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Uniform temporal convergence of numerical schemes for miscible flow through porous media

Convergence uniforme en temps de schémas numériques pour un écoulement miscible en milieu poreux

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ABSTRACT

The Hybrid Mimetic Mixed (HMM) family of schemes contains the Hybrid Finite Volume, Mimetic Finite Difference and Mixed Finite Volume methods. This note proves that HMM schemes, when applied to a model of the miscible displacement of one incompressible fluid by another through a porous medium, produce approximations of the concentration variable that converge uniformly towards the exact concentration.
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RÉSUMÉ

La famille de schémas hybrides mimétiques mixtes (HMM) englobe les méthodes volumes finis hybrides, différences finies mimétiques et volumes finis mixtes. Cette note prouve que les schémas HMM, appliqués à un modèle de déplacement incompressible miscible en milieu poreux d'un fluide par un autre, produisent des concentrations approchées qui convergent uniformément vers la concentration exacte.
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1. Introduction

The nonlinearly-coupled elliptic–parabolic system

$$\left.\begin{array}{ll} \operatorname{div}(\bar{\mathbf{u}}) = q^I - q^P \quad \text{in } \Omega \times (0,T), & \bar{\mathbf{u}} = -A(\cdot,\bar{c})\nabla\bar{p} \quad \text{in } \Omega\times(0,T), \\ \int_\Omega \bar{p}(x,\cdot)\,\mathrm{d}x = 0 \quad \text{on } (0,T), & \bar{\mathbf{u}}\cdot\mathbf{n} = 0 \quad \text{on } \partial\Omega\times(0,T), \end{array}\right\} \tag{1}$$

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$$\left.\begin{aligned}&\Phi\partial_t\bar{c}-\operatorname{div}(\mathbf{D}(\cdot,\bar{\mathbf{u}})\nabla\bar{c}-\bar{c}\,\bar{\mathbf{u}})=\hat{c}q^I-\bar{c}q^P\quad\text{in }\Omega\times(0,T),\\&\bar{c}(\cdot,0)=c_0\quad\text{in }\Omega,\\&\mathbf{D}(\cdot,\bar{\mathbf{u}})\nabla\bar{c}\cdot\mathbf{n}=0\quad\text{on }\partial\Omega\times(0,T),\end{aligned}\right\}\tag{2}$$

describes the single-phase, miscible displacement through a porous medium of one incompressible fluid by another, in the absence of gravity [9]. The unknowns are the pressure $\bar{p}$ of the fluid mixture, the Darcy velocity $\bar{\mathbf{u}}$ of the fluid mixture and the concentration $\bar{c}$ of the injected fluid in the medium. We assume that the porous medium Ω is an open, bounded, convex polygonal subset of $\mathbb{R}^d$, $d\geq 2$, and that the displacement occurs over the time interval $(0,T)$, $T>0$. The porosity $\Phi\in L^\infty(\Omega)$, and there is $\phi_*>0$ such that for a.e. $x\in\Omega$, $\phi_*\leq\Phi(x)\leq\phi_*^{-1}$. The coefficient $A:\Omega\times\mathbb{R}\to M_d(\mathbb{R})$ is a uniformly elliptic, bounded Carathéodory function that combines the absolute permeability of the medium and viscosity of the fluid mixture. The injected concentration $\hat{c}\in L^\infty(\Omega\times(0,T))$ satisfies $0\leq\hat{c}(x,t)\leq 1$ for a.e. $(x,t)\in\Omega\times(0,T)$, and the initial concentration $c_0\in L^\infty(\Omega)$ satisfies $0\leq c_0(x)\leq 1$ for a.e. $x\in\Omega$. The injection well source terms $q^I\in L^\infty(0,T;L^2(\Omega))$ and production well sink terms $q^P\in L^\infty(0,T;L^r(\Omega))$ (for some $r>2$) are nonnegative and satisfy $\int_\Omega q^I(x,t)\,\mathrm{d}x=\int_\Omega q^P(x,t)\,\mathrm{d}x$ for all $t\in(0,T)$. Finally, the diffusion–dispersion tensor $\mathbf{D}:\Omega\times\mathbb{R}^d\to M_d(\mathbb{R})$ is a Carathéodory function with positive constants $\alpha_{\mathbf{D}}$, $\Lambda_{\mathbf{D}}$ such that for a.e. $x\in\Omega$ and all $\zeta,\xi\in\mathbb{R}^d$,

$$\mathbf{D}(x,\zeta)\xi\cdot\xi\geq\alpha_{\mathbf{D}}(1+|\zeta|)|\xi|^2\text{ and }|\mathbf{D}(x,\zeta)|\leq\Lambda_{\mathbf{D}}(1+|\zeta|).\tag{3}$$

Our notion of weak solution to (1)–(2) is [2, Definition 1.1].

2. Preliminaries

The HMM family of discretisations includes Hybrid Finite Volumes (HFV) [8], Mimetic Finite Differences [1] and Mixed Finite Volumes (MFV) [3]. These three families are equivalent [5]. An implementation of the HMM method amounts to a choice of any of these three discretisations.

The variable of most interest is the concentration $\bar{c}$. To obtain meaningful approximations of this quantity, one must account for the (dominant) convective term $\operatorname{div}(\bar{c}\,\bar{\mathbf{u}})$ in (2). One therefore requires a good approximation of the normal flux of $\bar{\mathbf{u}}$ through the boundaries of control volumes of the mesh. We therefore follow [2] by choosing the MFV framework to discretise (1). The HFV framework enables direct approximations of $\bar{c}$ and $\nabla\bar{c}$ without first approximating the flux of $\mathbf{D}(\cdot,\bar{\mathbf{u}})\nabla\bar{c}$, which for our purposes herein would be redundant. For brevity, we omit the details of the MFV scheme for (1); interested readers should consult [2].

With the exception of the convective term, we present the HFV discretisation of (2) in the notation of *gradient schemes*, a framework introduced by [6]. Incorporating the discretisation of convective terms into the gradient schemes framework is the subject of future work.

We adopt the same notion of admissible mesh $\mathcal{D}=(\mathcal{M},\mathcal{E})$ of Ω and corresponding notation as [2], with the following exceptions: we write $|K|$ for the d-dimensional measure of a control volume K, and $|\sigma|$ for the $(d-1)$-dimensional measure of an edge σ. For all $K\in\mathcal{M}$ and $\sigma\in\mathcal{E}_K$, we denote by $d_{K,\sigma}$ the Euclidean distance between x_K and the hyperplane containing σ. Furthermore, we discretise the time interval by choosing a sequence $(t^{(n)})^{n=0,\ldots,N}$ such that $0=t^{(0)}<t^{(1)}<\cdots<t^{(N)}=T$. For $n=1,\ldots,N$, set $\delta t^{(n-\frac{1}{2})}=t^{(n)}-t^{(n-1)}$ and $\delta_{\mathcal{D}}=\max_{n=1,\ldots,N}\delta t^{(n-\frac{1}{2})}$.

The space of discrete unknowns is $X_{\mathcal{D}}:=\{v=((v_K)_{K\in\mathcal{M}},(v_\sigma)_{\sigma\in\mathcal{E}}):v_K\in\mathbb{R},v_\sigma\in\mathbb{R}\}$. For $v\in X_{\mathcal{D}}$ and $K\in\mathcal{M}$, define the operator $\Pi_{\mathcal{D}}:X_{\mathcal{D}}\to L^2(\Omega)$ by $\Pi_{\mathcal{D}}(v)=v_K$ on K. We employ the same discrete gradient operator $\nabla_{\mathcal{D}}:X_{\mathcal{D}}\to L^2(\Omega)^d$ as in [6, Eq. (5.8)]; see this reference for further details of the construction. The norm on $v\in X_{\mathcal{D}}$ is then $\|v\|_{X_{\mathcal{D}}}:=\|\Pi_{\mathcal{D}}v\|_{L^2(\Omega)}+\|\nabla_{\mathcal{D}}v\|_{L^2(\Omega)^d}$. To specify the initial condition in the scheme, we use a linear interpolation operator $\mathcal{I}_{\mathcal{D}}:L^2(\Omega)\to X_{\mathcal{D}}$. The space of discrete fluxes is $\mathcal{F}_{\mathcal{D}}:=\{G=(G_{K,\sigma})_{K\in\mathcal{M},\sigma\in\mathcal{E}_K}:G_{K,\sigma}\in\mathbb{R}\}$.

In the scheme below, we consider sequences $(c^{(n)})_{n=0,\ldots,N}\subset X_{\mathcal{D}}$ and $(F^{(n)})_{n=1,\ldots,N}\subset\mathcal{F}_{\mathcal{D}}$. For $n=1,\ldots,N$, $c_K^{(n)}$ approximates $\bar{c}$ on $K\times[t^{(n-1)},t^{(n)})$, and $F_{K,\sigma}^{(n)}$ approximates the flux $-\int_\sigma\bar{\mathbf{u}}\cdot\mathbf{n}_{K,\sigma}\,\mathrm{d}\gamma$ on $[t^{(n-1)},t^{(n)})$. As in [2], we denote by $\mathbf{u}$ the piecewise-constant, cell-centred approximation of $\bar{\mathbf{u}}$ coming from the MFV scheme for (1).

Following convention in the gradient schemes literature to date, we use the notation $\Pi_{\mathcal{D}}$ and $\nabla_{\mathcal{D}}$ for functions dependent on both space and time. Thus if $(v^{(n)})_{n=0,\ldots,N}\subset X_{\mathcal{D}}$, for a.e. $x\in\Omega$ we set $\Pi_{\mathcal{D}}v(x,0)=\Pi_{\mathcal{D}}v^{(0)}(x)$, and for all $n=1,\ldots,N$, all $t\in[t^{(n-1)},t^{(n)})$ and a.e. $x\in\Omega$, $\Pi_{\mathcal{D}}v(x,t)=\Pi_{\mathcal{D}}v^{(n)}(x)$, $\nabla_{\mathcal{D}}v(x,t)=\nabla_{\mathcal{D}}v^{(n)}(x)$ and

$$\delta_{\mathcal{D}}v(t)=\delta_{\mathcal{D}}^{(n-\frac{1}{2})}v:=\frac{v^{(n)}-v^{(n-1)}}{\delta t^{(n-\frac{1}{2})}}\in X_{\mathcal{D}}.$$

The scheme for (2) is then: find sequences $(c^{(n)})_{n=0,\ldots,N}\subset X_{\mathcal{D}}$ and $(F^{(n)})_{n=1,\ldots,N}\subset\mathcal{F}_{\mathcal{D}}$ such that

$$
\left.\begin{aligned}
&c^{(0)} = \mathcal{I}_{\mathcal{D}} c_0 \text{ and for all } \varphi = (\varphi^{(n)})_{n=1,\ldots,N} \subset X_{\mathcal{D}},\\
&\int_0^T \int_\Omega [\Phi(x)\Pi_{\mathcal{D}}\delta_{\mathcal{D}}c(x,t)\Pi_{\mathcal{D}}\varphi(x,t) + \mathbf{D}(x,\mathbf{u}(x,t))\nabla_{\mathcal{D}}c(x,t)\cdot\nabla_{\mathcal{D}}\varphi(x,t)]\,\mathrm{d}x\,\mathrm{d}t\\
&\quad + \sum_{n=1}^{N} \delta t^{(n-\frac{1}{2})} \sum_{K\in\mathcal{M}} \sum_{\substack{\sigma\in\mathcal{E}_K\cap\mathcal{E}_{\text{int}}\\ \sigma=K|L}} \left[(-F_{K,\sigma}^{(n)})^+ c_K^{(n)} - (-F_{K,\sigma}^{(n)})^- c_L^{(n)}\right]\varphi_K^{(n)}\\
&\quad + \int_0^T \int_\Omega \left[q^P(x,t)\Pi_{\mathcal{D}}c(x,t)\Pi_{\mathcal{D}}\varphi(x,t) - q^I(x,t)\hat{c}(x,t)\Pi_{\mathcal{D}}\varphi(x,t)\right]\mathrm{d}x\,\mathrm{d}t = 0,
\end{aligned}\right\} \quad (4)
$$

where $(-F_{K,\sigma}^{(n)})^+$ and $(-F_{K,\sigma}^{(n)})^-$ denote the positive and negative parts of $-F_{K,\sigma}^{(n)}$, respectively.

The following convergence is known:

Theorem 2.1. *(See Chainais-Hillairet–Droniou [2].) Take a sequence $(\mathcal{D}_m)_{m\in\mathbb{N}}$ of admissible meshes of $\Omega \times (0,T)$ satisfying the appropriate regularity hypotheses and such that* $\text{size}(\mathcal{D}_m) \to 0$ *and $\delta_{\mathcal{D}_m} \to 0$ as $m \to \infty$. Then, up to a subsequence,*

(i) *$\Pi_{\mathcal{D}_m} c \to \bar{c}$, a.e. on $\Omega \times (0,T)$, weakly-$*$ in $L^\infty(0,T;L^2(\Omega))$ and strongly in $L^p(0,T;L^q(\Omega))$ for all $p<\infty$ and all $q<2$;*
(ii) *$\nabla_{\mathcal{D}_m} c \rightharpoonup \nabla\bar{c}$ weakly in $L^2(\Omega\times(0,T))^d$, and*
(iii) *$\mathbf{u}_m \to \bar{\mathbf{u}}$, weakly-$*$ in $L^\infty(0,T;L^2(\Omega)^d)$ and strongly in $L^2(\Omega\times(0,T))^d$.*

The purpose of this note is to demonstrate that (i) holds with $p=\infty$ and $q=2$.

3. Uniform temporal convergence

In the following estimate, we employ the dual seminorm

$$
|v|_{\star,\mathcal{D}} = \sup\left\{\int_\Omega \Pi_{\mathcal{D}}v(x)\Pi_{\mathcal{D}}w(x)\,\mathrm{d}x : w\in X_{\mathcal{D}},\ \|\nabla_{\mathcal{D}}w\|_{L^{2d}(\Omega)^d} = 1\right\}.
$$

Lemma 3.1 *(Discrete time derivative estimate). Let $\mathcal{D}$ be an admissible mesh of $\Omega\times(0,T)$ and take a solution $(c^{(n)})_{n=0,\ldots,N}\subset X_{\mathcal{D}}$, $(F^{(n)})_{n=1,\ldots,N}\subset\mathcal{F}_{\mathcal{D}}$ to (4). Then there exists $C_1>0$, not depending upon the mesh, such that*

$$
\int_0^T |\delta_{\mathcal{D}}c(t)|_{\star,\mathcal{D}}^4\,\mathrm{d}t \le C_1. \quad (5)
$$

Sketch of proof. Let $w\in X_{\mathcal{D}}$ and $t\in(0,T)$. For $\zeta\in\mathbb{R}^d$, denote by $\mathbf{D}^{1/2}(\cdot,\zeta)$ the square root of the positive-definite matrix $\mathbf{D}(\cdot,\zeta)$. From (3), the estimates [2, Propositions 3.1, 3.2] on $\mathbf{u}$ and $\nabla_{\mathcal{D}}c$ and the bound $|\mathbf{D}^{1/2}(\cdot,\mathbf{u})| \le \Lambda_{\mathbf{D}}^{1/2}(1+|\mathbf{u}|)^{1/2}$, one can show that $\int_\Omega \mathbf{D}(\cdot,\mathbf{u})\nabla_{\mathcal{D}}c(t)\cdot\nabla_{\mathcal{D}}w \le C_2\|\nabla_{\mathcal{D}}w\|_{L^4(\Omega)^d}$. Use the conservativity $F_{K,\sigma}^{(n)} + F_{L,\sigma}^{(n)} = 0$ ($\sigma = K|L \in \mathcal{E}_{\text{int}}$) to gather by edges and write

$$
\begin{aligned}
&\sum_{K\in\mathcal{M}} \sum_{\substack{\sigma\in\mathcal{E}_K\cap\mathcal{E}_{\text{int}}\\ \sigma=K|L}} \left[(-F_{K,\sigma}^{(n)})^+ c_K^{(n)} - (-F_{K,\sigma}^{(n)})^- c_L^{(n)}\right] w_K^{(n)}\\
&\quad = \sum_{\sigma=K|L\in\mathcal{E}_{\text{int}}} (-F_{K,\sigma}^{(n)})^+ c_K^{(n)}(w_K^{(n)} - w_L^{(n)}) + \sum_{\sigma=K|L\in\mathcal{E}_{\text{int}}} (-F_{L,\sigma}^{(n)})^+ c_L^{(n)}(w_L^{(n)} - w_K^{(n)}).
\end{aligned}
$$

These terms are symmetric in K and L so it suffices to estimate only one of them. By squaring both sides of [5, Eq. (2.22)] and making the appropriate scalings, one can show that

$$
\sum_{\sigma=K|L\in\mathcal{E}_{\text{int}}} \frac{d_{K,\sigma}}{|\sigma|}\left|-F_{K,\sigma}^{(n)}\right|^2 \le C_3.
$$

Then Cauchy–Schwarz and Hölder give, for some $p>1$ to be determined,

$$\begin{aligned}&\left|\sum_{\sigma=K|L\in\mathcal{E}_{\text{int}}}(-F^{(n)}_{K,\sigma})^+c^{(n)}_K(w^{(n)}_K-w^{(n)}_L)\right|\\&\le C_4\left(\sum_{K\in\mathcal{M}}\sum_{\substack{\sigma\in\mathcal{E}_K\cap\mathcal{E}_{\text{int}}\\ \sigma=K|L}}|\sigma|d_{K,\sigma}\left|\frac{w^{(n)}_K-w^{(n)}_L}{d_{K,\sigma}}\right|^{2p'}\right)^{\frac{1}{2p'}}\left(\sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_K}(|\sigma|d_{K,\sigma})|c^{(n)}_K|^{2p}\right)^{\frac{1}{2p}}\\&\le C_5\left\|\nabla_{\mathcal{D}}w\right\|_{L^{2p'}(\Omega)^d}\left\|\Pi_{\mathcal{D}}c(t)\right\|_{L^{2p}(\Omega)},\end{aligned}$$

the last inequality coming from [8, Lemma 4.2] and the identity $\sum_{\sigma\in\mathcal{E}_K}|\sigma|d_{K,\sigma}=d|K|$. Applying a discrete Sobolev inequality [8, Lemma 5.3] to the estimates [2, Proposition 3.2], we can interpolate between the spaces $L^\infty(0,T;L^2(\Omega))$ and $L^2(0,T;L^{2^\star}(\Omega))$ to ensure that $\Pi_{\mathcal{D}}c$ is bounded in the $L^4(0,T;L^{\frac{2d}{d-1}}(\Omega))$ norm. We therefore set $p=\frac{d}{d-1}>1$, which gives $p'=d$, thereby justifying the choice of Lebesgue exponent in the dual seminorm definition above.

Denote by k the index such that $t\in[t^{(k-1)},t^{(k)})$. In (4), take $\varphi=(\varphi^{(n)})_{n=1,\dots,N}\subset X_{\mathcal{D}}$ satisfying $\varphi^{(k)}=w$ and $\varphi^{(n)}=0$ for $n\neq k$ to see that

$$\begin{aligned}\int_\Omega \Pi_{\mathcal{D}}\delta_{\mathcal{D}}c(t)\Pi_{\mathcal{D}}w\le C_6\left\|\nabla_{\mathcal{D}}w\right\|_{L^{2d}(\Omega)^d}\Bigg(1+\left\|\Pi_{\mathcal{D}}c(t)\right\|_{L^{\frac{2d}{d-1}}(\Omega)}\\+\left\|q^P\right\|_{L^\infty(0,T;L^r(\Omega))}\left\|\Pi_{\mathcal{D}}c(t)\right\|_{L^2(\Omega)}+\left\|q^I\right\|_{L^\infty(0,T;L^2(\Omega))}\Bigg).\end{aligned}$$

Then

$$\int_0^T|\delta_{\mathcal{D}}c(t)|^4_{\star,\mathcal{D}}\,\mathrm{d}t\le C_7\left\|\Pi_{\mathcal{D}}c(t)\right\|_{L^4(0,T;L^{\frac{2d}{d-1}}(\Omega))}\le C_1.\qquad\square$$

The key ideas for the following result are due to [4].

Theorem 3.2 *(Uniform temporal convergence of concentration). Assume the same hypotheses as Theorem 2.1. If, for all $T_0\in[0,T]$, $\bar{c}$ and $\bar{\mathbf{u}}$ satisfy the energy identity*

$$\frac12\int_\Omega\Phi\bar{c}(T_0)^2=\frac12\int_\Omega\Phi c_0^2+\int_0^{T_0}\int_\Omega\bar{c}\hat{c}q^I-\frac12\int_0^{T_0}\int_\Omega\bar{c}^2(q^I+q^P)-\int_0^{T_0}\int_\Omega\mathbf{D}(\cdot,\bar{\mathbf{u}})\nabla\bar{c}\cdot\nabla\bar{c},\tag{6}$$

then, up to a subsequence, $\Pi_{\mathcal{D}_m}c\to\bar{c}$ in $L^\infty(0,T;L^2(\Omega))$.

Remark 3.3. The identity (6) seems natural, but certainly not obvious. If $\mathbf{D}$ is uniformly bounded, its proof is straightforward; see for example the calculations in [7, Proposition 3.1].

Sketch of proof. Fix $T_0\in[0,T]$ and take a sequence $(T_m)_{m\in\mathbb{N}}\subset[0,T]$ with $T_m\to T_0$ as $m\to\infty$. Denote by $k_m\in[1,N_m]$ the index such that $T_0\in[t^{(k_m-1)},t^{(k_m)})$. Apply the uniform-in-time, weak-in-space discrete Aubin–Simon theorem [4, Theorem 3.1] with estimates [2, Proposition 3.2] and (5) to obtain $\Pi_{\mathcal{D}_m}c\to\bar{c}$ in $L^\infty(0,T;L^2(\Omega)\text{-w})$. This gives $\sqrt{\Phi}\Pi_{\mathcal{D}_m}c(T_m)\rightharpoonup\sqrt{\Phi}\bar{c}(T_0)$ weakly in $L^2(\Omega)$ and hence

$$\liminf_{m\to\infty}\int_\Omega\Phi\left(\Pi_{\mathcal{D}_m}c(T_m)\right)^2\ge\int_\Omega\Phi\left(\bar{c}(T_0)\right)^2.\tag{7}$$

Take $\varphi=(c^{(1)},\dots,c^{(k_m)},0,\dots,0)\subset X_{\mathcal{D}}$ in (4) and follow the calculations in [2, Proposition 3.2] to obtain

$$\begin{aligned}&\frac12\int_\Omega\Phi\left(\Pi_{\mathcal{D}_m}c(T_m)\right)^2+\int_0^{T_m}\int_\Omega\mathbf{D}(\cdot,\mathbf{u}_m)\nabla_{\mathcal{D}_m}c\cdot\nabla_{\mathcal{D}_m}c\\&\quad+\frac12\int_0^{T_m}\int_\Omega(\Pi_{\mathcal{D}_m}c)^2(q^I+q^P)\le\frac12\int_\Omega\Phi(\mathcal{I}_{\mathcal{D}_m}c_0)^2+\int_0^{t^{(k_m)}}\int_\Omega\Pi_{\mathcal{D}_m}c\hat{c}q^I.\end{aligned}$$

Using the fact that $\mathcal{I}_{\mathcal{D}_m} c_0 \to c_0$ in $L^2(\Omega)$, take the limit superior as $m \to \infty$:

$$\begin{aligned}\frac{1}{2}\limsup_{m\to\infty}\int_\Omega \Phi(\Pi_{\mathcal{D}_m} c(T_m))^2 \le{}& \frac{1}{2}\int_\Omega \Phi c_0^2 + \limsup_{m\to\infty}\int_0^{t^{(k_m)}}\int_\Omega \Pi_{\mathcal{D}_m} c\hat{c} q^I \\ &- \frac{1}{2}\liminf_{m\to\infty}\int_0^{T_m}\int_\Omega (\Pi_{\mathcal{D}_m} c)^2(q^I+q^P) - \liminf_{m\to\infty}\int_0^{T_m}\int_\Omega \mathbf{D}(\cdot,\mathbf{u}_m)\nabla_{\mathcal{D}_m} c\cdot\nabla_{\mathcal{D}_m} c \\ =:{}& \frac{1}{2}\int_\Omega \Phi c_0^2 + \limsup_{m\to\infty}\mathcal{S}_1^{(m)} - \frac{1}{2}\liminf_{m\to\infty}\mathcal{S}_2^{(m)} - \liminf_{m\to\infty}\mathcal{S}_3^{(m)}. \end{aligned} \tag{8}$$

Note that the $t^{(k_m)}$ such that $T_m \in [t^{(k_m-1)}, t^{(k_m)})$ converges to T_0 as $m \to \infty$. From Theorem 2.1 and Fatou's lemma,

$$\limsup_{m\to\infty}\mathcal{S}_1^{(m)} = \int_0^{T_0}\int_\Omega \bar{c}\hat{c}q^I \quad\text{and}\quad \liminf_{m\to\infty}\mathcal{S}_2^{(m)} \ge \int_0^{T_0}\int_\Omega \bar{c}^2(q^I+q^P).$$

A similar argument to [7, Remark 3.2] shows that $\mathbf{D}^{1/2}(\cdot,\mathbf{u}_m)\nabla_{\mathcal{D}_m} c \rightharpoonup \mathbf{D}^{1/2}(\cdot,\bar{\mathbf{u}})\nabla\bar{c}$ weakly in $L^2(0,T;L^2(\Omega)^d)$. Dominated convergence gives $\mathbf{1}_{[0,T_m]}\mathbf{D}^{1/2}(\cdot,\bar{\mathbf{u}})\nabla\bar{c} \to \mathbf{1}_{[0,T_0]}\mathbf{D}^{1/2}(\cdot,\bar{\mathbf{u}})\nabla\bar{c}$ in $L^2(0,T;L^2(\Omega)^d)$. Thus

$$\begin{aligned}\int_0^{T_0}\int_\Omega \mathbf{D}(\cdot,\bar{\mathbf{u}})\nabla\bar{c}\cdot\nabla\bar{c} &= \lim_{m\to\infty}\int_0^{T_m}\int_\Omega \left(\mathbf{D}^{1/2}(\cdot,\bar{\mathbf{u}})\nabla_{\mathcal{D}}\bar{c}\right)\left(\mathbf{D}^{1/2}(\cdot,\mathbf{u}_m)\nabla_{\mathcal{D}_m} c\right) \\ &\le \left\|\mathbf{D}^{1/2}(\cdot,\bar{\mathbf{u}})\nabla\bar{c}\right\|_{L^2(0,T;L^2(\Omega)^d)} \liminf_{m\to\infty}\left\|\mathbf{1}_{[0,T_m]}\mathbf{D}^{1/2}(\cdot,\mathbf{u}_m)\nabla_{\mathcal{D}_m} c\right\|_{L^2(0,T;L^2(\Omega)^d)},\end{aligned}$$

and so

$$\liminf_{m\to\infty}\mathcal{S}_3^{(m)} \ge \int_0^{T_0}\int_\Omega \mathbf{D}(\cdot,\bar{\mathbf{u}})\nabla\bar{c}\cdot\nabla\bar{c}.$$

Collecting these convergences, we see that the right-hand sides of (8) and (6) agree, giving

$$\limsup_{m\to\infty}\int_\Omega \Phi(\Pi_{\mathcal{D}_m} c(T_m))^2 \le \int_\Omega \Phi\bar{c}(T_0)^2. \tag{9}$$

Comparing (7) and (9) shows that $\lim_{m\to\infty}\left\|\sqrt{\Phi}\Pi_{\mathcal{D}_m} c(T_m)\right\|^2_{L^2(\Omega)} = \left\|\sqrt{\Phi}\bar{c}(T_0)\right\|^2_{L^2(\Omega)}$, which, thanks to the weak-$L^2(\Omega)$ convergence established earlier, gives $\sqrt{\Phi}\Pi_{\mathcal{D}_m} c(T_m) \to \sqrt{\Phi}\bar{c}(T_0)$ strongly in $L^2(\Omega)$. Apply the characterisation [4, Lemma 6.4] of uniform convergence and the uniform positivity of Φ to conclude the proof. □

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