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Parametric estimation in autoregressive processes under quasi-associated random errors ☆

Estimation paramétrique dans des processus autorégressifs sous des erreurs quasi associées

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ABSTRACT

In this paper, we study the consistence of a recurrent stochastic algorithm under quasi-associated random errors. Kholev's algorithm estimates an unknown non-zero parameter θ introduced in a nonlinear autoregressive model. We establish the complete convergence and deduce a confidence interval for θ.

RÉSUMÉ

Dans cette note, on étudie la consistance d'un algorithme récurrent sous des erreurs quasi-associées. L'algorithme de Kholev approxime un paramètre non nul θ introduit dans un modèle autorégressif. On établit la convergence complète et on déduit un intervalle de confiance pour θ.

1. Introduction

It is known that the modern method of time series analysis plays a major role in the evaluation of econometric models and statistics. The study of the autoregressive models constitutes one of the fundamental problems posed therein [9,10]. Roughly speaking, the analysis of the autoregressive models allows us to establish controls to facilitate the development of the forecast and lead to the reduction of the undesirable fluctuations. Classical regression and related methods are almost always used in parameter estimation and hypothesis testing.

Let us consider the following nonlinear autoregressive process

$$X_{n+1} = f_\theta(X_n) + g(X_n)\xi_{n+1} \tag{1}$$

where the sequence $(\xi_n)_n$ is a stationary and zero-mean white noise. The functions f_θ, g are real such that f is differentiable with respect to θ. The parameter θ is non-zero and unknown.

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The estimation of the parameter θ of the procedure (1) was initiated by Bondarev [2] in the case of independent random errors $(\xi_n)_n$. However, in some real-life situations, the random variables need not to be independent. Recently, Dahmani and Ait Saidi [1] have studied the procedure (1) under strong mixing (dependent) of the sequence $(\xi_n)_n$.

In this paper, we study the procedure (1) under another form of dependence, which is the quasi-association of the sequence $(\xi_n)_n$. We establish the complete convergence of the estimator of the parameter θ and we deduce a confidence interval thereof. The notion of quasi-association arises in widely different areas such as reliability theory, statistical mechanics [4], percolation theory [5], etc.

2. Definitions

A finite family of random variables $\{Y_i, 1 \le i \le n\}$ is said to be quasi-associated [3] if, for every disjoint subsets S_1 and S_2 of $\{1, \dots, n\}$ and every pair of Lipschitz functions k and h, the following inequality of covariance is satisfied:

$$\left|\mathrm{Cov}\left(k(Y_i,\ i \in S_1), h(Y_j,\ j \in S_2)\right)\right| \le \mathrm{Lip}(k)\,\mathrm{Lip}(h) \sum_{i \in S_1} \sum_{j \in S_2} \left|\mathrm{Cov}\left(Y_i, Y_j\right)\right|,$$

where

$$\mathrm{Lip}(h) = \sup_{x \neq y} \frac{|h(x) - h(y)|}{|x - y|}.$$

Similarly, an infinite family is quasi-associated if every finite subfamily is quasi-associated.

According to Hsu and Robbins [6], a sequence of real-valued random variables $\{T_n, n \ge 1\}$ converges completely to 0 if for every positive ε,

$$\sum_{n=1}^{\infty} P\{|T_n| > \varepsilon\} < +\infty.$$

3. Algorithm and assumptions

We aim to estimate the non-zero unknown parameter θ by a sequence $(\theta_n)_n$ and establish the complete convergence thereof. For this, we use Kholev's algorithm [8] defined as follows:

$$\theta_{n+1} = \theta_n + \frac{a}{n}\left(X_{n+1} - f_{\theta_n}(X_n)\right) \frac{\partial f_{\theta_n}}{\partial \theta_n}(X_n),\ n \in \mathbf{N}^*, a > 0. \tag{2}$$

If we set

$$H_{\theta_n}(x) = \frac{\partial f_{\theta_n}}{\partial \theta_n}(x)\, g(x), \qquad R_{\theta_n}(\theta, x) = \frac{f_{\theta_n}(x) - f_\theta(x)}{\theta_n - \theta} \frac{\partial f_{\theta_n}}{\partial \theta_n}(x),$$

we obtain

$$\theta_{n+1} - \theta = (\theta_n - \theta)\left(1 - \frac{a}{n} R_{\theta_n}(\theta, X_n)\right) + \frac{a}{n} H_{\theta_n}(X_n)\, \xi_{n+1}.$$

By iterating this formula, we have

$$\theta_n - \theta = (\theta_1 - \theta) \prod_{i=1}^{n}\left(1 - \frac{a}{i} R_{\theta_i}(\theta, X_i)\right) + \sum_{i=1}^{n}\left(\frac{a}{i} \prod_{k=i+1}^{n}\left(1 - \frac{a}{k} R_{\theta_k}(\theta, X_k)\right) H_{\theta_i}(X_i)\, \xi_{i+1}\right),$$

where θ_1 is the initial guess of θ. Let us now introduce some hypotheses:

(H1): $\forall x \in \mathbf{R}, \forall \theta \in \mathbf{R}^*, 0 < m \le R_\theta(\theta, x)$.
(H2): $\forall \theta \in \mathbf{R}^*, |H_\theta(x)| < B < \infty$.
(H3): θ verify $|\theta_1 - \theta| \le C < \infty$.

These hypotheses are classical, we can find them in [1,8].

Lemma 1. *(See [7].) Let $Y_1, \dots, Y_n$ be real-valued random variables with $EY_i = 0$ and $|Y_i| \le M < \infty$ a.s for all $i = 1, \dots, n$ and some $M < \infty$. Assume, furthermore, that there exist $K < \infty$ and $\gamma > 0$ such that, for all u-tuples $(s_1, \dots, s_u)$ and all v-tuples $(t_1, \dots, t_v)$ with*

$$1 \le s_1 \le \ldots \le s_u \le t_1 \le \ldots \le t_v \le n$$

the following inequality is fulfilled

$$\left|\mathrm{Cov}\left(\prod_{i=1}^{u} Y_{s_i}, \prod_{j=1}^{v} Y_{t_j}\right)\right| \le K^2 M^{u+v-2} v \exp(-\gamma(t_1 - s_u)).$$

Then

$$P\left\{\sum_{i=1}^{n} Y_i \ge t\right\} \le \exp\left(\frac{-t^2}{C_1 n + C_2 t^{\frac{5}{3}}}\right),$$

where $C_1 = \frac{4K^2}{1-\exp(-\gamma)}$ *and* $C_2 = \left[\frac{2\max(K,\ M)}{1-\exp(-\gamma)}\right]^{\frac{1}{3}}$.

4. Main results

Theorem 2. *Assume that* $\xi_1, \xi_2, \ldots$ *of the algorithm (2) are quasi-associated random variables, stationary with zero mean and they satisfy the following conditions:*

1. $\sup_{i\in\mathbf{N}} |\xi_i| \le M < \infty$ *a.s.*
2. $\exists\, d_0 > 0,\ d > 0$ *such that* $\left|\mathrm{Cov}(\xi_0, \xi_j)\right| \le d_0 \exp(-dj)$ *for every* j.

Then for every $\varepsilon > 0$*, we have:*

$$P\{|\theta_n - \theta| > \varepsilon\} \le 2\exp\left(-\frac{\varepsilon^2 n}{4\left(C_1 + C_2 n^{\frac{2}{3}} \left(\frac{\varepsilon}{2}\right)^{\frac{5}{3}}\right)}\right), \tag{3}$$

where $C_1 = \frac{4K^2}{1-\exp(-d)}$, $C_2 = \left[\frac{2\max(K,\ M)}{1-\exp(-d)}\right]^{\frac{1}{3}}$ *and* $K = \max\left(2aBM, 2aB\sqrt{\frac{d_0}{1-\exp(-d)}}\right)$.

Proof. For $\varepsilon > 0$, we have

$$\begin{aligned} P\{|\theta_n - \theta| > \varepsilon\} &= P\left\{\left|\begin{array}{l}(\theta_1 - \theta)\prod_{i=1}^{n}\left(1 - \frac{a}{i} R_{\theta_i}(\theta, X_i)\right) \\ + \sum_{i=1}^{n}\left(\frac{a}{i}\prod_{k=i+1}^{n}\left(1 - \frac{a}{k} R_{\theta_k}(\theta, X_k)\right) H_{\theta_i}(X_i)\,\xi_{i+1}\right)\end{array}\right| > \varepsilon\right\} \\ &\le P\left\{\begin{array}{l}\left|(\theta_1 - \theta)\prod_{i=1}^{n}\left(1 - \frac{a}{i} R_{\theta_i}(\theta, X_i)\right)\right| \\ + \left|\sum_{i=1}^{n}\left(\frac{a}{i}\prod_{k=i+1}^{n}\left(1 - \frac{a}{k} R_{\theta_k}(\theta, X_k)\right) H_{\theta_i}(X_i)\,\xi_{i+1}\right)\right| > \varepsilon\end{array}\right\}. \end{aligned}$$

For n sufficiently large, we obtain:

$$\left|(\theta_1 - \theta)\prod_{i=1}^{n}\left(1 - \frac{a}{i} R_{\theta_i}(\theta, X_i)\right)\right| \le C\left|\prod_{i=1}^{n}\left(1 - \frac{a}{i} m\right)\right| < \frac{\varepsilon}{2}.$$

Consequently,

$$P\{|x_n - \theta| > \varepsilon\} \le P\left\{\left|\sum_{i=1}^{n} n\left(\frac{a}{i}\prod_{k=i+1}^{n}\left(1 - \frac{a}{k} R_{\theta_k}(\theta, X_k)\right) H_{\theta_i}(X_i)\,\xi_{i+1}\right)\right| > \frac{n\varepsilon}{2}\right\}.$$

Let us set

$$Z_i = n\frac{a}{i}\prod_{k=i+1}^{n}\left(1 - \frac{a}{k} R_{\theta_k}(\theta, X_k)\right) H_{\theta_i}(X_i)\,\xi_{i+1}$$
$$\zeta_i = \xi_{i+1}.$$

Let $(s_1, \ldots, s_u)$, $(t_1, \ldots, t_v)$ be a u-tuples, v-tuples respectively such that

$$1 \le s_1 \le \ldots \le s_u \le t_1 \le \ldots \le t_v \le n.$$

In the case where $s_u < t_1$, then by using the quasi-association of the sequence $(\zeta_n)_n$, we obtain:

$$\begin{aligned}
\left|\mathrm{Cov}\left(\prod_{i=1}^{u} Z_{s_i}, \prod_{j=1}^{v} Z_{t_j}\right)\right| &= \left|\prod_{i=1}^{u}\left(n\frac{a}{s_i}\prod_{k=s_i+1}^{n} n\frac{a}{i}\left(1-\frac{a}{k}R_{\theta_k}(\theta, X_k)\right)H_{\theta_{s_i}}(X_{s_i})\right)\right| \\
&\quad \left|\prod_{j=1}^{v}\left(n\frac{a}{t_j}\prod_{k=t_j+1}^{n}\left(1-\frac{a}{k}R_{\theta_k}(\theta, X_k)\right)H_{\theta_{t_j}}(X_{t_j})\right)\right|\left|\mathrm{Cov}\left(\prod_{i=1}^{u}\zeta_{s_i}, \prod_{j=1}^{v}\zeta_{t_j}\right)\right| \\
&\le (naB)^{u+v}\prod_{i=1}^{u}\left(\frac{1}{s_i}\frac{s_i+1}{n}\right)\prod_{j=1}^{v}\left(\frac{1}{t_i}\frac{t_i+1}{n}\right)M^{u+v-2}\sum_{m=1}^{u}\sum_{l=1}^{v}\left|\mathrm{Cov}\left(\zeta_{s_m}, \zeta_{t_l}\right)\right| \\
&\le (naB)^{u+v}\left(\frac{2}{n}\right)^{u+v} M^{u+v-2}\sum_{m=1}^{u}\sum_{l=1}^{v}\left|\mathrm{Cov}\left(\zeta_{s_m}, \zeta_{t_l}\right)\right| \\
&\le uv\,(2aB)^{u+v} M^{u+v-2}\sum_{r\ge t_1-s_u}\left|\mathrm{Cov}(\zeta_0, \zeta_r)\right| \\
&\le (2aB)^2(4aBM)^{u+v-2}\frac{d_0}{1-\exp(-d)}\, v\exp(-d(t_1-s_u)).
\end{aligned}$$

In the second case, if $s_u = t_1$, we have

$$\left|\mathrm{Cov}\left(\prod_{i=1}^{u} Z_{s_i}, \prod_{j=1}^{v} Z_{t_j}\right)\right| \le 2(2aBM)^{u+v} \le v(2aBM)^2(4aBM)^{u+v-2}.$$

In both cases, the following inequality is fulfilled

$$\left|\mathrm{Cov}\left(\prod_{i=1}^{u} Z_{s_i}, \prod_{j=1}^{v} Z_{t_j}\right)\right| \le K^2(4aBM)^{u+v-2} v\exp(-d(t_1-s_u)),$$

where $K = \max\left(2aBM, 2aB\sqrt{\frac{d_0}{1-\exp(-d)}}\right)$.

From Lemma 1, we get:

$$P\left(\left|\sum_{i=1}^{n} Z_i\right| > \frac{n\varepsilon}{2}\right) \le 2\exp\left(-\frac{\varepsilon^2 n^2}{4\left(C_1 n + C_2\left(\frac{n\varepsilon}{2}\right)^{\frac{5}{3}}\right)}\right). \qquad \square$$

Corollary 3. *Under the assumptions of this theorem, the sequence $(\theta_n)_n$ converges completely toward the parameter θ of the procedure (1).*

Proof. The complete convergence follows from the inequality (3). $\square$

Corollary 4. *For a given significance threshold level $\sigma \in \,]0,1[$, we can find a natural integer n_σ such that the parameter θ belongs to the closed interval $[x_{n_\sigma} - \varepsilon,\ x_{n_\sigma} + \varepsilon]$ with a probability greater than or equal to $1-\sigma$.*

Proof. Since

$$\lim_{n\to+\infty} 2\exp\left(-\frac{\varepsilon^2 n}{4\left(C_1 + C_2 n^{\frac{2}{3}}\left(\frac{\varepsilon}{2}\right)^{\frac{5}{3}}\right)}\right) = 0$$

there exists a natural integer n_σ such that

$$n \geq n_\sigma \Rightarrow 2\exp\left(-\frac{\varepsilon^2 n}{4\left(C_1 + C_2 n^{\frac{2}{3}} \left(\frac{\varepsilon}{2}\right)^{\frac{5}{3}}\right)}\right) \leq \sigma$$

which gives

$$P\{|x_n - \theta| \leq \varepsilon\} \geq 1 - \sigma. \qquad \square$$

Remark 5. The results obtained in this paper remain true if the random errors $(\xi_n)_n$ are positively, negatively associated, associated or independent. Because these classes are quasi-associated random variables.

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