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Partial differential equations

# Relative entropy for compressible Navier–Stokes equations with density-dependent viscosities and applications

## *Entropie relative pour les équations de Navier–Stokes compressibles avec viscosités dépendant de la densité*

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A B S T R A C T

Recently A. Vasseur and C. Yu have proved (see A. Vasseur, C. Yu, Existence of global weak solutions for 3D degenerate compressible Navier–Stokes equations, arXiv:1501.06803, 2015) the existence of global entropy-weak solutions to the compressible Navier–Stokes equations with viscosities $\nu(\varrho) = \mu\varrho$ and $\lambda(\varrho) = 0$ and a pressure law under the form $p(\varrho) = a\varrho^\gamma$ with $a > 0$ and $\gamma > 1$ constants. In this note, we propose a non-trivial relative entropy for such system in a periodic box and give some applications. This extends, in some sense, results with constant viscosities recently initiated by E. Feireisl, B.J. Jin and A. Novotny in [*J. Math. Fluid Mech.* (2012)]. We present some mathematical results related to the weak–strong uniqueness, the convergence to a dissipative solution to compressible or incompressible Euler equations. As a by-product, this mathematically justifies the convergence of solutions to a viscous shallow-water system to solutions to the inviscid shallow-water system.



R É S U M É

Récemment, A. Vasseur et C. Yu ont prouvé (voir A. Vasseur, C. Yu, Existence of global weak solutions for 3D degenerate compressible Navier–Stokes equations, arXiv:1501.06803, 2015) l'existence globale de solutions faibles entropiques des équations de Navier–Stokes compressibles avec des viscosités $\nu(\varrho) = \mu\varrho$, $\lambda(\varrho) = 0$ et une pression du type $p(\varrho) = a\varrho^\gamma$, avec $a > 0$ et $\gamma > 1$ deux constantes. Dans cette note, on propose une entropie relative originale pour un tel système, avec cette dépendance des viscosités en la densité, et on donne quelques applications. Ceci étend les résultats avec viscosités constantes initiés par E. Feireisl, B.J. Jin and A. Novotny dans [*J. Math. Fluid Mech.* (2012)]. On présente quelques résultats liés à l'unicité faible–fort, la convergence vers une solution dissipative d'Euler

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compressible. Ceci justifie en particulier la convergence d'un système de Saint-venant avec viscosité vers son analogue non visqueux.


## Version française abrégée

Dans cette note, on prolonge à un cas de viscosités dépendant de la densité certains résultats maintenant connus dans le cas de viscosités constantes. Plus précisément, on montre que toute solution $\kappa$-entropique au sens de [3] satisfait une entropie relative qui permet d'obtenir, par exemple, un résultat d'unicité fort–faible, un résultat de convergence de Navier–Stokes compressible vers Euler compressible. On améliore ici l'entropie relative introduite dans [10], qui demandait l'hypothèse forte d'une viscosité proportionnelle à la pression et qui considérait la dimension un d'espace. Nous nous focaliserons sur le cas $\nu(\varrho) = \mu\rho$ et $\lambda(\varrho) = 0$ (associé aux équations de Saint-Venant), pour lequel un résultat complet d'existence globale de solutions $\kappa$-entropique existe : construction de solutions approchées et stabilité sans terme supplémentaire dans le système final. Plus précisément, A. Vasseur et C. Yu ont obtenu (cf. [14]) l'existence globale de solutions faibles de Navier–Stokes compressible satisfaisant la BD-entropie avec viscosité $\nu(\varrho) = \mu\varrho$ et $\lambda(\varrho) = 0$ : solution qui est $\kappa$-entropique pour tout $\kappa \in (0, 1)$ au sens donné dans [3]. On présente quelques résultats liés à l'unicité faible–fort, la convergence vers une solution dissipative d'Euler compressible. Ceci justifie, en particulier, le lien entre un système de type Saint-Venant avec viscosité et le système de Saint-Venant non visqueux. On mentionne également la convergence vers Euler incompressible. La principale difficulté est la dégénérescence de la viscosité quand la densité s'annule, qui ne permet pas le même contrôle sur la vitesse qu'avec des viscosités constantes. La modulation de la $\kappa$-entropie introduite par le premier auteur et ses collaborateurs demande aussi un traitement précis du terme de pression : voir ligne 3 de l'entropie relative (4) et relation (5). On renvoie le lecteur interessé à [4] pour le détail des preuves et une discussion sur l'extension au cas où les viscosités $\nu$ et $\lambda$ satisfont la relation algébrique $\lambda(\varrho) = 2(\nu'(\varrho)\varrho - \nu(\varrho))$ : relation introduite pour la première fois par le premier auteur et B. Desjardins dans [2].

## 1. Introduction

Since the pioneering work of C. Dafermos and of H.-T. Yau, relative entropy methods have become a popular and crucial tool in the study of asymptotic limits and large-time behavior for nonlinear PDEs. In a recent paper, E. Feireisl, B.J. Jin, A. Novotny (see [7]) have introduced relative entropies, suitable weak solutions and weak–strong uniqueness for the compressible Navier–Stokes equations with constant viscosities. The interested reader is referred to [12,6] and references cited therein. Based on such relative entropies, various papers have been dedicated to singular perturbations, see the interesting book [8], the articles [1,13] for instance. See also the recent interesting work by Th. Gallouët, R. Herbin, D. Maltese, A. Novotny in [9], where relative entropy techniques are developed to obtain error estimates for a numerical approximation of the compressible Navier–Stokes equation with constant viscosities.

Here we focus on an adaptation of the results found in [7,1] and [13] to the case of compressible Navier–Stokes equations with *degenerate viscosities* depending on the density and set in a periodic box. Relative entropy for the one-dimensional compressible Navier–Stokes equations with degenerate density dependent viscosity has been, for instance, recently used by B. Haspot in [10] under the strong assumption that the viscosity function $\nu(\varrho)$ is equal to the pressure law $p(\varrho)$ up to a multiplicative constant. This is due to the form of the modulated term in the relative entropy chosen for the quantity coming from the pressure law. The main objective is to get rid of such an assumption and to extend the result to the multi-dimensional in-space case. For that purpose, we will take advantage of the recent $\kappa$-entropy introduced recently by the first author, B. Desjardins and E. Zatorska in [3]. We introduce a relative entropy based on a new modulated quantity for the term involving the pressure, which allows us to relax the relation between the viscosity and the pressure required in [10]. By this way we cover the (physical) case $\nu(\varrho) = \mu\varrho$, $\lambda(\varrho) = 0$ and the general pressure $p(\varrho) = a\varrho^\gamma$ with $\gamma > 1$. This corresponds to the recent system for which A. Vasseur and C. Yu [14] have obtained recently global existence of entropy-weak solutions based on the BD-entropy, the Mellet–Vasseur estimates and some original renormalization technics linked to the momentum equations to get rid of extra terms used in previous works (drag and capillary terms). Note that such relative entropy will be used to prove convergence of appropriate schemes for the compressible Navier–Stokes equation with degenerate viscosities in the forthcoming paper [5]. Finally, we present some mathematical results related to the weak–strong uniqueness, the convergence to a dissipative solution to compressible Euler equations. As a by-product, this mathematically justifies the vanishing viscosity limit passage in viscous shallow-water equations systems.

## 2. The degenerate compressible Navier–Stokes equations and the $\kappa$-entropy

Let us recall the compressible Navier–Stokes equations with $\nu(\varrho) = \mu\rho$, $\lambda(\varrho) = 0$:

$$\begin{cases} \partial_t\varrho + \mathrm{div}(\varrho\mathbf{u}) = 0, \\ \partial_t(\varrho\mathbf{u}) + \mathrm{div}(\varrho\mathbf{u}\otimes u) + \nabla p(\varrho) - 2\mu\mathrm{div}(\varrho D(\mathbf{u})) = 0 \end{cases} \tag{1}$$

with $D(\mathbf{u}) = (\nabla\mathbf{v} + (\nabla\mathbf{v})^T)/2$. Recently, in [3], it has been observed that such a compressible Navier–Stokes system may be reformulated through an augmented system. Introducing the intermediate velocity $\mathbf{v} = \mathbf{u} + 2\kappa\mu\nabla\log(\varrho)$, a drift velocity $\mathbf{w} = 2\sqrt{\kappa(1-\kappa)}\mu\nabla\log(\varrho)$ and a mixture coefficient $\kappa$, it reads

$$\begin{cases} \partial_t\varrho + \mathrm{div}(\varrho\left(\mathbf{v} - 2\kappa\mu\nabla\log\varrho\right)) = 0, \\ \partial_t(\varrho\mathbf{v}) + \mathrm{div}(\varrho\mathbf{v}\otimes(\mathbf{v} - 2\kappa\mu\nabla\log\varrho) + \nabla p(\varrho) \\ \quad = \mu\mathrm{div}(2\varrho(1-\kappa)D(\mathbf{v})) + \mu\mathrm{div}(2\kappa\varrho A(\mathbf{v})) - \mu\mathrm{div}\left(2\sqrt{\kappa(1-\kappa)}\varrho\nabla\mathbf{w}\right), \\ \partial_t(\varrho\mathbf{w}) + \mathrm{div}(\varrho\mathbf{w}\otimes(\mathbf{v} - 2\kappa\mu\nabla\log\varrho)) = \mu\mathrm{div}(2\kappa\varrho\nabla\mathbf{w}) - \mu\mathrm{div}(2\sqrt{\kappa(1-\kappa)}\varrho(\nabla\mathbf{v})^T). \end{cases} \tag{2}$$

with $A(\mathbf{v}) = (\nabla\mathbf{v} - (\nabla\mathbf{v})^T)/2$. The associated $\kappa$-entropy reads for all $t \in [0,T]$:

$$\sup_{\tau\in[0,t]}\int_\Omega\Big[\varrho(\frac{|\mathbf{v}|^2}{2} + \frac{|\mathbf{w}|^2}{2}) + F(\varrho)\Big](\tau)\,\mathrm{d}x + 2\mu\int_0^t\int_\Omega\varrho\left(\kappa|A(\mathbf{v})|^2 + |D(\sqrt{1-\kappa}\mathbf{v}) - \nabla(\sqrt{\kappa}\mathbf{w})|^2\right)\mathrm{d}x\,\mathrm{d}s$$
$$+ 2\kappa\mu\int_0^t\int_\Omega\frac{p'(\varrho)}{\varrho}|\nabla\varrho|^2\,\mathrm{d}x\,\mathrm{d}s \le \int_\Omega\Big[\varrho(\frac{|\mathbf{v}|^2}{2} + \frac{|\mathbf{w}|^2}{2}) + F(\varrho)\Big](0) \tag{3}$$

with $sF'(s) - F(s) = p(s)$. This $\kappa$-entropy is obtained by taking the scalar product of the equation satisfied by $\mathbf{v}$ and $\mathbf{w}$ respectively with $\mathbf{v}$ and $\mathbf{w}$, adding the results and using the mass equation. Note that the more general case with $\nu(\varrho)$ arbitrary and $\lambda(\varrho) = 2(\nu'(\varrho)\varrho - \nu(\varrho))$ for the compressible Navier–Stokes equations is covered in [3] (but with extra drag or cold pressure terms for the existence). This allows to define appropriate global solutions named $\kappa$-entropy solutions.

## 3. Relative $\kappa$-entropy for compressible Navier–Stokes equations with degenerate viscosities

Let us consider the relative energy functional, denoted $E(\rho,\mathbf{v},\mathbf{w}|r,V,W)$, defined by

$$E(\rho,\mathbf{v},\mathbf{w}|r,V,W) = \frac{1}{2}\int_\Omega\varrho(|\mathbf{w}-W|^2 + |\mathbf{v}-V|^2)\,\mathrm{d}x + \int_\Omega(F(\varrho) - F(r) - F'(r)(\varrho - r))\,\mathrm{d}x$$

which measures the distance between a $\kappa$-entropic weak solution $(\varrho,\mathbf{v},\mathbf{w})$ to any smooth enough test function $(r,V,W)$. We can prove that any weak solution $(\rho,\mathbf{v},\mathbf{w})$ of the augmented system satisfies the following so-called relative entropy inequality

$$\begin{aligned} & E(\rho,v,w|r,V,W)(\tau) - E(\rho,v,w|r,V,W)(0) \\ & \quad + 2\kappa\mu\int_0^\tau\int_\Omega\varrho[A(\mathbf{v}-V)|^2 + 2\mu\int_0^\tau\int_\Omega\varrho|D(\sqrt{(1-\kappa)}(\mathbf{v}-V) - \sqrt{\kappa}(\mathbf{w}-W))|^2 \\ & \quad + 2\kappa\mu\int_0^\tau\int_\Omega\varrho\Big[p'(\varrho)\nabla\log\varrho - p'(r)\nabla\log r\Big]\cdot\Big[\nabla\log\varrho - \nabla\log r\Big] \\ & \le \int_0^\tau\int_\Omega\varrho\Big(((\mathbf{v} - \sqrt{\frac{\kappa}{(1-\kappa)}}\mathbf{w})\cdot\nabla W)\cdot(W-\mathbf{w}) + ((\mathbf{v} - \sqrt{\frac{\kappa}{(1-\kappa)}}\mathbf{w})\cdot\nabla V)\cdot(V-\mathbf{v})\Big) \\ & \quad + \int_0^\tau\int_\Omega\varrho\Big(\partial_t W\cdot(W-\mathbf{w}) + \partial_t V\cdot(V-\mathbf{v})\Big) \\ & \quad + \int_0^\tau\int_\Omega\partial_t F'(r)(r-\varrho) - \int_0^\tau\int_\Omega\nabla F'(r)\cdot\Big[\varrho(\mathbf{v} - \sqrt{\frac{\kappa}{(1-\kappa)}}\mathbf{w}) - r(V - \sqrt{\frac{\kappa}{(1-\kappa)}}W)\Big] \\ & \quad + \int_0^\tau\int_\Omega(p(r) - p(\varrho))\,\mathrm{div}(V - \sqrt{\frac{\kappa}{(1-\kappa)}}W) - \kappa\int_0^\tau\int_\Omega p'(\varrho)\nabla\varrho\cdot[2\mu\frac{\nabla r}{r} - \frac{1}{\sqrt{(1-\kappa)\kappa}}W] \\ & \quad + 2\mu\int_0^\tau\int_\Omega\varrho\Big(D(\sqrt{(1-\kappa)}V) - \nabla(\sqrt{\kappa}W)\Big):\Big(D(\sqrt{(1-\kappa)}(V-\mathbf{v})) - \nabla(\sqrt{\kappa}(W-\mathbf{w}))\Big) \end{aligned}$$

$$
+ 2\kappa\mu \int_0^\tau \int_\Omega \varrho A(V) : A(V - \mathbf{v}) + 2\kappa\mu \int_0^\tau \int_\Omega \frac{\varrho}{r} p'(r) \nabla r \cdot (\frac{\nabla r}{r} - \frac{\nabla \varrho}{\varrho})
$$

$$
+ 2\sqrt{\kappa(1-\kappa)}\mu \int_0^\tau \int_\Omega \varrho \Big[ A(W) : A(\mathbf{v} - V) - A(\mathbf{w} - W) : A(V) \Big] \tag{4}
$$

for all $\tau \in [0, T]$ and for any pair of test functions

$$
r \in C^1([0,T] \times \overline{\Omega}), \qquad r > 0, \qquad V, W \in C^1([0,T] \times \overline{\Omega}).
$$

By using a density argument, we can of course relax the regularity on the test functions using the regularity of the $\kappa$-entropy solutions as it was done in [7] for the constant viscosities. Remark that here, we do not assume $W$ to be a gradient. The third line is also original compared to [10] and allows us to relax the strong constraint imposed in [10] that the viscosity is proportional to the pressure law and covering now the physically founded case of the shallow-water equations.

## 4. Some applications

Using the relative entropy (4) and the identity

$$
\begin{aligned}
&\varrho[p'(\varrho)\nabla \log \varrho - p'(r)\nabla \log r] \cdot [\nabla \log \varrho - \nabla \log r] \\
&\quad = \varrho p'(\varrho) |\nabla \log \varrho - \nabla \log r|^2 + \nabla[p(\varrho) - p(r) - p'(r)(\varrho - r)] \cdot \nabla \log r \\
&\qquad - \big[\varrho(p'(\varrho) - p'(r)) - p''(r)(\varrho - r) r\big] |\nabla \log r|^2,
\end{aligned} \tag{5}
$$

we can justify several mathematical results. The interested reader is referred to the forthcoming paper [4] for details of the proof. Let us mention two of them that extend to the density-dependent viscosities the well-known results for constant viscosities.

*I) Weak–strong uniqueness.* Let us consider a $\kappa$-entropy solution $(\varrho, \mathbf{u})$ and recall $\mathbf{v} = \mathbf{u} + 2\kappa\mu\nabla\log\varrho$ and $\mathbf{w} = 2\mu\sqrt{\kappa(1-\kappa)}\nabla\log\varrho$. Assume that $(r, W, V)$ satisfies the augmented system with the regularity written before and assume that $W = 2\mu\sqrt{\kappa(1-\kappa)}\nabla\log r$. Then we prove that $(\varrho, \mathbf{v}, \mathbf{w}) = (r, V, W)$, which means the weak–strong uniqueness property: this gives $(\varrho, \mathbf{u}) = (r, U)$ with $U = V - \sqrt{\kappa} W / \sqrt{(1-\kappa)}$. More precisely, we have the following result.

**Theorem 1.** *Let $\Omega$ be a periodic box. Suppose that $p(\varrho) = a\varrho^\gamma$ with $\gamma > 1$. Let $(\varrho, \mathbf{u})$ be a $\kappa$-entropy solution to the compressible Navier–Stokes system (1). Assume that there exists a strong solution $(r, U)$ of the compressible Navier–Stokes equations (1) such that the terms in (4) are defined with $r > 0$ and $r \in L^2(0,T; W^{1,\infty}(\Omega)) \cap L^1(0,T; W^{2,\infty}(\Omega))$. Then we have the weak–strong uniqueness result: $(\varrho, \mathbf{u}) = (r, U)$.*

*II) Inviscid limit: convergence to dissipative solution.* Let us recall the definition of a dissipative solution of compressible Euler equations. Such a concept has been introduced by P.-L. Lions in the incompressible setting: see for instance [11]. The reader is referred to [6] and [1] for the extension to the compressible framework with constant viscosities. Here we deal with an example of density-dependent viscosities with a dissipative solution target. Of course the target could be the local strong solution to the compressible Euler equations similarly to [13].

**Definition.** The pair $(\overline{\varrho}, \overline{u})$ is a dissipative solution to the compressible Euler equations if and only if $(\overline{\varrho}, \overline{u})$ satisfies the relative energy inequality

$$
E(\overline{\varrho}, \overline{u}, 0 | r, U, 0)(t) \le E(\overline{\varrho}, \overline{u}, 0 | r, U, 0)(0) \exp\Big[c_0(r) \int_0^t \|\mathrm{div} U(\tau)\|_{L^\infty(\Omega)} \, \mathrm{d}\tau\Big]
$$

$$
+ \int_0^t \exp\Big[c_0(r) \int_s^t \|\mathrm{div} U(\tau)\|_{L^\infty(\Omega)}\Big] \int_\Omega \varrho E(r, U) \cdot (U - \overline{u}) \, \mathrm{d}x \, \mathrm{d}s
$$

for all smooth test functions $(r, U)$ defined on $[0, T] \times \overline{\Omega})$ so that $r$ is bounded above and below away from zero and $(r, U)$ satisfies

$$
\partial_t r + \mathrm{div}(rU) = 0, \qquad \partial_t U + U \cdot \nabla U + \nabla F'(r) = E(r, U)
$$

for some residual $E(r, U)$. We prove the following result.

**Theorem 2.** *Let $(\varrho_\varepsilon, \mathbf{u}_\varepsilon)$ be any finite $\kappa$-entropy solution to the compressible Navier–Stokes equations (1) in the periodic setting replacing $\mu$ by $\varepsilon$. Then, any weak limit $(\overline{\varrho}, \overline{u})$ of $(\varrho_\varepsilon, \mathbf{u}_\varepsilon)$ in the sense*

$$\varrho_\varepsilon \to \overline{\varrho} \text{ weakly in } L^\infty(0,T;L^\gamma(\Omega)),$$
$$\varrho_\varepsilon|\mathbf{v}_\varepsilon|^2 \to \overline{\varrho}\,|\overline{u}|^2 \text{ weakly in } L^\infty(0,T;L^1(\Omega))$$
$$\varrho_\varepsilon|\mathbf{w}_\varepsilon|^2 \to 0 \text{ weakly in } L^\infty(0,T;L^1(\Omega))$$

*with $\mathbf{v}_\varepsilon = \mathbf{u}_\varepsilon + 2\varepsilon\kappa\nabla\log\varrho_\varepsilon$ and $\mathbf{w}_\varepsilon = 2\varepsilon\sqrt{\kappa(1-\kappa)}\log\varrho_\varepsilon$ as $\varepsilon$ tends to zero, is a dissipative solution to the compressible Euler equations.*

As a by-product, this justifies the limit between a viscous shallow-water system and the inviscid shallow-water system. Using the relative entropy, it is possible to prove the convergence of the viscous shallow-water system to the incompressible Euler equations: low Froude and inviscid limit using the mean velocity plus the oscillating part as target functions (see [6] for the constant viscosity case).

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