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Algebraic geometry

Finiteness of Lagrangian fibrations with fixed invariants

Finitude des fibrations lagrangiennes à invariants fixes

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ABSTRACT

We prove finiteness of the deformation classes of hyperkähler Lagrangian fibrations in any fixed dimension with fixed Fujiki constant and discriminant of the Beauville–Bogomolov–Fujiki lattice. We also prove there are only finitely many deformation classes of hyperkähler Lagrangian fibrations with an ample line bundle of a given degree on the general fibre of the fibration.

RÉSUMÉ

Nous démontrons la finitude des classes de déformation des fibrations lagrangiennes hyperkählériennes, de dimension quelconque, avec constante de Fujiki et discriminant du réseau de Beauville–Bogomolov–Fujiki fixes. Nous montrons également qu'il n'y a qu'un nombre fini de classes de déformation des fibrations lagrangiennes hyperkählériennes avec un fibré en droite ample de degré donné sur la fibre générale de la fibration.

1. Introduction

For a hyperkähler manifold M, the Fujiki constant and the discriminant of the Beauville–Bogomolov–Fujiki lattice are topological invariants. It is very natural to fix them and ask for finiteness of hyperkähler manifolds with these invariants. In this paper we establish finiteness of Lagrangian fibrations of hyperkähler manifolds with fixed topological invariants as above.

Theorem 1.1. *There are at most finitely many deformation classes of Lagrangian fibrations $\pi : M \to \mathbb{C}P^n$ with a fixed Fujiki constant c and a given discriminant of the Beauville–Bogomolov–Fujiki lattice (Λ, q).*

Francois Charles has the following boundedness result for families of hyperkähler varieties up to deformation. He drops the assumption that L is ample in Kollár–Matsusaka's theorem applied for hyperkähler manifolds and replaces it with the assumption that $q(L) > 0$.

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Theorem 1.2. *(See Charles [4].) Let n and r be two positive integers. Then there exists a scheme S of finite type over $\mathbb{C}$, and a projective morphism $\mathcal{M} \longrightarrow S$ such that if M is a complex hyperkähler variety of dimension 2n and L is a line bundle on M with $c_1(L)^{2n} = r$ and $q(L) > 0$, where q is the Beauville–Bogomolov form, then there exists a complex point s of S such that $\mathcal{M}_s$ is birational to M.*

In our case, there is a natural line bundle L associated with the Lagrangian fibration. Using Fujiki's formula, it is a straightforward observation that $q(L) = 0$, while F. Charles deals with the case when $q(L) > 0$ (in which case M is projective by a result of D. Huybrechts: Theorem 3.11 in [8]).

In the proof of our main theorems we use F. Charles's finiteness result applied to an ample line bundle with minimal positive square of the Beauville–Bogomolov–Fujiki form. Since we are interested in a finiteness result up to deformation equivalence, one can obtain an ample line bundle after deforming a given Lagrangian fibration to a projective one. We also use lattice theory estimates applied to the Beauville–Bogomolov–Fujiki form.

In [13], Sawon proved a finiteness theorem for Lagrangian fibrations with a lot of natural assumptions on the fibration, such as existence of a section, fixed polarization type of a very ample line bundle, semi-simple degenerations as the general singular fibres, and a maximal variation of the fibres. We give the precise statement of Sawon's theorem in Section 2. Using the techniques in our proofs, one can also drop most of the other conditions in Sawon's theorem. We prove the following generalization.

Theorem 1.3. *Consider a Lagrangian fibration $\pi : M \longrightarrow \mathbb{C}P^n$ such that there is a line bundle P on M with $q(P) > 0$ and with a given P-degree d on the general fibre F of π, i.e., $P^n \cdot F = d$. Then there are at most finitely many deformation classes of hyperkähler manifolds M as above, i.e., they form a bounded family.*

For completeness of the exposition, we also mention Huybrechts' classical finiteness results.

Theorem 1.4. *(See Huybrechts [10].) If the second integral cohomology $H^2(\mathbb{Z})$ and the homogeneous polynomial of degree $2n - 2$ on $H^2(Z)$ defined by the first Pontrjagin class are given, then there exist at most finitely many diffeomorphism types of compact hyperkähler manifolds of real dimension 4n realizing this structure.*

Theorem 1.5. *(See Huybrechts [10].) Let M be a fixed compact manifold. Then there exist at most finitely many different deformation types of irreducible holomorphic symplectic complex structures on M.*

Using Theorem 1.5, the author and Misha Verbitsky established the following finiteness results in [11].

Theorem 1.6. *(See Kamenova–Verbitsky [11].) Let M be a fixed compact manifold. Then there are only finitely many deformation types of hyperkähler Lagrangian fibrations $(M, I) \longrightarrow \mathbb{C}P^n$, for all complex structures I on M.*

In the main theorem of this paper, we prove the finiteness of deformation classes of the total space M of the Lagrangian fibration $M \longrightarrow \mathbb{C}P^n$ with fixed dimension, Fujiki constant and discriminant of the Beauville–Bogomolov–Fujiki lattice. As a corollary of Theorem 1.6, one also obtains the finiteness of the deformation classes of the Lagrangian fibration $M \longrightarrow \mathbb{C}P^n$.

2. Hyperkähler geometry: preliminary results

Definition 2.1. A **hyperkähler manifold** is a compact Kähler holomorphic symplectic manifold.

Definition 2.2. A hyperkähler manifold M is called **simple** if $H^1(M, \mathbb{C}) = 0$ and $H^{2,0}(M) = \mathbb{C}$.

Remark 2.3. From now on, we assume that all hyperkähler manifolds are simple.

Remark 2.4. The following two notions are equivalent: a holomorphic symplectic Kähler manifold and a manifold with a *hyperkähler structure*, that is, a triple of complex structures satisfying the quaternionic relations and parallel with respect to the Levi-Civita connection. In the compact case, the equivalence between these two notions is provided by Yau's solution to Calabi's conjecture [2]. In this paper, we assume compactness and we use the complex algebraic point of view.

Definition 2.5. Let M be a compact complex manifold and $\mathrm{Diff}^0(M)$ the connected component of the identity of its diffeomorphism group. Denote by Comp the space of complex structures on M, equipped with a structure of Fréchet manifold. The *Teichmüller space* of M is the quotient $\mathrm{Teich} := \mathrm{Comp} / \mathrm{Diff}^0(M)$. For a hyperkähler manifold M, the Teichmüller space is finite-dimensional [3]. Let $\mathrm{Diff}^+(M)$ be the group of orientable diffeomorphisms of a complex manifold M. The *mapping class group* $\Gamma := \mathrm{Diff}^+(M)/\mathrm{Diff}^0(M)$ acts naturally on Teich. For $I \in \mathrm{Teich}$, let Γ_I be the subgroup of Γ which fixes the connected component of complex structure I. The *monodromy group* is the image of Γ_I in $\mathrm{Aut}\, H^2(M, \mathbb{Z})$.

Theorem 2.6. *(See Fujiki [6].) Let $\eta \in H^2(M)$, and $\dim M = 2n$, where M is hyperkähler. Then $\int_M \eta^{2n} = c \cdot q(\eta, \eta)^n$, for some integral quadratic form q on $H^2(M)$, where $c > 0$ is a constant depending on the topological type of M. The constant c in Fujiki's formula is called the* ***Fujiki constant****.* □

Definition 2.7. This form is called the **Beauville–Bogomolov–Fujiki form**.

Remark 2.8. The form q has signature $(3, b_2 - 3)$. It is negative definite on primitive forms, and positive definite on the space $\langle \Omega, \overline{\Omega}, \omega \rangle$ where Ω is the holomorphic symplectic form and ω is a Kähler form (see, e.g., [14], Theorem 6.1, or [9], Corollary 23.9).

Definition 2.9. Let $[\eta] \in H^{1,1}(M)$ be a real (1,1)-class on a hyperkähler manifold M. We say that $[\eta]$ is **parabolic** if $q([\eta], [\eta]) = 0$. A line bundle L is called **parabolic** if the class $c_1(L)$ is parabolic.

Remark 2.10. If L is a parabolic class and $P \in H^2(M)$ is any class, then after we substitute $\eta = P + tL$ into Fujiki's formula in Theorem 2.6, and compare the coefficients of t^n on both sides, we obtain $\binom{2n}{n} P^n L^n = c 2^n q(P, L)^n$.

Theorem 2.11. *(See Matsushita [12].) Let $\pi : M \to B$ be a surjective holomorphic map from a hyperkähler manifold M to a base B, with $0 < \dim B < \dim M$. Then $\dim B = 1/2 \dim M$, and the fibres of π are holomorphic Lagrangian (this means that the symplectic form vanishes on the fibres).*

Definition 2.12. Such a map is called **a holomorphic Lagrangian fibration**.

Definition 2.13. A line bundle L is called **semiample** if L^N is generated by its holomorphic sections that have no common zeros.

Remark 2.14. From semiampleness, it trivially follows that L is nef. Indeed, let $\pi : M \to \mathbb{P}H^0(L^N)^*$ be the standard map. Since the sections of L have no common zeros, π is holomorphic. Then $L \cong \pi^* \mathcal{O}(1)$, and the curvature of L is the pullback of a Kähler form on $\mathbb{C}P^n$. However, a nef bundle is not necessarily semiample (see e.g. [5, Example 1.7]).

Remark 2.15. Let $\pi : M \to B$ be a holomorphic Lagrangian fibration, and ω_B a Kähler class on B. Then $\eta := \pi^* \omega_B$ is semiample and parabolic. The converse is also true, by Matsushita's theorem: if L is semiample and parabolic, L induces a Lagrangian fibration.

Our main results in this paper rely on the following theorem.

Theorem 2.16. *(See Charles [4].) Let n and r be two positive integers. Then there exists a scheme S of finite type over $\mathbb{C}$, and a projective morphism $\mathcal{M} \longrightarrow S$ such that if M is a complex hyperkähler variety of dimension $2n$ and L is a line bundle on M with $c_1(L)^{2n} = r$ and $q(L) > 0$, where q is the Beauville–Bogomolov form, then there exists a complex point s of S such that $\mathcal{M}_s$ is birational to M.*

We would also like to mention the following theorem in the recent literature.

Theorem 2.17. *(See Sawon [13].) Fix positive integers n and $d_1, \dots, d_n$, with $d_1 | d_2 | \cdots | d_n$. Consider Lagrangian fibrations $\pi : M \to \mathbb{C}P^n$ that satisfy:*

(1) $\pi : M \to \mathbb{C}P^n$ admits a global section,

(2) there is a very ample line bundle on M which gives a polarization of type $(d_1, \dots, d_n)$ when restricted to a generic smooth fibre M_t,

(3) over a generic point t of the discriminant locus the fibre M_t is a rank-one semi-stable degeneration of abelian varieties,

(4) a neighbourhood U of a generic point $t \in \mathbb{C}P^n$ describes a maximal variation of abelian varieties.

Then there are finitely many such Lagrangian fibrations up to deformation.

Remark 2.18. If there is a section $\sigma : \mathbb{C}P^n \longrightarrow M$, this means that $\sigma(\mathbb{C}P^n)$ would be a Lagrangian subvariety in M. Finding Lagrangian $\mathbb{C}P^n$'s in a hyperkähler manifold is itself a very interesting task (for example, see [7]). Moreover, the Lagrangian $\sigma(\mathbb{C}P^n)$ would have to intersect the general fibre of π in one point.

3. Main results

Consider a lattice Λ, i.e., a free $\mathbb{Z}$-module of finite rank equipped with a non-degenerate symmetric bilinear from q with values in $\mathbb{Z}$. If $\{e_i\}$ is a basis of Λ, the *discriminant* of Λ is defined as $\mathrm{discr}(\Lambda) = \det(e_i \cdot e_j)$.

Lemma 3.1. *Let (Λ, q) be an indefinite lattice and $v \in \Lambda$ be an isotropic non-zero primitive vector. Then there exists a positive vector $w \in \Lambda$ such that $0 < q(w, v) \leqslant |discr(\Lambda)|$ and $0 < q(w, w) \leqslant 2|discr(\Lambda)|$.*

Proof. Let w_0 be a vector with minimal positive intersection $q(w_0, v)$. Then by Lemma 3.7. in [11], $q(w_0, v)$ divides $N = |\mathrm{discr}(\Lambda)|$. Therefore, $0 < q(w_0, v) \leqslant N$. Let α be the smallest integer such that $q(w_0 + \alpha v, w_0 + \alpha v) > 0$. Since $q(v, v) = 0$, the square of the vector $w = w_0 + \alpha v$ is: $q(w_0 + \alpha v, w_0 + \alpha v) = q(w_0, w_0) + 2\alpha q(w_0, v)$. Then w is a positive vector with $0 < q(w, v) = q(w_0, v) \leqslant N$. Notice that automatically $0 < q(w, w) = q(w_0 + \alpha v, w_0 + \alpha v) = q(w_0, w_0) + 2\alpha q(w_0, v) \leqslant 2N = 2|\mathrm{discr}(\Lambda)|$. □

We recall the following result from a paper of the author's together with Misha Verbitsky, Theorem 3.6 in [11].

Theorem 3.2. *Consider the action of the monodromy group Γ_I on $H^2(M, \mathbb{Z})$, and let $S \subset H^2(M, \mathbb{Z})$ be the set of all classes which are parabolic and primitive. Then there are only finitely many orbits of Γ_I on S.*

Our main result is the following finiteness theorem.

Theorem 3.3. *There are at most finitely many deformation classes of Lagrangian fibrations $\pi : M \to \mathbb{C}P^n$ with a fixed Fujiki constant c and a given discriminant of the Beauville–Bogomolov–Fujiki lattice (Λ, q).*

Proof. As in Remark 2.15, Lagrangian fibrations correspond to parabolic semiample classes. Now consider $S \subset H^2(M, \mathbb{Z})$ defined above, the set of all classes that are parabolic and primitive, which is possibly larger than the set of parabolic semiample classes. By Theorem 3.2, there are only finitely many orbits of the monodromy group Γ_I on S.

Let L be a nef parabolic class ($q(L) = 0$) coming from the Lagrangian fibration. Deform the Lagrangian fibration preserving the fibration structure, i.e. preserving the class of L to a projective hyperkähler Lagrangian fibration. Since we are interested in finiteness results up to deformation, we are going to work in the projective setting. By Huybrechts result (Theorem 3.11 in [8]), there exists a line bundle with positive square. Apply Lemma 3.1 for $(\Lambda, q) = (H^2(X, \mathbb{Z}), q)$ and $v = L$. There exists a positive vector w with $0 < q(w, v) \leqslant |\mathrm{discr}(\Lambda)| = N$. We could choose w to be a vector with the smallest positive square $q(w, w) > 0$. From the lemma we see that $0 < q(w, w) \leqslant 2|\mathrm{discr}(\Lambda)|$, which is bounded since we consider a fixed discriminant.

Now we can apply F. Charles's Theorem 2.16 to the case when the first Chern class is w, in which case, by Fujiki's formula, $0 < r = w^{2n} = c \cdot q(w, w)^n \leqslant c \cdot (2|\mathrm{discr}(\Lambda)|)^n$ is bounded. For each r in this interval we obtain only finitely many deformation classes of the total space M. □

Since the families of hyperkähler manifolds as above form a bounded family, there are only finitely many choices of the second Betti number, which plays an important role in studying the geometry of hyperkähler manifolds. We obtain the following.

Corollary 3.4. *In the assumptions of Theorem 3.3, the second Betti number $b_2(M)$ is bounded.*

Using similar methods as above together with F. Charles's Theorem 2.16, we generalize Sawon's Theorem 2.17 by dropping most of the assumptions.

Theorem 3.5. *Consider a Lagrangian fibration $\pi : M \longrightarrow \mathbb{C}P^n$ such that there is a line bundle P on M with $q(P) > 0$ and with a given P-degree d on the general fibre F of π, i.e., $P^n \cdot F = d$. Then there at most finitely many deformation classes of hyperkähler manifolds M as above, i.e., they form a bounded family.*

Proof. Let L be a nef parabolic class ($q(L) = 0$) coming from the Lagrangian fibration (e.g., as the pullback of a hyperplane class on $\mathbb{C}P^n$). The fundamental class $[F]$ of the general fibre of π is L^n. By assumption, $P^n \cdot L^n = d$ is fixed. Define $v = L/m$, where $m \in \mathbb{Z}_{>0}$ is the divisibility of L, and therefore v is a primitive class. Since P is in the interior of the positive cone $\mathcal{C}$ and v is on the boundary of $\mathcal{C}$, it follows that $q(P, v) > 0$ (Corollary 7.2 in [1]). Now we shall follow the proof of Lemma 3.1. Let k be the smallest integer such that $q(P + kv) > 0$. Then $q(P + kv) \leqslant 2q(P, v)$ and

$$(P + kv)^{2n} = c \cdot q(P + kv)^n \leqslant c2^n q(P, v)^n = \binom{2n}{n} P^n \cdot v^n = \binom{2n}{n} \frac{P^n \cdot L^n}{m^n} = \binom{2n}{n} \frac{d}{m^n} \leqslant \binom{2n}{n} d.$$

Here we applied Fujiki's formula twice (as in Theorem 2.6 and Remark 2.10), where c is the Fujiki constant. We apply F. Charles's Theorem 2.16 to obtain a bounded family of such M, which implies finiteness of deformations of M. □

Remark 3.6. In Theorem 3.3 and Theorem 3.5, we prove the finiteness of deformation classes of the total space M of the Lagrangian fibration. However, in Theorem 1.6 the author together with Misha Verbitsky prove that for a fixed compact manifold M there are only finitely many deformation types of hyperkähler Lagrangian fibrations with total space M.

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