C. R. Acad. Sci. Paris, Ser. I 354 (2016) 191–194

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Optimal control

Pointwise second-order necessary conditions for optimal control problems evolved on Riemannian manifolds

Conditions nécessaires ponctuelles du second ordre pour des problèmes de contrôle optimal évolués sur une variété riemannienne

Qing Cui [a], Li Deng [a], Xu Zhang [b]

[a] *School of Mathematics, Southwest Jiaotong University, Chengdu 611756, Sichuan Province, China*
[b] *School of Mathematics, Sichuan University, Chengdu 610064, Sichuan Province, China*

A R T I C L E I N F O

Article history:
Received 29 August 2015
Accepted 23 September 2015
Available online 10 December 2015

Presented by Jean-Michel Coron

A B S T R A C T

In this Note, we study an optimal control problem on a Riemannian manifold. The control set in our problem is assumed to be a general Polish space, and therefore the classical variation technique fails. We obtain a pointwise second-order optimality condition, for which the curvature tensor of the manifold appears explicitly in the second-order dual equation.

R É S U M É

Dans cette Note, nous étudions un problème du contrôle optimal sur une variété riemannienne. Dans ce problème, l'ensemble des contrôles est un espace de Polish général ; ainsi, la technique de variation classique ne s'applique pas ici. On obtient une condition d'optimalité ponctuelle du second ordre, pour laquelle le tenseur de courbure de la variété apparaît explicitement dans l'équation duale du second ordre.

1. Introduction

Let M be a complete simply connected $n(\in \mathbb{N})$ dimensional Riemannian manifold with metric g. Let ∇ be the Levi-Civita connection on M related to g, $\rho(\cdot,\cdot)$ be the distance function on M, T_xM be the tangent space of the manifold M at $x \in M$, T_x^*M be the cotangent space at x, and $\exp_x : T_xM \to M$ be the exponential map at x. Denote by $\langle\cdot,\cdot\rangle$ and $|\cdot|$ respectively the inner product and the norm over T_xM related to g. Also, denote by TM, T^*M and $C^\infty(M)$ the tangent bundle, the cotangent bundle and the set of smooth functions on M, respectively. Let $T > 0$, U be a Polish space, $f : [0,T] \times M \times U \to TM$ and $f^0 : [0,T] \times M \times U \to \mathbb{R}$ be two given maps. Given $y_0 \in M$, we consider the following control system

$$\begin{cases} \dot{y}(t) = f(t, y(t), u(t)), \quad a.e.\, t \in [0,T], \\ y(0) = y_0, \end{cases} \tag{1}$$

E-mail addresses: cuiqing@home.swjtu.edu.cn (Q. Cui), dengli@home.swjtu.edu.cn (L. Deng), zhang_xu@scu.edu.cn (X. Zhang).

http://dx.doi.org/10.1016/j.crma.2015.09.032
1631-073X/

where the control $u(\cdot) \in \mathcal{U}_{ad} \equiv \{u(\cdot) : [0,T] \to U; \quad u(\cdot) \text{ is measurable}\}$. The cost functional associated with (1) is $J(u(\cdot)) = \int_0^T f^0(t, y(t), u(t))\mathrm{d}t$. Our problem is to find a $\bar{u}(\cdot) \in \mathcal{U}_{ad}$ such that

$$J(\bar{u}(\cdot)) = \min_{u(\cdot)\in\mathcal{U}_{ad}} J(u(\cdot)). \tag{2}$$

We call such a $\bar{u}(\cdot)$ an optimal control, the corresponding solution $\bar{y}(\cdot)$ to (1) an optimal trajectory, and $(\bar{y}(\cdot), \bar{u}(\cdot))$ an optimal pair.

The above problem can be viewed as an optimal control problem with state constrained on a submanifold of the Euclidean space. One of the central topics in control theory is to establish necessary conditions for optimal controls. As that in calculus, one can derive the first-order necessary condition for optimal controls, as done in the classical monograph [12], even for some situation of state constrains. Clearly, for some optimal control problems, it may happen that the first-order necessary conditions turn out to be trivial. In this case, the first-order necessary condition cannot provide enough information for the theoretical analysis and numerical computation, and therefore one needs to study the second- (or even higher-) order necessary conditions for optimal controls.

In this note, we aim to give a second-order necessary condition for the optimal control problem (1)–(2). There are many references devoted to the second-order necessary conditions for optimal control problems in Euclidean spaces, such as [2,4,6–11] and the references therein. For the control system whose state is constrained to a manifold, there are also some literatures (e.g., [1,3,13] and the references therein) devoted to the second-order necessary conditions for optimal controls when the control set is an open subset of some manifold (hence the classical variation technique can be employed). Nevertheless, to the best of our knowledge, there is no publication addressing the second-order optimality condition for the above control problem with a general control set U for which the classical variation technique fails.

2. The main result

We begin with some notations. Denote by $i(x)$ the injectivity radius of a point $x \in M$. For any two points $x_1, x_2 \in M$ with $\rho(x_1, x_2) < \min\{i(x_1), i(x_2)\}$, there exists a unique geodesic starting from x_1 and ending at x_2. Denote by $L_{x_1x_2}$ the parallel translation along this geodesic. Denote by R the curvature tensor of the Riemannian manifold M, and write

$$R(X,Y,Z,W) = \langle R(X,Y)Z, W\rangle, \quad \forall\, X,Y,Z,W \in TM.$$

Denote by $\mathcal{E}$ the tensor contraction map satisfying

$$\mathcal{E}(v_1 \otimes v_2 \otimes v_3 \otimes v_4) = \begin{cases} v_2(v_3)v_1 \otimes v_4, & \text{if } v_1, v_2, v_4 \in T^*M,\ v_3 \in TM; \\ v_3(v_2)v_1 \otimes v_4, & \text{if } v_1, v_3, v_4 \in T^*M,\ v_2 \in TM. \end{cases}$$

We need the following assumptions:

(C1) *The maps $f\ (= f(t,x,u)) : [0,T] \times M \times U \to TM$ and $f^0\ (= f^0(t,x,u)) : [0,T] \times M \times U \to \mathbb{R}$ are measurable in t, continuous in (x,u), and C^1 in x. Moreover, there exists a constant $L > 1$ such that,*

$$|f^0(s,x_1,u) - f^0(s,x_2,u)| \le L\rho(x_1,x_2), \qquad |L_{x_1x_2} f(s,x_1,u) - f(s,x_2,u)| \le L\rho(x_1,x_2),$$
$$|f^0(s,x_0,u)| \le L, \qquad |f(s,x_0,u)| \le L$$

for all $s \in [0,T]$, $u \in U$, and $x_1, x_2 \in M$ with $\rho(x_1,x_2) \le \min\{i(x_1), i(x_2)\}$, where $x_0 \in M$ is fixed.

(C2) *The maps f and f^0 are C^2 in their second argument. Furthermore, it holds that*

$$|\nabla_x f(t,x_1,u) - L_{x_2x_1}\nabla_x f(t,x_2,u)| \le L\rho(x_1,x_2), \quad |\nabla_x f^0(t,x_1,u) - L_{x_2x_1}\nabla_x f^0(t,x_2,u)| \le L\rho(x_1,x_2),$$

for all $x_1, x_2 \in M$ with $\rho(x_1,x_2) < \min\{i(x_1), i(x_2)\}$ and $(t,u) \in [0,T] \times U$, where $\nabla_x f(t,\cdot,u)$ and $\nabla_x f^0(t,\cdot,u)$ are respectively the covariant derivatives of $f(t,\cdot,u)$ and $f^0(t,\cdot,u)$ w.r.t. the state variable.

In the sequel, we fix an optimal pair $(\bar{y}(\cdot), \bar{u}(\cdot))$ for the optimal control problem (1)–(2). Let $\psi(t) \in T^*_{\bar{y}(t)}M$ be the solution to the following first-order dual equation:

$$\begin{cases} \nabla_{\dot{\bar{y}}(t)}\psi = -\nabla_x f(t,\bar{y}(t),\bar{u}(t))(\psi(t),\cdot) + d_x f^0(t,\bar{y}(t),\bar{u}(t)), & t \in [0,T), \\ \psi(T) = 0, \end{cases} \tag{3}$$

where $d_x f^0$ denotes the exterior derivative of f^0, and $\nabla_x f(t,\bar{y}(t),\bar{u}(t))(\psi(t),\cdot)$ is a tensor satisfying

$$\nabla_x f(t,\bar{y}(t),\bar{u}(t))(\psi(t),\cdot)(X) \equiv \nabla_x f(t,\bar{y}(t),\bar{u}(t))(\psi(t),X) = \nabla_{X(\bar{y}(t))} f(t,\cdot,\bar{u}(t))(\psi(t)), \quad \forall\, X \in TM.$$

Moreover, let $w(t)$ be a tensor of order 2 at $\bar{y}(t)$, and satisfy the following second-order dual equation:

$$\begin{cases} \nabla_{\dot{\bar{y}}(t)} w + \mathcal{E}\Big((\nabla_x f)^\top(t,\bar{y}(t),\bar{u}(t)) \otimes w(t)\Big) + \mathcal{E}\Big(w(t) \otimes \nabla_x f(t,\bar{y}(t),\bar{u}(t))\Big) \\ \qquad\qquad + \nabla_x^2 H(t,\bar{y}(t),\psi(t),\bar{u}(t)) - R\Big(\tilde{\psi}(t),\cdot, f(t,\bar{y}(t),\bar{u}(t)),\cdot\Big) = 0, \quad t \in [0,T), \\ w(T) = 0, \end{cases} \tag{4}$$

where $\nabla_x f^\top(t,\bar{y}(t),\bar{u}(t))$ is the transpose of $\nabla_x f(t,\bar{y}(t),\bar{u}(t))$, given by

$$\nabla_x f^\top(t,\bar{y}(t),\bar{u}(t))(X,\eta) \equiv \nabla_x f(t,\bar{y}(t),\bar{u}(t))(\eta,X), \quad \forall \eta \in T^*M,\ X \in TM;$$

the Hamiltonian function $H : [0,T] \times T^*M \times U \to \mathbb{R}$ is defined by

$$H(t,x,p,u) \equiv p(f(t,x,u)) - f^0(t,x,u), \quad \forall\,(t,x,p,u) \in [0,T] \times T^*M \times U; \tag{5}$$

$\tilde{\psi}(t)$ is the dual vector of $\psi(t)$, given by

$$\langle \tilde{\psi}(t), v\rangle = \psi(t)(v), \quad \forall\, v \in T_{\bar{y}(t)}M;$$

and $\nabla_x^2 H(t,\bar{y}(t),\psi(t),\bar{u}(t))$ and $R(\tilde{\psi}(t),\cdot, f(t,\bar{y}(t),\bar{u}(t)),\cdot)$ are tensors given respectively by

$$\begin{aligned} &\nabla_x^2 H(t,\bar{y}(t),\psi(t),\bar{u}(t))(X,Y) \equiv \nabla_x^2 f(t,\bar{y}(t),\bar{u}(t))(\psi(t),X,Y) - \nabla_x^2 f^0(t,\bar{y}(t),\bar{u}(t))(X,Y) \\ &\text{and} \quad R(\tilde{\psi}(t),\cdot, f(t,\bar{y}(t),\bar{u}(t)),\cdot)(X,Y) \equiv R(\tilde{\psi}(t),X, f(t,\bar{y}(t),\bar{u}(t)),Y), \qquad \forall\, X,Y \in TM. \end{aligned}$$

Put

$$\overline{U}(t) \equiv \{u \in U;\ \ H(t,\bar{y}(t),\psi(t),\bar{u}(t)) = H(t,\bar{y}(t),\psi(t),u)\}.$$

We have the following second-order necessary condition for the optimal pair $(\bar{y}(\cdot),\bar{u}(\cdot))$:

Theorem 2.1. *Assume that* $(C1)$ *and* $(C2)$ *hold. Then, for a.e.* $t \in [0,T]$ *and any* $v \in \overline{U}(t)$,

$$\begin{aligned} &\frac{1}{2}\Big(w(t) + w^\top(t)\Big)\Big(f(t,\bar{y}(t),\bar{u}(t)) - f(t,\bar{y}(t),v),\ f(t,\bar{y}(t),\bar{u}(t)) - f(t,\bar{y}(t),v)\Big) \\ &\quad + \Big(\nabla_x H(t,\bar{y}(t),\psi(t),\bar{u}(t)) - \nabla_x H(t,\bar{y}(t),\psi(t),v)\Big)\Big(f(t,\bar{y}(t),\bar{u}(t)) - f(t,\bar{y}(t),v)\Big) \le 0, \end{aligned} \tag{6}$$

where the 2*-form* $w^\top(t)$ *is the transpose of* $w(t)$*, given by* $w^\top(t)(X,Y) \equiv w(t)(Y,X)$ *for all* $X,Y \in TM$*; and* $\nabla_x H(t,\bar{y}(t),\psi(t),u)$ *with* $(t,u) \in [0,T] \times U$ *is a tensor satisfying*

$$\nabla_x H(t,\bar{y}(t),\psi(t),u)(X) \equiv \nabla_x f(t,\bar{y}(t),u)(\psi(t),X) - \langle \nabla_x f^0(t,\bar{y}(t),u), X(\bar{y}(t))\rangle, \qquad \forall\, X \in TM.$$

We refer to [5] for a detailed proof of Theorem 2.1 and other results in this context.

Our Theorem 2.1 is different from [1, Theorem 20.6, p. 300 and Proposition 20.11, p. 310] in two aspects. First, the control set in the present work is a Polish space (recall that all nonempty closed sets and open sets in $\mathbb{R}^m$ ($m \in \mathbb{N}$) are polish spaces), while the control set in [1] is an open subset of a manifold. Second, our second-order necessary condition depends on the curvature tensor (see equation (4)), which does not appear explicitly in [1]. Also, it should be mentioned that, compared to the corresponding result in the Euclidean space [11], the presence of the curvature tensor in the second-order necessary condition shows the very difference between a curved space and the flat space from the viewpoint of optimal control.

Acknowledgements

This work is supported by the National Science Foundation of China under grants 11401491, 11221101 and 11231007, and Fundamental Research Funds for the Central Universities under grant 2682014CX052.

References

[1] A.A. Agrachev, Y.L. Sachkov, Control Theory from the Geometric Viewpoint, Encyclopaedia Math. Sci., vol. 87, Springer-Verlag, Berlin, 2004.
[2] D.J. Bell, D.H. Jacobson, Singular Optimal Control Problems, Academic Press, London, New York, 1975.
[3] B. Bonnard, J.B. Caillau, E. Trélat, Second order optimality conditions in the smooth case and applications in optimal control, ESAIM Control Optim. Calc. Var. 13 (2007) 207–236.
[4] D.J. Clements, B.D.O. Anderson, Singular Optimal Control: The Linear-Quadratic Problem, Springer-Verlag, Berlin, New York, 1978.
[5] Q. Cui, L. Deng, X. Zhang, Second order necessary conditions for optimal control problems on Riemannian manifolds, arXiv:1508.06175 [math.OC].
[6] H. Frankowska, D. Tonon, Pointwise second-order necessary optimality conditions for the Mayer problem with control constraints, SIAM J. Control Optim. 51 (2013) 3814–3843.
[7] R.F. Gabasov, F.M. Kirillova, High order necessary conditions for optimality, SIAM J. Control 10 (1972) 127–168.
[8] B.S. Goh, Necessary conditions for singular extremals involving multiple control variables, SIAM J. Control 4 (1966) 716–731.
[9] H.-W. Knobloch, Higher Order Necessary Conditions in Optimal Control Theory, Springer-Verlag, Berlin, New York, 1981.

[10] A.J. Krener, The high order maximal principle and its application to singular extremals, SIAM J. Control Optim. 15 (1977) 256–293.
[11] H. Lou, Second-order necessary/sufficient conditions for optimal control problems in the absence of linear structure, Discrete Contin. Dyn. Syst., Ser. B 14 (2010) 1445–1464.
[12] L.S. Pontryagin, V.G. Boltyanskii, R.V. Gamkrelidze, E.F. Mischenko, Mathematical Theory of Optimal Processes, Wiley, New York, 1962.
[13] H. Schättler, U. Ledzewicz, Geometric Optimal Control, Theory, Methods and Examples, Interdiscip. Appl. Math., vol. 38, Springer, New York, 2012.

Further reading

[14] J.F. Bonnans, A. Hermant, Second-order analysis for optimal control problems with pure state constraints and mixed control-state constraints, Ann. Inst. Henri Poincaré, Anal. Non Linéaire 26 (2009) 561–598.
[15] D. Hoehener, Variational approach to second-order optimality conditions for control problems with pure state constraints, SIAM J. Control Optim. 50 (2012) 1139–1173.
[16] Z. Páles, V. Zeidan, Optimal control problems with set-valued control and state constraints, SIAM J. Optim. 14 (2003) 334–358.