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Number theory

On two problems of Ljujić and Nathanson ☆

Sur deux problèmes de Ljujić et Nathanson

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ABSTRACT

Let $\mathbf{N}$ be the set of all nonnegative integers. For $A, M \subseteq \mathbf{N} \setminus \{0\}$ and $n \in \mathbf{N}$, let $p(n, A, M)$ denote the number of representations of n in the form $n = \sum_{a \in A} m_a a$, where $m_a \in M \cup \{0\}$ for all $a \in A$. Recently, by using the probabilistic method, Alon answered two questions of Ljujić and Nathanson affirmatively by proving that, for $A = \{n!\}_{n \ge 1}$ or for $A = \{n^n\}_{n \ge 1}$, there exists n_0 and an infinite set M of positive integers so that $0 < p(n, A, M) < n^{8+o(1)}$ for all $n > n_0$. In this note, by an explicit construction, as a corollary of our main result, it is proved that, for $A = \{n!\}_{n \ge 1}$ or for $A = \{n^n\}_{n \ge 1}$, there exists an explicit infinite set M of positive integers so that $0 < p(n, A, M) \le n^{2+o(1)}$ for all $n \ge 1$. Several open questions are posed for further research.

RÉSUMÉ

Soit $\mathbf{N}$ l'ensemble des entiers positifs ou nul. Pour $A, M \subset \mathbf{N} \setminus \{0\}$ et $n \in \mathbf{N}$, notons $p(n, A, M)$ le nombre de représentations de n sous la forme $n = \sum_{a \in A} m_a a$, avec $m_a \in M \cup \{0\}$ pour tout $a \in A$. Récemment, utilisant une méthode probabiliste, Alon a répondu positivement à deux questions de Ljujić et Nathanson. Il a montré que, pour $A = \{n!\}_{n \ge 1}$ ou $A = \{n^n\}_{n \ge 1}$, il existe n_0 et un ensemble infini M d'entiers positifs tel que $0 < p(n, A, M) < n^{8+o(1)}$ pour tout $n > n_0$. Dans cette Note, par une construction explicite et comme corollaire de notre résultat principal, nous montrons que, pour $A = \{n!\}_{n \ge 1}$ ou $A = \{n^n\}_{n \ge 1}$, il existe un ensemble infini explicite M d'entiers positifs tel que $0 < p(n, A, M) < n^{2+o(1)}$ pour tout $n \ge 1$. Plusieurs questions ouvertes sont proposées pour de futures recherches.

1. Introduction

Let $\mathbf{N}$ be the set of all nonnegative integers. In 2012, the following variation of the classical partition problem is studied by Canfield and Wilf [2]: for $A, M \subseteq \mathbf{N} \setminus \{0\}$ and $n \in \mathbf{N}$, let $p(n, A, M)$ denote the number of representations of n in the

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form $n=\sum_{a\in A} m_a a$, where $m_a \in M\cup\{0\}$ for all $a\in A$, and $m_a\in M$ for only finitely many a. An arithmetic function f has polynomial growth if there is a positive integer k and an integer $N_0(k)$ such that $1\le f(n)\le n^k$ for all $n\ge N_0(k)$.

Ljujić and Nathanson [3] proved the following nice result: If $A(x)\ge c\log x$ for some constant $c>0$ and all $x\ge x(A)$, then there is no any infinite set M of positive integers such that $p(n,A,M)$ has polynomial growth. Ljujić and Nathanson [3] also posed the following two questions:

Question 1.1. *(See Ljujić and Nathanson [3].) Let $A=\{n!\}_{n=1}^{\infty}$. Does there exist an infinite set M of positive integers so that $p(n,A,M)>0$ for all sufficiently large n and p has polynomial growth?*

Question 1.2. *(See Ljujić and Nathanson [3].) Let $A=\{n^n\}_{n=1}^{\infty}$. Does there exist an infinite set M of positive integers so that $p(n,A,M)>0$ for all sufficiently large n and p has polynomial growth?*

Recently, by an explicit construction, Alon [1] answered a question of Canfield and Wilf [2] by proving the following nice result: there are two explicit infinite sets of positive A and M so that $p(n,A,M)=1$ for all $n\ge 1$. By using the probabilistic method, Alon [1] answered the above two questions affirmatively by proving that, for $A=\{n!\}_{n\ge 1}$ or for $A=\{n^n\}_{n\ge 1}$, there exists n_0 and an infinite set M of positive integers so that $0<p(n,A,M)<n^{8+o(1)}$ for all $n>n_0$

In this note, by an explicit construction, a stronger result is proved.

Theorem 1.3. *Let $A=\{1=a_1<a_2<\cdots\}$ be an infinite set of positive integers such that*

$$c_1(n+1)^{\theta_1}a_n\le a_{n+1}\le c_2(n+1)^{\theta_2}a_n,\quad n=1,2,\ldots,$$

where $c_2>c_1>0$ and $\theta_2\ge\theta_1>0$ are constants. Then

$$0<p(n,A,M)\le n^{(\theta_2+1)/\theta_1+o(1)}$$

for all $n\ge 1$, where

$$M=\{k2^{n-1}:1\le k\le \max\{2c_2,1\}(n+1)^{\theta_2},\ n=1,2,\ldots\}.$$

We have the following corollary, which can be applied to both $A=\{n!\}_{n\ge1}$ and $A=\{n^n\}_{n\ge1}$.

Corollary 1.4. *Let $A=\{1=a_1<a_2<\cdots\}$ be an infinite set of positive integers such that*

$$c_1(n+1)a_n\le a_{n+1}\le c_2(n+1)a_n,\quad n=1,2,\ldots$$

for two constants $c_2>c_1>0$. Then $0<p(n,A,M)\le n^{2+o(1)}$ for all $n\ge 1$, where

$$M=\{k2^{n-1}:1\le k\le \max\{2c_2,1\}(n+1),\ n=1,2,\ldots\}.$$

Remark 1.5. For $A=\{n!\}_{n\ge1}$, we have $a_{n+1}=(n+1)a_n$. Thus for

$$M=\{k2^{n-1}:1\le k\le 2(n+1),\ n=1,2,\ldots\},$$

we have $0<p(n,A,M)\le n^{2+o(1)}$ for all $n\ge 1$.

For $A=\{n^n\}_{n\ge1}$, we have

$$a_{n+1}=(n+1)^{n+1}=(n+1)(n+1)^n=(n+1)\left(1+\frac{1}{n}\right)^n n^n\le e(n+1)a_n$$

and $a_{n+1}>(n+1)a_n$. Thus for

$$M=\{k2^{n-1}:1\le k\le 2e(n+1),\ n=1,2,\ldots\},$$

we have $0<p(n,A,M)\le n^{2+o(1)}$ for all $n\ge 1$.

2. Proof of Theorem 1.3

First we prove that $p(n,A,M)\ge 1$ for all $n\ge 1$.

It is enough to prove that any nonnegative integer n can be written in the form

$$n=\sum_{i=1}^{\infty}k_i 2^{i-1}a_i,\quad 0\le k_i\le \max\{2c_2,1\}(i+1)^{\theta_2},\ i=1,2,\ldots.$$

We prove this by induction on n. It is trivial for $n = 0, 1$. Now we assume that $n > 1$ and the conclusion is true for all nonnegative integers less than n. Let m be the positive integer such that $2^{m-1}a_m \le n < 2^m a_{m+1}$. Let k_m be the integer with $k_m 2^{m-1}a_m \le n < (k_m+1)2^{m-1}a_m$. It is clear that $k_m \ge 1$ and $0 \le n - k_m 2^{m-1}a_m < 2^{m-1}a_m$. Since

$$k_m 2^{m-1}a_m \le n < 2^m a_{m+1} \le 2^m c_2 (m+1)^{\theta_2} a_m,$$

it follows that $1 \le k_m < 2c_2(m+1)^{\theta_2}$.

By the inductive hypothesis, $n - k_m 2^{m-1}a_m$ can be written in the form

$$n - k_m 2^{m-1}a_m = \sum_{i=1}^{\infty} k_i' 2^{i-1}a_i, \quad 0 \le k_i' \le \max\{2c_2, 1\}(i+1)^{\theta_2}, \ i = 1, 2, \ldots.$$

Since $n - k_m 2^{m-1}a_m < 2^{m-1}a_m$, it follows that $k_i' = 0$ for all $i \ge m$. Let $k_i = k_i'$ for all $1 \le i < m$ and $k_i = 0$ for all $i > m$. Then

$$n = \sum_{i=1}^{\infty} k_i 2^{i-1}a_i, \quad 0 \le k_i \le \max\{2c_2, 1\}(i+1)^{\theta_2}, \ i = 1, 2, \ldots.$$

Thus we have proved that $p(n, A, M) \ge 1$ for all $n \ge 1$.

Now we prove that $p(n, A, M) \le n^{2+o(1)}$.

We may assume that $n > 10$. Then there exist two positive integers m and l such that

$$a_m \le n < a_{m+1}, \quad 2^l \le n < 2^{l+1}.$$

Since

$$n \ge a_m \ge c_1 m^{\theta_1} a_{m-1} \ge c_1^{m-1}(m!)^{\theta_1} a_1 = c_1^{m-1}(m!)^{\theta_1},$$

it follows that $\log n \ge \theta_1 \log m! + (m-1)\log c_1 = (\theta_1 + o(1)) m \log m$. Thus

$$|A \cap [1, n]| = m \le (\theta_1^{-1} + o(1))\frac{\log n}{\log\log n}, \text{ as } n \to \infty.$$

We also have

$$|M \cap [1, n]| \le \sum_{i=0}^{l} \max\{2c_2, 1\}(i+2)^{\theta_2} \le c_3 l^{\theta_2+1} \le c_4 (\log n)^{\theta_2+1} - 1,$$

for two positive constants c_3 and c_4. Thus

$$\begin{aligned}\sum_{s=1}^{n} p(s, A, M) &\le (|M \cap [1, n]| + 1)^{|A\cap[1,n]|} \\ &\le \left(c_4 (\log n)^{\theta_2+1}\right)^{(\theta_1^{-1}+o(1))\log n/\log\log n} \\ &= n^{(\theta_2+1)/\theta_1+o(1)}, \text{ as } n \to \infty.\end{aligned}$$

Therefore, $p(n, A, M) \le n^{(\theta_2+1)/\theta_1+o(1)}$. This completes the proof of Theorem 1.3.

3. Final remarks

An arithmetic function f has logarithm polynomial growth if there is a positive integer k and an integer $N_0(k)$ such that $1 \le f(n) \le (\log n)^k$ for all $n \ge N_0(k)$. Now we pose several questions here.

Question 3.1. *Let $A = \{n!\}_{n=1}^{\infty}$. Does there exist an infinite set M of positive integers so that $p(n, A, M) > 0$ for all sufficiently large n and p has logarithm polynomial growth?*

Question 3.2. *Let $A = \{n^n\}_{n=1}^{\infty}$. Does there exist an infinite set M of positive integers so that $p(n, A, M) > 0$ for all sufficiently large n and p has logarithm polynomial growth?*

Furthermore, we pose the following questions:

Question 3.3. *Let $A = \{n!\}_{n=1}^{\infty}$. Does there exist an infinite set M of positive integers and a constant c so that $0 < p(n, A, M) < c$ for all sufficiently large n?*

Question 3.4. *Let $A = \{n^n\}_{n=1}^{\infty}$. Does there exist an infinite set M of positive integers and a constant c so that $0 < p(n, A, M) < c$ for all sufficiently large n?*

Motivated by Theorem 1.3, we pose the following question:

Question 3.5. *Do there exist two infinite sets $A = \{a_n\}_{n=1}^{\infty}$ and M of positive integers such that*

$$\lim_{n\to\infty} \frac{\log a_{n+1} - \log a_n}{\log n} = +\infty$$

and $p(n, A, M) > 0$ for all sufficiently large n and p has polynomial growth?

References

[1] N. Alon, Restricted integer partition functions, Integers 13 (2013) A16.
[2] E.R. Canfield, H.S. Wilf, On the growth of restricted integer partition functions, Dev. Math. 23 (2012) 39–46.
[3] Z. Ljujić, M. Nathanson, On a partition problem of Canfield and Wilf, Integers 12A (2012) A11.