C. R. Acad. Sci. Paris, Ser. I 353 (2015) 1067–1073

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Number theory/Ordinary differential equations

On the mock-theta behavior of Appell–Lerch series

Sur le comportement mock thêta de séries d'Appell–Lerch

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ARTICLE INFO

Article history:
Received 2 April 2015
Accepted 29 September 2015
Available online 31 October 2015

Presented by Jean-Pierre Ramis

ABSTRACT

The goal of this paper is to find one natural way to write the first order Appell–Lerch series in terms of two functions whose asymptotic behavior becomes simple. It is shown that such writing exists, using only theta-like functions and functions having a Gevrey asymptotic expansion. In order of simplify the presentation, we introduce three types of theta-like functions that will be called theta-type, false theta-type and mock theta-type.
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RÉSUMÉ

Le but de cette Note est de trouver une manière naturelle d'écrire chaque série d'Appell–Lerch du premier ordre au moyen de deux fonctions dont le comportement asymptotique devient plus simple. On démontre qu'une telle écriture existe, avec seulement des fonctions de type thêta et celles qui ont un développement asymptotique Gevrey. Afin de faciliter l'exposé, nous introduisons trois types de fonctions du genre thêta, qui seront appelés type thêta, type faux thêta et type mock thêta.
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Version française abrégée

Inspirés, d'une part, par l'article historique [16] de G.N. Watson et surtout son commentaire à la page 78 de celui-ci et, de l'autre, par le comportement asymptotique en chaque racine unité et en zéro des fonctions classiques thêta de Jacobi, nous proposons la définition suivante.

Définition. Soit $\zeta = \mathrm{e}^{2\pi \mathrm{i} r}$, $r \in \mathbb{Q}$ et soit $q = \mathrm{e}^{2\pi \mathrm{i}\tau}$, avec τ dans le demi-plan supérieur $\Im\tau > 0$. Soit f une fonction analytique dans un sector V de sommet en ζ, symétrique par rapport au rayon $O\zeta$ et situé intégralement à l'intérieur du disque unité.

(1) On dit que f est une fonction de type thêta pour $q \to \zeta$ dans V si l'on peut trouver un quadruplet $(\nu, \lambda, I, \gamma)$, composé de $(\nu, \lambda) \in \mathbb{Q} \times \mathbb{R}$, d'une suite croissante de nombres réels I et d'une application $\gamma : I \to \mathbb{C}$, tel que, pour tout entier positif ou nul N, on ait :

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http://dx.doi.org/10.1016/j.crma.2015.09.028
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$$f(q) = \left(\frac{\mathrm{i}}{\hat\tau}\right)^{\nu} e(\lambda\hat\tau)\Big(\sum_{k\in I\cap]-\infty,N[} \gamma(k)\, q_1^k + o(q_1^N)\Big)$$

lorsque $\hat\tau = \tau - r \to 0$, avec $q \in V$ et $q_1 = e(-\frac{1}{\hat\tau})$.
Lorsque ζ est remplacé par zéro, on écrit $r = \mathrm{i}\infty$, $\hat\tau = 1/\tau$, et V sera un secteur de sommet au point à l'origine du plan et symétrique par rapport au demi-axe réel positif ;

(2) f sera dite de type thêta si elle est de type thêta lorsque q s'approche d'une racine unité quelconque ζ ou zéro dans un certain secteur V ;
(3) f sera dite de type presque thêta pour $q \to \zeta$ dans V s'il existe une fonction f_ζ ayant un développement asymptotique Gevrey en ζ dans V telle que $f - f_\zeta$ est de type thêta ;
(4) f sera dite de type faux thêta si f n'est pas de type thêta et qu'il existe une fonction ϑ de type thêta partout, y compris en zéro, telle que la fonction $f - \vartheta$ admet un développement asymptotique Gevrey en chaque racine unité et aussi en zéro et que ses limites aux racines unités constituent un ensemble borné ;
(5) f sera dite de type mock thêta si f est de type presque thêta partout, y compris en zéro, mais tout en n'étant, ni de type thêta, ni de type faux thêta.

L'objet de cette Note est de montrer comment comprendre, d'un point de vue de la théorie analytique des équations aux q-différences [13], l'énoncé ci-dessous.

Théorème. *Si R_1 et θ sont des fonctions définies respectivement par (1) et (4), alors pour tout couple $(z, w) = (a + b\tau, c + d\tau)$ avec $a, c \in (-\frac12, \frac12]$ et $b, d \in [-\frac12, \frac12)$, la fonction $\frac{R_1(-z+w-\frac{\tau}{2}+\frac12, w\,|\,\tau)}{\theta(-z+w+\frac{\tau}{2}\,|\,\tau)}$ est de type mock thêta, sauf si z vaut l'une des valeurs $\frac12$, $\frac12 - \frac12\tau$ et $-\frac12\tau$. Dans ces derniers cas, la fonction est, soit constante, soit de type faux thêta.*

La série R_1 évoquée dans le théorème appartient à la famille des séries d'Appell–Lerch, étroitement liée au monde des fonctions mock-thêta de Ramanujan. Elle sera interprétée, à un θ-facteur près, à la somme d'un q-analogue de la série d'Euler le long d'un chemin spiral, ce qui nous conduira naturellement à une relation du type modulaire utilisant une intégrale gaussienne, dite aussi de type Mordell. Un point clef consistera à établir l'asymptotique de cette dernière intégrale en la valeur unité (Lemme 3.1), les transformées modulaires fondées sur les fractions continues permettant de ramener chaque racine unité en celle-ci. Mentionnons enfin que toutes les séries d'Appell–Lerch peuvent se décomposer en un nombre fini de séries d'Appell–Lerch considérées dans la présente Note.

1. Introduction

The Ramanujan's mock-theta functions have an ultimate link with Appell–Lerch series. For that, one can see the short papers [11] and [6] of K. Ono and W. Duke in the Notices of the AMS or, more extensively, the references such as [16,18,7,22], etc. See also [4] and [8] for more recent results. The goal of our paper is to study the asymptotic behavior of these series near every unity root, in the hope of contributing later to a best possible knowledge of the world of the Ramanujan's mock-theta functions.

We shall give some general ideas on our approaches. Let R_1 to be the following Appell–Lerch series [9]:

$$R_1(z, w\,|\,\tau) = \sum_{n=-\infty}^{\infty} \frac{q^{n^2/2}\, \mathrm{e}^{2nz\pi\mathrm{i}}}{1 - q^n\, \mathrm{e}^{2w\pi\mathrm{i}}} = \sum_{n=-\infty}^{\infty} \frac{e(\frac12 n^2\tau + nz)}{1 - e(n\tau + w)}\,. \tag{1}$$

Here, $z \in \mathbb{C}$, $w \in \mathbb{C} \setminus (\mathbb{Z} \oplus \tau\mathbb{Z})$, $\tau \in \mathcal{H}$ and $e(.)$ is defined by the relation $e(\alpha) = \mathrm{e}^{2\pi\mathrm{i}\alpha}$ for all $\alpha \in \mathbb{C}$. By direct computation, one finds that

$$R_1(z, w\,|\,\tau) - e(w)\, R_1(z + \tau, w\,|\,\tau) = \sum_{n=-\infty}^{\infty} e(\frac12 n^2\tau + nz) = \vartheta_3(z\,|\,\tau)\,. \tag{2}$$

Thus, $R_1(z, w\,|\,\tau)$ is a *special* solution to some functional equations using the difference operator $z \mapsto z + \tau$. Letting $x = e(z)$ transforms the difference operator $z \mapsto z + \tau$ into the q-difference operator $x \mapsto qx$, and (2) will be transformed into a non-Fuchsian linear q-difference equation. So, apart from a factor of theta function, R_1 will be exactly the sum-function along a spiral for the q-Euler series $\hat E(x; q)$ ([14,20]):

$$\hat E(x; q) = \sum_{n=0}^{\infty} q^{-n(n-1)/2}\, (-x)^n\,. \tag{3}$$

Moreover, with this same divergent series is associated one other family of sum-functions by means of Gaussian kernel [19], which contains the well-known Mordell integrals [10]; see also [1]. Comparing different sum-functions yields exponentially small functions, that are in fact of modular-like type. So, modular-like transforms, exponential-scales changes or Stokes phenomena may come together. This implies the role that the singularities analysis of q-difference equation may play inside the theory of theta-like functions.

Beside, G.N. Watson has written several papers about Ramanujan's mock-theta functions, and one of the most popular may be [16]. At p. 78 of this paper, he said: *It can be proved that these expansions possess the property that (for α complex) the error due to stopping at any term never exceeds in absolute value the first term neglected; in addition, for α positive, the error is of the same sign as that term**. The footnote here is the following: *This property... It is the fact that these expansions are asymptotic (and not terminating series) which shows that mock theta-functions are of a more complex character than ordinary theta-functions.*

This quotation shows that G.N. Watson knew well the natural role played by the asymptotic expansions for the mock-theta functions of Ramanujan. In the following, we will make use of the theory of Gevery asymptotic expansions, whose origin can go back to G.N. Watson [15]. In fact, the theory of Gevrey asymptotic expansion is an exponential-type asymptotical analysis, and such an approach provides a framework around which the exponentially smallness appears in a natural fashion; see [3] or [12].

Organization of the paper The rest will be divided into two sections. In Section 2, we propose some classes of functions that have a behavior such as that of a θ-function at any unit root. In Section 3, we consider the asymptotic behavior of the Mordell integrals and Appell–Lerch series; Lemma 3.1 will play a central role in this paper.

2. Notations, definitions and main result

Let $\mathbb{D}$ be the open unit disc $|q|<1$, $\mathbb{D}^*=\mathbb{D}\setminus\{0\}$, and let $\mathcal{H}$ to denote the Poincaré's half-plane $\Im z>0$. By the map $e:\tau\mapsto e(\tau)=\mathrm{e}^{2\pi\mathrm{i}\tau}$, the analytic space $\mathcal{H}$ is identified with the universal covering of $\mathbb{D}^*$, and $\mathbb{U}=e(\mathbb{Q})\subset\partial\mathbb{D}$, where $\mathbb{U}$ denotes the set of all unity roots. By convention, $e(\alpha)=0$ when $\alpha=\mathrm{i}\infty$.

2.1. Modular group, radial convergence and exponential smallness

Given $a\in\partial\mathbb{D}^*$, by *radially symmetric sector at a inside* $\mathbb{D}^*$ we mean any sector $V_a(d,r)$ defined for $d\in(0,\pi)$ and $r\in(0,1)$ in the following way: if $a\neq 0$, $V_a(d,r)=\{q\in\mathbb{D}^*:|\arg(1-\frac{q}{a})|<\frac{d}{2},\ |q-a|<r\}$; otherwise, one writes $V_0(d,r)=\{q\in\mathbb{D}^*:|\arg q|<\frac{d}{2},\ |q|<r\}$. We shall say that q *almost radially tends towards* $a\in\partial\mathbb{D}^*$ and we write $q\xrightarrow{a.r.}a$, if $q\to a$ within some radially symmetric sector at a inside $\mathbb{D}^*$.

Let $\alpha\in\partial\mathcal{H}\cup\{i\infty\}$, with $a=e(\alpha)$. A *vertically symmetric sector at α in $\mathcal{H}$* is, by definition, the reciprocal image of some $V_a(d,r)$ by the map $\tau\mapsto e(\tau)$. We shall say that τ *almost vertically tends towards* α and we write $\tau\xrightarrow{a.v.}\alpha$, if $\tau\to\alpha$ inside some vertically symmetric sector at α in $\mathcal{H}$.

The modular group $SL(2;\mathbb{Z})$ acts naturally on $\mathcal{H}$ by the linear fractional transformation $\tau\mapsto M\tau$ for all $M\in S(2;\mathbb{Z})$. This action is compatible with the above-introduced convergence notions. Namely, let $a\in\partial\mathbb{D}^*$, $\alpha\in\partial\mathcal{H}\cup\{i\infty\}$, and assume that $a=e(\alpha)$. Then the following conditions are equivalent:

(i) $q\xrightarrow{a.r.}a$ or $\tau\xrightarrow{a.v.}\alpha$;
(ii) $M\tau\xrightarrow{a.v.}M\alpha$ for any $M\in SL(2;\mathbb{Z})$.

Let $\mathcal{A}_a^r$ and $\mathcal{A}_\alpha^v$ to denote respectively the sheaves of germs of analytic functions in radially symmetric sector $V_a(d,r)$ and in vertically symmetric sector $V_\alpha(\delta\,|\,\rho)$, with $\mathcal{A}_a^r\ni f\mapsto\check{f}=f\circ e\in\mathcal{A}_\alpha^v$. The local coordinate $\tau-\alpha$ equals $1/\tau$ if $\alpha=\mathrm{i}\infty$.

2.2. Definition of (almost) theta-type functions

Given $a\in\partial\mathbb{D}\cup\{0\}$, we let $\mathfrak{G}_a^r$ to be the space of $f\in\mathcal{A}_a^r$ that has a Gevrey asymptotic expansion as $q\xrightarrow{a.r.}a$, this implies particularly that $f(q)=O(1)$ as $q\xrightarrow{a.r.}a$. Using the map $f\mapsto\check{f}$, one defines the sub-space $\mathfrak{G}_\alpha^v$ inside $\mathcal{A}_\alpha^v$. If no confusion is possible, we will drop the upper indices r and v and simply write $\mathfrak{G}_a$ and $\mathfrak{G}_\alpha$ there.

For $z\in\mathbb{C}$ and $\tau\in\mathcal{H}$, we let $\theta(z\,|\,\tau)$ to be the following Jacobi theta-function:

$$\theta(z\,|\,\tau):=\sum_{n\in\mathbb{Z}}e\Big(\frac{n(n-1)}{2}\tau+nz\Big). \tag{4}$$

Let $(a,b)\in\mathbb{R}^2$ and $f(q)=\theta(a+b\tau\,|\,\tau)$. The classical θ-modular formula allows one to obtain the asymptotic behavior for q near any unity root, and this leads us to the following

Définition 2.1. Let $q=e(\tau)$, $\tau\in\mathcal{H}$, $a=e(\alpha)\in\mathbb{U}\cup\{0\}$, and let $f\in\mathcal{A}_a^r$.

(i) One says that $f(q)$ is of **theta-type as** $q\xrightarrow{a.r.}a$ and one writes $f\in\mathfrak{T}_a$, if for all $N\in\mathbb{Z}_{\geq 0}$, there exist $(\upsilon,\lambda)\in\mathbb{Q}\times\mathbb{R}$, a finite set $I=I_{a,N}\subset\mathbb{R}$ and a $\mathbb{C}^*$-valued map $\gamma=\gamma_{a,N}$ on I such that the following relation holds for $\tau\xrightarrow{a.v.}\alpha$:

$$f(q) = \left(\frac{i}{\hat\tau}\right)^{\upsilon} e(\lambda\hat\tau)\Big(\sum_{k\in I}\gamma(k)\,q_1^k + o(q_1^N)\Big), \tag{5}$$

where $\hat\tau = \tau - \alpha$ and $q_1 = e(-\frac{1}{\hat\tau})$.

(ii) One says that f is of **almost theta-type as** $q \xrightarrow{a.r.} a$ and one writes $f \in \tilde{\mathfrak{T}}_a$, if there exists $\vartheta \in \mathfrak{T}_a$ such that $f - \vartheta \in \mathfrak{G}_a$.

(iii) One says that f is a **theta-type function** and one writes $f \in \mathfrak{T}$, if $f \in \mathfrak{T}_\zeta$ for all $\zeta \in \mathbb{U}\cup\{0\}$.

(iv) One says that f is a **false theta-type function** and one writes $f \in \mathfrak{F}$, if $f \notin \mathfrak{T}$ and there exists $\vartheta \in \mathfrak{T}$ such that $f(q) - \vartheta(q) \in \mathfrak{G}_\zeta$ for all $\zeta \in \mathbb{U}\cup\{0\}$ and that, furthermore,

$$\sup_{\zeta\in\mathbb{U}}\Big|\lim_{q\xrightarrow{a.v.}\zeta}(f(q) - \vartheta(q))\Big| < \infty. \tag{6}$$

(v) One says that f is a **mock theta-type function** and one writes $f \in \mathfrak{M}$, if $f \in \tilde{\mathfrak{T}}_\zeta$ for all $\zeta = \mathbb{U}\cup\{0\}$ and, moreover, f is not a false theta-type function.

The uniqueness of the theta-type part of an almost theta-type function is specified by the first assertion of the following

Remark 1.

(i) Given f as in (5), if $I_{a,N} \neq \emptyset$ for some non-negative integer N, then the pair (υ, λ) is uniquely determined, independently of N. Moreover, $I_{a,N} \subset I_{a,M}$ and $\gamma_{a,M}\big|_{I_{a,N}} = \gamma_{a,N}$ for all $M > N$.

(ii) For every $f \in \mathfrak{G}_a \cap \mathfrak{T}_a$, one can find $n \in \mathbb{Z}_{\geq 0}$, $c \in \mathbb{C}$, $\lambda \in \mathbb{C}$ and $\kappa > 0$ such that $f(q) = \hat\tau^n e(\lambda\hat\tau)\big(c + o(\mathrm{e}^{-\kappa/|\hat\tau|})\big)$ for $\hat\tau \xrightarrow{a.v.} 0$.

2.3. Some commentaries on (almost) theta-type functions

In [7, p. 98–99], are given the definition of a *mock θ-function* and that of a *strong mock θ-function*; see also [2]. To avoid the confusion, we shall make use of *theta-type* instead of *theta*.

Lemma 2.1.

(i) *Any theta-type function that is analytic at any point $\tau = \tau_0 \in \mathbb{Q} \subset \mathbb{C}$ equals necessarily to $Ce(\lambda\tau)$ for suitable $C \in \mathbb{C}^*$ and $\lambda \in \mathbb{R}$.*

(ii) *The sets $\mathfrak{T}_a$ and $\mathfrak{T}$ constitute multiplicative groups. Moreover, $\mathfrak{T}$ is stable by the ramification operator $q \mapsto q^\nu$ for all $\nu \in \mathbb{Q}_{>0}$.*

(iii) *The four classical Jacobi's theta-functions $\vartheta_1(z\,|\,\tau), \ldots, \vartheta_4(z\,|\,\tau)$ belong to $\mathfrak{T}\cup\{0\}$ for all $z \in \mathbb{R}\oplus\tau\mathbb{R}$.*

For the definition of the ϑ-functions, we are referred to [17, p. 464 and p. 487], where πz and $\pi\tau$ need to be read as z and τ, respectively. See also (2) for ϑ_3.

One of the simple false theta-type functions may be $f(q) = 1 + e(\tau)$. If $\lambda \in \mathbb{R}_{<0}$ and $f(q) = 1 + \tau\,e(\lambda\tau)$ or $f(q) = \tau + e(-\lambda\tau)$, then $f \in \mathfrak{M}$, so f belongs to neither $\mathfrak{T}$ nor $\mathfrak{F}$. More surprisingly, both $e(\frac{1}{\tau})$ and $e(\tau + \frac{1}{\tau})$ really belong to $\mathfrak{M}$, contrarily to what happens for the theta-type function $e(\tau)$.

2.4. Main result

In what follows, we write $\Omega = (-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2})$ and $\Omega_\tau = \{a + b\tau : (a,b) \in \Omega\}$, where $\tau \in \mathcal{H}$. The goal of this paper is to obtain the following

Theorem 2.2. *Let $z, w \in \Omega_\tau$. Assume that $w \neq 0$ and $w \neq z$, and consider*

$$f(q) = \frac{R_1(-z + w - \frac{\tau}{2} + \frac{1}{2}, w\,|\,\tau)}{\theta(-z + w + \frac{\tau}{2}\,|\,\tau)}, \tag{7}$$

where $R_1(z, w\,|\,\tau)$ is defined by (1). Then f is a mock theta-type function except the following cases:

(i) *$z \in \{\frac{1}{2}, \frac{1}{2} - \frac{\tau}{2}, -\frac{\tau}{2}\}$ and $w \in \{\frac{1}{2}, \frac{1}{2} - \frac{\tau}{2}, -\frac{\tau}{2}\}$, in which case f is a constant function;*

(ii) *$z \in \{\frac{1}{2}, \frac{1}{2} - \frac{\tau}{2}, -\frac{\tau}{2}\}$ and $w \notin \{\frac{1}{2}, \frac{1}{2} - \frac{\tau}{2}, -\frac{\tau}{2}\}$, in which case f is a false theta-type function.*

Very clearly, the above function $f(q)$ admits all unit roots as singularities. This gives the following

Corollary 2.3. *The function $f(q)$ of Theorem 2.2 is a strong mock θ-function in the sense of [2].*

3. Mordell integrals and Appell–Lerch series

From the analytic classification viewpoint of differential and q-difference equations, the series $\hat{E}(x;q)$ given in (3) is a q-analog of the Euler series $\sum_{n\ge 0} n!\,(-x)^n$. It satisfies the q-difference equation

$$y(qx) + qx\,y(x) = 1\,, \tag{8}$$

that admits $x = 0$ as non-Fuchsian singular point; see [13]. As the moment problem of the sequence $(q^{-n^2/2})_n$ is undetermined, the sum-function of this q-Euler series is not unique.

3.1. Mordell integral as sum-function of q-Euler series

When $q \in (0,1)$, a q-analog of Borel–Laplace summation, called *Gq-summation*, is introduced in [19]. Applying this to $\hat{E}(x;q)$ yields the following solution to (8) over the Riemann surface $\tilde{\mathbb{C}}^*$ of logarithm:

$$G(x;q) = \int_0^\infty \frac{\omega(\xi/x;q)}{1+\xi}\,\frac{\mathrm{d}\xi}{\xi}\,, \tag{9}$$

where the integration path is any straight-line starting from 0 to infinity in $\mathbb{C}\setminus(-\infty,0]$ and where $\omega(u;q) = \frac{1}{\sqrt{2\pi\,\ln(1/q)}}\,\mathrm{e}^{\frac{\log^2(u/\sqrt{q})}{2\ln q}}$ for $u \in \tilde{\mathbb{C}}^*$.

Putting $\xi = \mathrm{e}^\sigma$ in (9) gives that $G(x;q) = \frac{2\,q^{-3/8}}{\sqrt{2\pi x\ln(1/q)}}\int_{\mathbb{R}} \frac{\mathrm{e}^{(\sigma-\log(qx))^2/2\ln q}}{\cosh(\sigma/2)}\,\mathrm{d}\sigma$. Thus, the integral appearing in (9) is of Mordell type; see [10,1,21]. Moreover, in [22] one considers $h(z;\tau) = \int_{\mathbb{R}} \frac{\mathrm{e}^{\pi i\tau t^2 - 2\pi zt}}{\cosh \pi t}\,\mathrm{d}t$. If $G(z\,|\,\tau) = G(e(z);e(\tau))$, it follows that $G(z\,|\,\tau) = 2\sqrt{\frac{\mathrm{i}}{\tau}}\,e\big(\frac{(z+\frac{\tau}{2})^2}{2\tau}\big)\,h\big(\frac{z}{\tau}+1;-\frac{1}{\tau}\big)$.

3.2. Analytic continuation of Mordell integrals

In (9), $G(x;q)$ is defined for $x \in \tilde{\mathbb{C}}^*$ and $q \in (0,1)$. If φ denotes an analytic function at $t = 0 \in \mathbb{C}$ with $\varphi(0) \ne 0$, we fix an argument for $\varphi(0)$, consider φ as an analytic function valued in $\tilde{\mathbb{C}}^*$ and get the composed function $G(\varphi(\epsilon);\mathrm{e}^{-\epsilon})$ for $\epsilon \sim 0^+$. If $R > 0$, we write $\Delta_R = \{\epsilon \in \tilde{\mathbb{C}}^* : |\epsilon| < R, |\arg\epsilon| < 3\pi/2\}$.

Lemma 3.1. *Given an analytic function φ at $\epsilon = 0 \in \mathbb{C}$ such that $\varphi(0) \ne 0$, if $\Phi(\epsilon) = G(\varphi(\epsilon);\mathrm{e}^{-\epsilon})$ for $\epsilon \sim 0^+$, then Φ can be continued to be an analytic function in some Δ_R with the following properties.*

(i) *If $\arg(\varphi(0)) \in (-\pi,\pi]$ and $\varphi(0) \ne -1$, then $\Phi(\epsilon)$ admits a Gevrey asymptotic expansion as $\epsilon \to 0$ in Δ_R and $\Phi(\epsilon) = \frac{1}{1+\varphi(0)} + O(\epsilon)$.*

(ii) *If $\varphi(0) = -1 = \mathrm{e}^{\mathrm{i}\pi}$, $\varphi(\epsilon) = -\mathrm{e}^{\epsilon(\psi(\epsilon)+\frac{1}{2})}$ and $\tilde{\Phi}(\epsilon) = \Phi(\epsilon) - \mathrm{i}\sqrt{\frac{\pi}{2\epsilon}}\,\mathrm{e}^{-\frac{\epsilon}{2}(\psi(\epsilon))^2}$, then $\tilde{\Phi}(\epsilon)$ admits a Gevrey asymptotic expansion as $\epsilon \to 0$ in Δ_R and $\tilde{\Phi}(\epsilon) = 1 + \varphi'(0) + O(\epsilon)$.*

The asymptotic expansions of Φ or $\tilde{\Phi}$ are divergent, except the cases specified by the following theorem.

Theorem 3.2.

(i) *Φ becomes analytic at $\epsilon = 0$ if, and only if, $\varphi(\epsilon) = \mathrm{e}^{(n+\frac{1}{2})\epsilon}$ for some $n \in \mathbb{Z}$.*

(ii) *When $\varphi(0) = \mathrm{e}^{\mathrm{i}\pi}$, $\tilde{\Phi}$ becomes analytic at $\epsilon = 0$ if, and only if, $\varphi(\epsilon) = \mathrm{e}^{\pi\mathrm{i}+\frac{n}{2}\epsilon}$ for some $n \in \mathbb{Z}$.*

3.3. Modular formula viewed as a Stokes phenomenon

For $a \in \mathbb{C}^*$, let $[a;q] = a\,q^{\mathbb{Z}}$; let $\mu \in \mathbb{C}^* \setminus [1;q]$. Following [20] and [14], we denote by $L(x,\mu;q)$ the sum-function of $\hat{E}(x;q)$ along the q-spiral $[-\mu;q]$: $L(x,\mu;q) = \sum_{\xi\in[-\mu;q]} \frac{1}{1+\xi}\,\frac{1}{\theta(\xi/x)}$, where $\theta(e(z)) = \theta(e(z);q) = \theta(z\,|\,\tau)$; see (4) for the definition of θ. Let z and $w \in \mathbb{C}$. If none of w and $w-z$ does belong to the lattice $\mathbb{Z}\oplus\tau\mathbb{Z}$, we let $L(z,w\,|\,\tau) = L(e(z),e(w);e(\tau))$. By (1), it follows that

$$L(z,w\,|\,\tau) = \frac{R_1(-z+w-\frac{\tau}{2}+\frac{1}{2}, w\,|\,\tau)}{\theta(-z+w+\frac{1}{2}\,|\,\tau)}\,. \tag{10}$$

This is related to the function $\mu(u,v;\tau)$ of [22] as follows: $\mu(u,v;\tau) = e\big(\frac{1}{2}(u-v) - \frac{1}{8}\tau\big)\,L(u-v-\tau, u\,|\,\tau)$.

The functions $L(x,\mu;q)$ and $G(x;q)$ are both sum-functions of the same power series, so their difference satisfies the homogeneous q-difference equation $y(qx)+qxy(x)=0$. It follows that [21] (see also [22])

$$L(z,w\,|\,\tau)=G(z\,|\,\tau)+C(z\,|\,\tau)\left(L(\frac{z}{\tau},\frac{w}{\tau}\,|\,-\frac{1}{\tau})-1\right), \tag{11}$$

where $C(z\,|\,\tau)$ has the following alternative expressions:

$$C(z\,|\,\tau)=-i\sqrt{\frac{\mathrm{i}}{\tau}}\,e(\frac{(z+\frac{\tau}{2}-\frac{1}{2})^2}{2\tau})=\frac{\theta(-\frac{z}{\tau}+\frac{1}{2}\,|\,-\frac{1}{\tau})}{\tau\,\theta(-z+\frac{1}{2}\,|\,\tau)}. \tag{12}$$

By gathering Lemma 3.1 with (1), (10) and (11), one observes the following

Proposition 3.1. *Let $z\in\Omega_\tau$ and $w\in\Omega_\tau\setminus\{0,z\}$. If $f(q)=L(z,w\,|\,\tau)$, then $f\in\mathfrak{T}_0\cap\tilde{\mathfrak{T}}_1$.*

Comparing the sum $L(z,w\,|\,\tau)$ to an other sum $L(z,w'\,|\,\tau)$ consists also of a Stokes phenomenon, where the q-integration path changes. It follows that [5, Theorem 3.13 & Remark 3.14]

$$L(z,w\,|\,\tau)-L(z,w'\,|\,\tau)=-\frac{(\tau\,|\,\tau)_\infty^3\,\theta(w-w'+\frac{1}{2},w+w'-z+\frac{1}{2}\,|\,\tau)}{\theta(w+\frac{1}{2},w-z+\frac{1}{2},-w'+\frac{1}{2},w'-z+\frac{1}{2}\,|\,\tau)}. \tag{13}$$

Let $\Lambda_\tau=\{\frac{1}{2},\frac{1}{2}-\frac{\tau}{2},-\frac{\tau}{2}\}$. One finds that $L(\frac{1}{2},w\,|\,\tau)=1$, $L(-\frac{\tau}{2},w\,|\,\tau)=L(\frac{1}{2}-\frac{\tau}{2},w\,|\,\tau)=\frac{1}{2}$ for z, $w\in\Lambda_\tau$ with $w\neq z$. This, together with (13), gives the following theorem.

Theorem 3.3. *Given $z\in\Lambda_\tau$ and $w\in\Omega_\tau\setminus\{0,z\}$, $L(z,w\,|\,\tau)$ is constant or is a false theta-type function.*

3.4. Continued fraction and modular transforms associated with a rational number

Let $[a]$ to denote the integral part of any real a, *i.e.* $[a]\in(a-1,a]\cap\mathbb{Z}$. Given a rational number $r\in(0,1)\cap\mathbb{Q}$, we write $r_0=r$ and, for $n\geq 0$, $m_n=-[\frac{-1}{r_n}]$, $r_{n+1}=\frac{-1}{r_n}+m_n$ Let $\nu\in\mathbb{Z}_{>0}$ such that $r_{\nu-1}\neq 0$ and $r_\nu=0$. Let $\tau_0=\tau$, and define $\tau_{n+1}=\frac{-1}{\tau_n}+m_n$ for $n\in[0,\nu]\cap\mathbb{Z}$. It is worth noting that

$$\tau\xrightarrow{a.v.}r\quad\Longleftrightarrow\tau_n\xrightarrow{a.v.}r_n\quad\Longleftrightarrow\tau_\nu\xrightarrow{a.v.}r_\nu=0. \tag{14}$$

If $r=\frac{p}{m}$, $(p,m)\in\mathbb{Z}_{>0}^2$ and $p\wedge m=1$, there exists $\alpha\in\mathbb{Z}$ such that $\alpha\wedge m=1$ and

$$\tau_0\,\tau_1\,\ldots\,\tau_\nu=m\big(\tau-r\big),\quad \tau_{\nu+1}=-\frac{1}{m^2(\tau-r)}+\frac{\alpha}{m}. \tag{15}$$

Given $(z,w)\in\Omega_\tau\times\Omega_\tau$, we write $z_0=z$, $w_0=w$ and, for $n\in[1,\nu+1]\cap\mathbb{Z}$, $(z_n,w_n)\in\Omega_{\tau_n}$ with

$$z_n=\frac{z_{n-1}}{\tau_{n-1}}\pmod{\mathbb{Z}},\quad w_n=\frac{w_{n-1}}{\tau_{n-1}}\pmod{\mathbb{Z}}. \tag{16}$$

Lemma 3.4. *In (16), $z_n=0$ (resp. $z_n\in\Lambda_{\tau_n}$) if, and only if, $z_{n+1}=0$ (resp. $z_{n+1}\in\Lambda_{\tau_{n+1}}$).*

Now, let $\tau\xrightarrow{a.v.}r$, and consider $z\in\Omega_\tau\setminus\{0\}$ and $w\in\Omega_\tau\setminus\{0,z\}$, the case of $z=0$ being related with the first expression of $C(z\,|\,\tau)$ in (12). One iterates (11), makes use of (z_n,w_n) given in (16), and writes $\delta=\frac{1}{2}$ (resp. 0) if $z_\nu=\frac{1}{2}+b_\nu\tau_\nu$ (resp. otherwise). Let $L(z,w\,|\,\tau)=g_\nu(z\,|\,\tau)+L_\nu(z,w\,|\,\tau)$, with

$$L_\nu(z,w\,|\,\tau)=\frac{\theta(-z_{\nu+1}+\frac{1}{2}\,|\,\tau_{\nu+1})}{m(\tau-r)\,\theta(-z+\frac{1}{2}\,|\,\tau)}\left(L(z_{\nu+1},w_{\nu+1}\,|\,\tau_{\nu+1})-\delta\right).$$

By considering (14) and Lemma 3.1, one finds that $g_\nu(z\,|\,\tau)\in\mathfrak{G}_\zeta$, where $\zeta=e(r)$. By Proposition 3.1, one finds that L_ν satisfies (5), with $\nu=-\frac{1}{2}$; thus, Remark 1 implies that this theta-type part is unique modulo exponential smallness. Moreover, L_ν is of theta-type only at $\tau=r$, excepted some cases where it takes a closed form with the boundedness condition in (6). This, together with Theorems 3.2 and 3.3 plus Lemma 3.4, gives Theorem 2.2.

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