \frac{e^{-\frac{(u-v)^2}{4s}}}{2\sqrt{\pi s}}$ be the Euclidean heat kernel. The key observation is that

$$\int_\mathbb{R} g(x)V_{A,\varepsilon}(p_s(u, .))(x)A(x)\,dx = \int_\mathbb{R} {}^tV_{A,\varepsilon}g(x)p_s(x, u)\,dx = ({}^tV_{A,\varepsilon}g * q_s)(u) \to {}^tV_{A,\varepsilon}g(u)$$

as $s \to 0$, where $q_s(r) := p_s(r, 0)$ and $*$ is the Euclidean convolution product. Thus, the positivity of ${}^tV_{A,\varepsilon}g$ reduces to the positivity of $V_{A,\varepsilon}(p_s(u, \cdot))$. Now, by (2.4) and (2.3) we prove that for every $s > 0$ and $u, x \in \mathbb{R}$, we have

$$V_{A,\varepsilon}(p_s(u, \cdot))(x) = \frac{1}{2\pi}\int_\mathbb{R} \Psi_{A,\varepsilon}(-\lambda, x)e^{-s\lambda^2}\, e^{i\lambda u}\,d\lambda,$$

which allowed us to show that $V_{A,\varepsilon}(p_s(u, \cdot))(x) \geq 0$. □

3. L^p-Fourier analysis

For $f \in L^1(\mathbb{R}, A(x)\,dx)$ put

$$\mathscr{F}_{A,\varepsilon}f(\lambda) = \int_\mathbb{R} f(x)\Psi_\varepsilon(\lambda, -x)A(x)\,dx, \tag{3.1}$$

which is well defined by Theorem 1.1.II.1

For $-1 \leq \varepsilon \leq 1$ and $0 < p \leq \frac{2}{1+\sqrt{1-\varepsilon^2}}$, set $\vartheta_{p,\varepsilon} := \frac{2}{p} - 1 - \sqrt{1-\varepsilon^2}$. Observe that $1 \leq \frac{2}{1+\sqrt{1-\varepsilon^2}} \leq 2$. We introduce the tube domain

$$\mathbb{C}_{p,\varepsilon} := \{\lambda \in \mathbb{C} \mid |\operatorname{Im}\lambda| \leq \varrho\,\vartheta_{p,\varepsilon}\}.$$

Theorem 3.1. *Let $f \in L^p(\mathbb{R}, A(x)\,dx)$ with $1 \leq p \leq \frac{2}{1+\sqrt{1-\varepsilon^2}}$. Then the following properties hold.*

1) *For $p > 1$, the Fourier transform $\mathscr{F}_{A,\varepsilon}(f)(\lambda)$ is well defined for all λ in $\mathring{\mathbb{C}}_{p,\varepsilon}$, the interior of $\mathbb{C}_{p,\varepsilon}$. Moreover, for all $\lambda \in \mathring{\mathbb{C}}_{p,\varepsilon}$, we have $|\mathscr{F}_{A,\varepsilon}(f)(\lambda)| \leq c\|f\|_p$. For $p = 1$, we may replace above the open domain $\mathring{\mathbb{C}}_{p,\varepsilon}$ by $\mathbb{C}_{p,\varepsilon}$.*
2) *The function $\mathscr{F}_{A,\varepsilon}(f)$ is holomorphic on $\mathring{\mathbb{C}}_{p,\varepsilon}$.*
3) (Riemann–Lebesgue lemma) *We have* $\lim_{\lambda \in \mathring{\mathbb{C}}_{p,\varepsilon}, |\lambda|\to\infty} |\mathscr{F}_{A,\varepsilon}(f)(\lambda)| = 0.$
4) *The Fourier transform $\mathscr{F}_{A,\varepsilon}$ is injective on $L^p(\mathbb{R}, A(x)\,dx)$ for $1 \leq p \leq \frac{2}{1+\sqrt{1-\varepsilon^2}}$.*

Sketch of Proof. The first two statements follow from the estimate of $\Psi_{A,\varepsilon}(\lambda, x)$ given in Theorem 1.1.II.4 (with $N = 0$), the fact that $A(x) \leq c|x|^\beta e^{2\varrho|x|}$ (a consequence of the hypothesis (C3) on the function A), the fact that $\Psi_{A,\varepsilon}(\lambda, \cdot)$ is holomorphic in λ, and Morera's theorem. To extend the first statement from $\mathring{\mathbb{C}}_{p,\varepsilon}$ to $\mathbb{C}_{p,\varepsilon}$ when $p = 1$, in addition, we show that $|\Psi_{A,\varepsilon}(\lambda, x)| \leq 2$ for all $\lambda \in \mathbb{C}_{1,\varepsilon}$ and for all $x \in \mathbb{R}$. The proof uses the maximum modulus principle and the fact that $|\Psi_{A,\varepsilon}(\lambda, x)| \leq \Psi_{A,\varepsilon}(i\operatorname{Im}\lambda, x)$. For the Riemann–Lebesgue lemma, a classical proof for the Euclidean Fourier transform carries over. The forth statement is based on the following steps:

Step 1) For $f \in L^p(\mathbb{R}, A(x)\,dx)$ et $g \in \mathscr{D}(\mathbb{R})$, we show, by means of Hölder's inequality and the first statement, that the mappings $f \mapsto (f, g)_A := \int_\mathbb{R} f(x)g(-x)A(x)\,dx$ and $f \mapsto (\mathscr{F}_{A,\varepsilon}(f), \mathscr{F}_{A,\varepsilon}(g))_{\pi_\varepsilon} := \int_\mathbb{R} \mathscr{F}_{A,\varepsilon}(f)(\lambda)\mathscr{F}_{A,\varepsilon}(g)(\lambda)\Big(1 - \frac{\varepsilon\varrho}{i\lambda}\Big)\pi_\varepsilon(d\lambda)$ are continuous functionals on $L^p(\mathbb{R}, A(x)\,dx)$. Here π_ε is a positive measure with support $\mathbb{R} \setminus]-\sqrt{1-\varepsilon^2}\varrho, \sqrt{1-\varepsilon^2}\varrho[$.

Step 2) We show that $(f, g)_A = (\mathscr{F}_{A,\varepsilon}(f), \mathscr{F}_{A,\varepsilon}(g))_{\pi_\varepsilon}$ for all $f, g \in \mathscr{D}(\mathbb{R})$. Thus, by Step 1), $(f, g)_A = (\mathscr{F}_{A,\varepsilon}(f), \mathscr{F}_{A,\varepsilon}(g))_{\pi_\varepsilon}$ for all $f \in L^p(\mathbb{R}, A(x)\,dx)$.

Hence, if we assume that $f \in L^p(\mathbb{R}, A(x)\,dx)$ and that $\mathscr{F}_{A,\varepsilon}(f) = 0$, then for all $g \in \mathscr{D}(\mathbb{R})$, we have $(f, g)_A = 0$ and therefore $f = 0$. □

For $-1 \le \varepsilon \le 1$ and $0 < p \le \frac{2}{1+\sqrt{1-\varepsilon^2}}$, denote by $\mathscr{S}_p(\mathbb{R})$ the space consisting of all functions $f \in C^\infty(\mathbb{R})$ such that

$$\sigma^{(p)}_{s,k}(f) := \sup_{x\in\mathbb{R}} (|x| + 1)^s\, e^{\frac{2}{p}\varrho|x|}\, |f^{(k)}(x)| < \infty \tag{3.2}$$

for any $s, k \in \mathbb{N}$. The topology of $\mathscr{S}_p(\mathbb{R})$ is defined by the seminorms $\sigma^{(p)}_{s,k}$. The space $\mathscr{D}(\mathbb{R})$ of smooth functions with compact support on $\mathbb{R}$ is a dense subspace of $\mathscr{S}_p(\mathbb{R})$; see for instance [9, Appendix A].

Let $\mathscr{S}(\mathbb{C}_{p,\varepsilon})$ be the Schwartz space consisting of all complex valued functions h that are analytic in the interior of $\mathbb{C}_{p,\varepsilon}$, and such that h together with all its derivatives extend continuously to $\mathbb{C}_{p,\varepsilon}$ and satisfy

$$\tau^{(\vartheta_{p,\varepsilon})}_{t,\ell}(h) := \sup_{\lambda\in\mathbb{C}_{p,\varepsilon}} (|\lambda| + 1)^t\, |h^{(\ell)}(\lambda)| < \infty \tag{3.3}$$

for any $t, \ell \in \mathbb{N}$. The topology of $\mathscr{S}(\mathbb{C}_{p,\varepsilon})$ is defined by the seminorms $\tau^{(\vartheta_{p,\varepsilon})}_{t,\ell}$.

Using Anker's approach [1], we prove the following result:

Theorem 3.2. *Let $-1 \le \varepsilon \le 1$ and $0 < p \le \frac{2}{1+\sqrt{1-\varepsilon^2}}$. Then the Fourier transform $\mathscr{F}_{A,\varepsilon}$ is a topological isomorphism between $\mathscr{S}_p(\mathbb{R})$ and $\mathscr{S}(\mathbb{C}_{p,\varepsilon})$.*

Sketch of Proof. The proof is based on the following steps:

Step 1) The transform $\mathscr{F}_{A,\varepsilon}$ maps $\mathscr{S}_p(\mathbb{R})$ continuously into $\mathscr{S}(\mathbb{C}_{p,\varepsilon})$ and is injective.
Step 2) The inverse Fourier transform $\mathscr{F}^{-1}_{A,\varepsilon} : PW(\mathbb{C}) \longrightarrow \mathscr{D}(\mathbb{R})$ given by

$$\mathscr{F}^{-1}_{A,\varepsilon} h(x) = c \int_{\mathbb{R}} h(\lambda)\Psi_{A,\varepsilon}(\lambda, x) \left(1 - \frac{\varepsilon\varrho}{i\lambda}\right) \pi_\varepsilon(d\lambda)$$

is continuous for the topologies induced by $\mathscr{S}(\mathbb{C}_{p,\varepsilon})$ and $\mathscr{S}_p(\mathbb{R})$. Here $PW(\mathbb{C})$ is the space of entire functions on $\mathbb{C}$ which are of exponential type and rapidly decreasing, and π_ε is a positive measure with support $\mathbb{R} \setminus]-\sqrt{1-\varepsilon^2}\varrho, \sqrt{1-\varepsilon^2}\varrho[$. We pin down that $PW(\mathbb{C})$ is dense in $\mathscr{S}(\mathbb{C}_{p,\varepsilon})$.

For Step 1), we prove that $\mathscr{F}_{A,\varepsilon}(f)$ is well defined for all $f \in \mathscr{S}_p(\mathbb{R})$. This is due to the growth estimates for $\Psi_{A,\varepsilon}(\lambda, x)$ stated in Theorem 1.1.II.4. Moreover, since the map $\lambda \mapsto \Psi_{A,\varepsilon}(\lambda, x)$ is holomorphic on $\mathbb{C}$, it follows that for all $f \in \mathscr{S}_p(\mathbb{R})$, the function $\mathscr{F}_{A,\varepsilon}(f)$ is analytic in the interior of $\mathbb{C}_{p,\varepsilon}$, and continuous on $\mathbb{C}_{p,\varepsilon}$. Finally, we prove that given a continuous seminorm τ on $\mathscr{S}(\mathbb{C}_{p,\varepsilon})$, there exists a continuous seminorm σ on $\mathscr{S}_p(\mathbb{R})$ such that $\tau(\mathscr{F}_{A,\varepsilon}(f)) \le c\sigma(f)$ for all $f \in \mathscr{S}_p(\mathbb{R})$. Indeed, by means of the growth estimates for $\partial^\ell_\lambda \Psi_{A,\varepsilon}(\lambda, x)$ stated in Theorem 1.1.II.4, we show first that

$$\left|\left\{(i\lambda)^r \mathscr{F}_{A,\varepsilon}(f)(\lambda)\right\}^{(\ell)}\right| \le c \int_{\mathbb{R}} |\Lambda^r_{A,\varepsilon} f(x)|\, (|x| + 1)^{\ell+1}\, e^{(|\operatorname{Im}\lambda| - \varrho(1-\sqrt{1-\varepsilon^2}))\,|x|}\, A(x)\,dx,$$

and then we prove that $|\Lambda^r_{A,\varepsilon} f(x)|$ is bounded by finite sums of the derivatives of f. Thus $\tau(\mathscr{F}_{A,\varepsilon}(f)) \le c \sum_{\text{finite}} \sigma(f)$ for all $f \in \mathscr{S}_p(\mathbb{R})$. The injectivity of $\mathscr{F}_{A,\varepsilon}$ on $\mathscr{S}_p(\mathbb{R})$ follows from Theorem 3.1.4 and the fact that $\mathscr{S}_p(\mathbb{R}) \subset L^q(\mathbb{R}, A(x)\,dx)$ for all $q < \infty$ so that $p \le q$.

For Step 2), we start by proving a Paley–Wiener theorem for $\mathscr{F}_{A,\varepsilon}$, i.e. we prove that $\mathscr{F}_{A,\varepsilon}$ is a linear isomorphism between the space $\mathscr{D}_R(\mathbb{R})$ of smooth compactly supported functions with support inside $[-R, R]$ and the space $PW_R(\mathbb{C})$ of entire functions that are of R-exponential type and rapidly decreasing. We note that $PW(\mathbb{C}) = \cup_{R>0} PW_R(\mathbb{C})$.

Next, we take $f \in \mathscr{D}(\mathbb{R})$ and $h \in PW(\mathbb{C})$ so that $f = \mathscr{F}^{-1}_{A,\varepsilon}(h)$. Denote by g the image of h by the inverse Euclidean Fourier transform $\mathscr{F}^{-1}_{\text{euc}}$. Making use of the Paley–Wiener theorem for $\mathscr{F}_{A,\varepsilon}$ and the classical Paley–Wiener theorem for $\mathscr{F}_{\text{euc}}$, we have the following support conservation property: $\operatorname{supp}(f) \subset I_R := [-R, R] \Leftrightarrow \operatorname{supp}(g) \subset I_R$.

For $j \in \mathbb{N}_{\ge1}$, let $\omega_j \in C^\infty(\mathbb{R})$ with $\omega_j = 0$ on I_{j-1} and $\omega_j = 1$ outside of I_j. Assume that ω_j and all its derivatives are bounded, uniformly in j. We write $g_j = \omega_j g$, and define $h_j := \mathscr{F}_{\text{euc}}(g_j)$ and $f_j := \mathscr{F}^{-1}_{A,\varepsilon}(h_j)$. Note that $g_j = g$ outside I_j. Hence, by the above support property, $f_j = f$ outside I_j.

In view of the growth estimate for $\partial^k_x \Psi_{A,\varepsilon}(\lambda, x)$ stated in Theorem 1.1.II.3, we prove that for all $j \in \mathbb{N}_{\ge1}$,

$$\sup_{x\in I_{j+1}\setminus I_j} (|x| + 1)^s\, e^{\frac{2}{p}\varrho|x|}\, |f^{(k)}_j(x)| \le c \sum_{r=0}^{s+3} \tau^{(\vartheta_{p,\varepsilon})}_{t,r}(h),$$

for some integer $t > 0$. For I_1, we show first that there exists an integer $m_k \geq 1$ such that

$$|\partial_x^k \Psi_{A,\varepsilon}(\lambda, x)| \leq c(|\lambda| + 1)^{m_k} (|x| + 1)\, e^{-\varrho|x|}, \tag{3.4}$$

for $\lambda \in \mathbb{R}$ such that $|\lambda| \geq \sqrt{1 - \varepsilon^2}\varrho$. Then, using the compactness of I_1, we prove that

$$\sup_{x \in I_1} (|x| + 1)^s\, e^{\frac{2}{p}\varrho|x|}\, |f^{(k)}(x)| \leq c\tau_{t,0}^{(0)}(h),$$

for some integer $t > 0$. □

Details of this paper will be given in other articles [3] and [4].

References

[1] J.-Ph Anker, The spherical Fourier transform of rapidly decreasing functions. A simple proof of a characterization due to Harish-Chandra, Helgason, Trombi, and Varadarajan, J. Funct. Anal. 96 (2) (1991) 331–349.
[2] S. Ben Said, A. Boussen, M. Sifi, Uncertainty principles and characterization of the heat kernel for certain differential-reflection operators, Adv. Pure Appl. Math. 6 (4) (2015) 215–239.
[3] S. Ben Said, A. Boussen, M. Sifi, Intertwining operators associated to a family of differential-reflection operators, preprint.
[4] S. Ben Said, A. Boussen, M. Sifi, L^p-Fourier analysis associated to a family of differential-reflection operators, preprint.
[5] W.R. Bloom, H. Heyer, Harmonic Analysis of Probability Measures on Hypergroups, de Gruyter Studies in Mathematics, vol. 20, Walter de Gruyter, Berlin, 1995, vi+601 pp.
[6] H. Chébli, Sur un théorème de Paley–Wiener associé à la décomposition spectrale d'un opérateur de Sturm–Liouville sur $(0, \infty)$, J. Funct. Anal. 17 (1974) 447–461.
[7] H. Chébli, Opérateurs de translation généralisée et semi-groupes de convolution, in: Théorie du Potentiel et Analyse Harmonique, Journées Soc. Math. France, Inst. Recherche Math. Avancée, Strasbourg, 1973, in: Lecture Notes in Math., vol. 404, Springer, Berlin, 1974, pp. 35–59.
[8] I. Cherednik, A unification of Knizhnik–Zamolodchnikov equations and Dunkl operators via affine Hecke algebras, Invent. Math. 106 (1991) 411–432.
[9] P. Delorme, Espace de Schwartz pour la transformation de Fourier hypergómt́rique, Appendix A by Mustapha Tinfou, J. Funct. Anal. 168 (1) (1999) 239–312.
[10] C. Dunkl, Differential-difference operators associated to reflection groups, Trans. Amer. Math. Soc. 311 (1989) 167–183.
[11] G.J. Heckman, An elementary approach to the hypergeometric shift operators of Opdam, Invent. Math. 103 (2) (1991) 341–350.
[12] T.H. Koornwinder, Jacobi functions and analysis on noncompact semisimple Lie groups, in: Special Functions: Group Theoretical Aspects and Applications, in: Math. Appl., Reidel, Dordrecht, The Netherlands, 1984, pp. 1–85.
[13] J.L. Lions, Opérateurs de Delsarte et problèmes mixtes, Bull. Soc. Math. Fr. 84 (1956) 9–95.
[14] E.M. Opdam, Harmonic analysis for certain representations of graded Hecke algebras, Acta Math. 175 (1995) 75–121.
[15] M. Thyssen, Sur certains opérateurs de transmutation particuliers, Mem. Soc. R. Sci. Liege (5) 6 (3) (1961) 32 pp.
[16] K. Trimèche, Generalized Wavelets and Hypergroups, Gordon and Breach Science Publishers, Amsterdam, 1997, xii+354 pp.
[17] K. Trimèche, Positivity of the transmutation operators associated with a Cherednik type operator on the real line, Adv. Pure Appl. Math. 3 (4) (2012) 361–376.
[18] K. Trimèche, The transmutation operators relating to a Dunkl type operator on $\mathbb{R}$ and their positivity, Mediterr. J. Math. 12 (2) (2015) 349–369.