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Equivariant trace formula mod p

Formule des traces équivariantes modulo p

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ABSTRACT

We give an equivariant version of Anderson's trace formula of L-function module p. As an application, we can prove the Stark's conjecture of Artin–Goss L-values of Drinfeld modules.

RÉSUMÉ

Nous donnons une version équivariante de la formule des traces d'Anderson pour les L-fonctions modulo p. Comme application, nous montrons la conjecture de Stark pour les valeurs de fonctions L de Artin–Gross des modules de Drinfeld.

1. Introduction and statement of the main results

In this paper, let k be a finite field of characteristic p, G a finite Abelian group of order prime to p. Let R and A be two k-algebras.

Definition 1.1. Let M be an $R[G]$-module. For any $n \geq 1$, an $A[G]$-linear operator ϕ on $M \otimes_k A$ is called n-th Frobenius (resp. n-th Cartier) over A on R if

$$\phi(rm) = r^{q^n}\phi(m) \ \ (\text{resp. } \phi(r^{q^n}m) = r\phi(m)) \text{ for any } r \in R \text{ and } m \in M \otimes_k A.$$

In this note, we mainly prove the following theorem.

Theorem 1.2. *Let R be a finitely generated regular domain over k of Krull dimension r. Let $\Omega_R = \wedge^r \Omega^1_{R/k}$. By [1, 2.6], we have a Cartier operator C on Ω_R. Let M be a finitely generated projective $R[G]$-module. For each $n \geq 1$, let $\tau_n : M \otimes_k A \to M \otimes_k A$ be an n-th Frobenius operator over A on R. Let $\widehat{M} = \mathrm{Hom}_R(M, \Omega_R)$. Define the adjoint operator C_n on $\widehat{M} \otimes_k A \simeq \mathrm{Hom}_{R\otimes_k A}(M \otimes_k A, \Omega_R \otimes_k A)$ of τ_n by the rule*

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$(C_n\hat{m})(m) = (C^{n[k:\mathbb{F}_p]} \otimes \mathrm{id}_A)(\hat{m}(\tau_n(m)))$ *for any* $m \in M \otimes_k A$ *and* $\hat{m} \in \widehat{M} \otimes_k A$.

Let I *be a maximal ideal of* R *and let* $d = [R/I : k]$. *Then*

$$\det_{A[G][[t]]}\Big(1 - \sum_{n=1}^{\infty} t^n \tau_n,\ M/IM \otimes_k A[[t]]\Big)^{-1} \in 1 + t^d A[G][[t^d]].$$

We have

$$\prod_{I \in \mathrm{Max}(R)} \det_{A[G][[t]]}\Big(1 - \sum_{n=1}^{\infty} t^n \tau_n,\ M/IM \otimes_k A[[t]]\Big)^{-1} = \det_{A[G][[t]]}\Big(1 - \sum_{n=1}^{\infty} t^n C_n,\ \widehat{M} \otimes_k A[[t]]\Big)^{(-1)^{r-1}}.$$

All determinants will be defined in the next section.

Remark 1.3. This theorem gives an equivariant version of Anderson's trace formula (see [1]). As an application, we prove an equivariant trace formula of crystals over function fields which generalizes the class number formula in [3] and [5]. This result is an analog of Stark's conjecture about special values of Artin–Goss L-functions of Drinfeld modules (see [4]).

2. Proof of Theorem 1.2

Lemma 2.1. *Let* M *be an* $R[G]$*-module. Define* $R[G]$*-linear maps*

$$\pi : R[G] \otimes_R M \to M, \qquad \bigoplus_{g \in G} g \otimes m_g \mapsto \sum_{g \in G} g m_g;$$

$$i : M \to R[G] \otimes_R M, \qquad m \mapsto \frac{1}{|G|} \sum_{g \in G} g \otimes g^{-1} m.$$

We have $\pi \circ i = \mathrm{id}_M$ *and* $R[G] \otimes_R M = i(M) \oplus N$ *for* $N = \ker(\pi)$. *Hence*

$$M \otimes_k A \oplus N \otimes_k A \simeq (M \oplus N) \otimes_k A \simeq R[G] \otimes_R M \otimes_k A = M \otimes_k A[G].$$

(1) Then M *is a projective* $R[G]$*-module if and only if* M *is projective as an* R*-module.*

(2) For any $\phi \in \mathrm{End}_{A[G]}(M \otimes_k A)$, $\phi \oplus 0 : (M \otimes_k A) \oplus (N \otimes_k A) \to (M \otimes_k A) \oplus (N \otimes_k A)$ *induces an* $A[G]$*-linear map* $\Phi : M \otimes_k A[G] \to M \otimes_k A[G]$. *If* $\dim_k M < \infty$, *then* $M \otimes_k A[G][t]$ *is a free* $A[G][t]$*-module of finite rank. In this case, we can define*

$$\det_{A[G][t]}(1 - t\phi,\ M \otimes_k A[t]) = \det_{A[G][t]}(1 - t\Phi,\ M \otimes_k A[G][t]) \in 1 + A[G][t].$$

Definition 2.2. Let M be a $k[G]$-module and let $\phi \in \mathrm{End}_{A[G]}(M \otimes_k A)$. A finitely generated $k[G]$-submodule M_0 of M is called a nucleus of ϕ if there exists an exhaustive increasing filtration of M by finitely generated $k[G]$-submodules $M_0 \subset M_1 \subset M_2 \subset \cdots$ such that $\phi(M_{i+1} \otimes_k A) \subset M_i \otimes_k A$ for any $i \geq 0$. One can show that $\det_{A[G][t]}(1 - t\phi, M_0 \otimes_k A[t])$ does not depend on the choice of the nucleus M_0 of ϕ. Define

$$\det_{A[G][t]}(1 - t\phi, M \otimes_k A[t]) = \det_{A[G][t]}(1 - t\phi,\ M_0 \otimes_k A[t]).$$

Definition 2.3. Let R be a finitely generated k-algebra and M a finitely generated $R[G]$-module. For any $n \geq 1$, let $C_n : M \otimes_k A \to M \otimes_k A$ be an n-th Cartier operator over A on R. By [1, Proposition 6], $C_1, C_2, \ldots, C_n$ has a common nucleus M_n for any $n \geq 1$. Then

$$\det_{A[G][t]}\Big(1 - \sum_{i=1}^{n} t^i C_i,\ M_n \otimes_k A[t]\Big) \in 1 + tA[G][t]$$

does not depend on the choice of M_n. Define

$$\det_{A[G][[t]]}\Big(1 - \sum_{n=1}^{\infty} t^n C_n,\ M \otimes_k A[[t]]\Big) = \lim_{n \to \infty} \det_{A[G][t]}\Big(1 - \sum_{i=1}^{n} t^i C_i,\ M_n \otimes_k A[t]\Big) \in 1 + tA[G][[t]].$$

We are ready to prove Theorem 1.2 now.

By Lemma 2.1, we may assume $G = \{1\}$. For the sequence $\{\tau_n\}_{n \geq 1}$, there exists a unique n-th Frobenius operator τ'_n on $M \otimes_k A$ over A on R for each n such that

$$1-\sum_{n=1}^{\infty} t^n\tau_n = (1-t\tau'_1)(1-t^2\tau'_2)(1-t^3\tau'_3)\cdots.$$

Let $C'_n : \widehat{M}\otimes_k A \to \widehat{M}\otimes_k A$ be the adjoint of τ'_n. We have

$$1-\sum_{n=1}^{\infty} t^n C_n = \cdots(1-t^3C'_3)(1-t^2C'_2)(1-tC'_1).$$

For any $n \geq 1$, let k_n be the extension field of k of degree n. For any object $\mathcal{F}$ over k, let $\mathcal{F}_n = k_n \otimes_k \mathcal{F}$. For any k-linear map $\phi : \mathcal{F} \to \mathcal{F}$, denote the k_n-linear map $\mathrm{id}_{k_n}\otimes\phi : \mathcal{F}_n \to \mathcal{F}_n$ also by ϕ. Then $\tau'_n : M_n \otimes_{k_n} A_n \to M_n \otimes_{k_n} A_n$ is a 1-th Frobenius operator over A_n on R_n. By [1, Theorem 1], we have

$$\prod_{J\in\mathrm{Max}(R_n)} \det{}_{A_n[[t]]}\Big(1-t\tau'_n,\ M_n/JM_n\otimes_{k_n} A_n[[t]]\Big)^{-1} = \det{}_{A_n[[t]]}\Big(1-tC'_n,\ \widehat{M}_n\otimes_{k_n} A_n[[t]]\Big)^{(-1)^{r-1}}.$$

For any maximal ideal I of R of degree d, $k_n\otimes_k R/I \simeq \prod_{I\subset J\in\mathrm{Max}(R_n)} R_n/J$ and $[R_n/J : k_n] = \frac{d}{(n,d)}$. By [2, Lemma 8.1.4], we have

$$\begin{aligned}
&\det{}_{A[[t]]}\Big(1-t\tau'_n,\ M/IM\otimes_k A[[t]]\Big)\\
&=\det{}_{A_n[[t]]}\Big(1-t\tau'_n,\ M_n/IM_n\otimes_{k_n} A_n[[t]]\Big)\\
&=\prod_{I\subset J\in\mathrm{Max}(R_n)} \det{}_{A_n[[t]]}\Big(1-t\tau'_n,\ M_n/JM_n\otimes_{k_n} A_n[[t]]\Big) \in 1+t^{\frac{d}{(n,d)}}A[[t^{\frac{d}{(n,d)}}]].
\end{aligned}$$

Substituting t by t^n, we get

$$\begin{aligned}
&\det{}_{A[[t]]}\Big(1-\sum_{n=1}^{\infty} t^n\tau_n,\ M/IM\otimes_k A[[t]]\Big)^{-1}\\
&=\prod_{n=1}^{\infty}\det{}_{A[[t]]}\Big(1-t^n\tau'_n,\ M/IM\otimes_k A[[t]]\Big)^{-1} \in 1+t^{\frac{nd}{(n,d)}}A[[t^{\frac{nd}{(n,d)}}]] \subset 1+t^dA[[t^d]]
\end{aligned}$$

and

$$\begin{aligned}
&\prod_{I\in\mathrm{Max}(R)} \det{}_{A[[t]]}\Big(1-t^n\tau'_n,\ M/IM\otimes_k A[[t]]\Big)^{-1}\\
&=\prod_{I\in\mathrm{Max}(R)}\ \prod_{I\subset J\in\mathrm{Max}(R_n)} \det{}_{A_n[[t]]}\Big(1-t^n\tau'_n,\ M_n/JM_n\otimes_{k_n} A_n[[t]]\Big)^{-1}\\
&=\prod_{J\in\mathrm{Max}(R_n)} \det{}_{A_n[[t]]}\Big(1-t^n\tau'_n,\ M_n/JM_n\otimes_{k_n} A_n[[t]]\Big)^{-1}\\
&=\det{}_{A_n[[t]]}\Big(1-t^nC'_n,\ \widehat{M}_n\otimes_{k_n} A_n[[t]]\Big)^{(-1)^{r-1}}\\
&=\det{}_{A[[t]]}\Big(1-t^nC'_n,\ \widehat{M}\otimes_k A[[t]]\Big)^{(-1)^{r-1}}.
\end{aligned}$$

Then we have

$$\begin{aligned}
&\prod_{I\in\mathrm{Max}(R)} \det{}_{A[[t]]}\Big(1-\sum_{n=1}^{\infty} t^n\tau_n,\ M/IM\otimes_k A[[t]]\Big)^{-1}\\
&=\prod_{I\in\mathrm{Max}(R)}\prod_{n=1}^{\infty}\det{}_{A[[t]]}\Big(1-t^n\tau'_n,\ M/IM\otimes_k A[[t]]\Big)^{-1}\\
&=\prod_{n=1}^{\infty}\prod_{I\in\mathrm{Max}(R)}\det{}_{A[[t]]}\Big(1-t^n\tau'_n,\ M/IM\otimes_k A[[t]]\Big)^{-1}
\end{aligned}$$

$$= \prod_{n=1}^{\infty} \det_{A[[t]]}\left(1 - t^n C_n', \widehat{M} \otimes_k A[[t]]\right)^{(-1)^{r-1}}$$

$$= \det_{A[[t]]}\left(1 - \sum_{n=1}^{\infty} t^n C_n, \widehat{M} \otimes_k A[[t]]\right)^{(-1)^{r-1}}.$$

This completes the proof of Theorem 1.2.

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