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Conditionally Gaussian stochastic integrals

Intégrales stochastiques conditionnellement gaussiennes

Nicolas Privault, Qihao She

Division of Mathematical Sciences, School of Physical and Mathematical Sciences, Nanyang Technological University, 21 Nanyang Link, Singapore 637371, Singapore

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ABSTRACT

We derive conditional Gaussian type identities of the form

$$E\left[\exp\left(\mathrm{i}\int_0^T u_t\,\mathrm{d}B_t\right)\Bigg|\int_0^T |u_t|^2\mathrm{d}t\right]=\exp\left(-\frac{1}{2}\int_0^T |u_t|^2\mathrm{d}t\right),$$

for Brownian stochastic integrals, under conditions on the process $(u_t)_{t\in[0,T]}$ specified using the Malliavin calculus. This applies in particular to the quadratic Brownian integral $\int_0^t AB_s\,\mathrm{d}B_s$ under the matrix condition $A^\dagger A^2=0$, using a characterization of Yor [6].

RÉSUMÉ

Nous obtenons des identités gaussiennes conditionnelles de la forme

$$E\left[\exp\left(\mathrm{i}\int_0^T u_t\,\mathrm{d}B_t\right)\Bigg|\int_0^T |u_t|^2\mathrm{d}t\right]=\exp\left(-\frac{1}{2}\int_0^T |u_t|^2\mathrm{d}t\right),$$

pour les intégrales stochastiques browniennes, sous des conditions sur le processus $(u_t)_{t\in[0,T]}$ exprimées à l'aide du calcul de Malliavin. Ces résultats s'appliquent en particulier à l'intégrale brownienne quadratique $\int_0^t AB_s\,\mathrm{d}B_s$ sous la condition matricielle $A^\dagger A^2=0$, en utilisant une caractérisation de Yor [6].

E-mail address: nprivault@ntu.edu.sg (N. Privault).

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1. Introduction

Let $(B_t)_{t\in[0,T]}$ be a d-dimensional Brownian motion generating the filtration $(\mathcal{F}_t)_{t\in[0,T]}$. When A is a $d \times d$ skew-symmetric matrix, the identity

$$E\left[\exp\left(\mathrm{i}\int_0^T AB_s\,\mathrm{d}B_s\right)\middle| B_t\right] = E\left[\exp\left(-\frac{1}{2}\int_0^T |AB_s|^2\,\mathrm{d}s\right)\middle| B_t\right], \tag{1}$$

$0 \le t \le T$, has been proved in Theorem 2.1 of [1], extending a formula of [7] for the computation of the characteristic function of Lévy's stochastic area in case $d = 2$.

This approach is connected to a result of Yor [6] stating that when $A^\dagger A^2 = 0$, the filtration $(\mathcal{F}_t^k)_{t\in[0,T]}$ of $t \mapsto \int_0^t AB_s\,\mathrm{d}B_s$ is generated by k independent Brownian motions, where k is the number of distinct eigenvalues of $A^\dagger A$.

In this Note, we derive conditional versions of the identity (1) for the stochastic integral $\int_0^T u_t\,\mathrm{d}B_t$ of an $(\mathcal{F}_t)$-adapted process $(u_t)_{t\in[0,T]}$ in Theorem 1, under conditions formulated in terms of the Malliavin calculus, using the cumulant–moment formula of [3,4]. In particular we provide conditions for $\int_0^T u_t\,\mathrm{d}B_t$ to be Gaussian $\mathcal{N}\left(0, \int_0^T |u_t|^2\,\mathrm{d}t\right)$-distributed given $\int_0^T |u_t|^2\,\mathrm{d}t$, cf. Theorem 2. This holds for example when $(u_t)_{t\in[0,T]} = (AB_t)_{t\in[0,T]}$ under Yor's condition $A^\dagger A^2 = 0$, cf. Corollary 3. We also consider a weakening of this condition to $A^\dagger A^2$ skew-symmetric, provided that $A^\dagger A$ is proportional to a projection, cf. Corollary 6.

2. Conditional characteristic functions

Let D denote the Malliavin gradient with domain $\mathbb{D}_{2,1}$ on the d-dimensional Wiener space, cf. § 1.2 of [2] for definitions. Taking $H = L^2([0,T];\mathbb{R}^d)$ for some $T > 0$ and u in the domain $\mathbb{D}_{k,1}(H)$ of D in $L^k(\Omega; H)$, we let

$$(Du)^k u_t := \int_0^T \cdots \int_0^T (D_{t_k}u_t)^\dagger (D_{t_{k-1}}u_{t_k})^\dagger \cdots (D_{t_1}u_{t_2})^\dagger u_{t_1}\,\mathrm{d}t_1\cdots\mathrm{d}t_k, \quad t\in[0,T], \quad k \ge 1.$$

Theorem 1. *Let $u \in \bigcap_{k\ge 1}\mathbb{D}_{k,1}(H)$ be an $(\mathcal{F}_t)$-adapted process such that*

$$\langle u_t, (Du)^k u_t\rangle_{\mathbb{R}^d} = 0, \qquad t\in[0,T], \quad k\ge 1.$$

We have

$$E\left[\exp\left(\mathrm{i}\int_0^T u_t\,\mathrm{d}B_t\right)\middle| (|u_t|)_{t\in[0,T]}\right] = \exp\left(-\frac{1}{2}\int_0^T |u_t|^2\,\mathrm{d}t\right), \tag{2}$$

provided that $\frac{1}{2}\int_0^T |u_t|^2\,\mathrm{d}t$ is exponentially integrable.

Proof. For any $F \in \mathbb{D}_{2,1}$ and $k \ge 1$, let

$$\Gamma_k^u F := \mathbb{1}_{\{k\ge 2\}} F \int_0^T \langle u_t, (Du)^{k-2}u_t\rangle_{\mathbb{R}^d}\,\mathrm{d}t + \int_0^T \langle D_tF, (Du)^{k-1}u_t\rangle_{\mathbb{R}^d}\,\mathrm{d}t.$$

Recall that for any $u \in \mathbb{D}_{2,1}(H)$ such that $\Gamma_{l_1}^u\cdots\Gamma_{l_k}^u\mathbb{1}$ has finite expectation for all $l_1 + \cdots + l_k \le n$, $k = 1,\ldots,n$, by Theorem 1 of [3] or Proposition 4.3 of [4] we have

$$E\left[F\left(\int_0^T u_t\,\mathrm{d}B_t\right)^n\right] = n!\sum_{a=1}^n \sum_{\substack{l_1+\cdots+l_a=n\\ l_1\ge 1,\ldots,l_a\ge 1}} \frac{E\left[\Gamma_{l_1}^u\cdots\Gamma_{l_a}^u F\right]}{l_1(l_1+l_2)\cdots(l_1+\cdots+l_a)}, \tag{3}$$

for $F \in \mathbb{D}_{2,1}$. Next, for any $f \in \mathcal{C}_b^1(\mathbb{R})$ and $k \ge 1$ we have

$$\begin{aligned}
&\Gamma_k^u f\left(\int_a^b |u_t|^2\,\mathrm{d}t\right)\\
&= \mathbb{1}_{\{k=2\}}\int_0^T |u_t|^2\,\mathrm{d}t f\left(\int_a^b |u_t|^2\,\mathrm{d}t\right) + f'\left(\int_a^b |u_t|^2\,\mathrm{d}t\right)\int_0^T \left\langle D_t\int_a^b |u_s|^2\,\mathrm{d}s, (Du)^{k-1}u_t\right\rangle_{\mathbb{R}^d}\mathrm{d}t,\\
&= \mathbb{1}_{\{k=2\}}\int_0^T |u_t|^2\,\mathrm{d}t f\left(\int_a^b |u_t|^2\,\mathrm{d}t\right) + 2f'\left(\int_a^b |u_t|^2\,\mathrm{d}t\right)\int_a^b \langle u_s, (Du)^k u_s\rangle_{\mathbb{R}^d}\,\mathrm{d}s,\\
&= \mathbb{1}_{\{k=2\}}\int_0^T |u_t|^2\,\mathrm{d}t f\left(\int_a^b |u_t|^2\,\mathrm{d}t\right), \qquad 0\le a\le b.
\end{aligned}$$

By induction, this yields

$$\Gamma_{l_1}^u\cdots\Gamma_{l_a}^u F = \mathbb{1}_{\{l_1=\cdots=l_a=2\}}\left(\int_0^T |u_t|^2\ \mathrm{d}t\right)^a F, \qquad l_1,\ldots,l_a\ge 1, \quad a\ge 1, \tag{4}$$

for any random variable F of the form

$$F = f\left(\int_{a_1}^{b_1} |u_t|^2\ \mathrm{d}t,\ldots,\int_{a_m}^{b_m} |u_t|^2\ \mathrm{d}t\right), \quad 0\le a_i\le b_i\le T, \quad i=1,\ldots,m,$$

where $f\in\mathcal{C}_b^1(\mathbb{R}^m)$, and by (3) and (4) we find

$$E\left[\left(\int_0^T u_t\,\mathrm{d}B_t\right)^{2n} F\right] = \frac{(2n)!}{2^n n!}E\left[\left(\int_0^T |u_t|^2\,\mathrm{d}t\right)^n F\right], \tag{5}$$

and $E\left[\left(\int_0^T u_t\,\mathrm{d}B_t\right)^{2n+1} F\right] = 0$ for all $n\in\mathbb{N}$. □

The following result is obtained by an argument similar to the proof of Theorem 1.

Theorem 2. *Let $u\in\bigcap_{k\ge 1}\mathbb{D}_{k,1}(H)$ be an $(\mathcal{F}_t)$-adapted process such that*

$$\langle u, (Du)^k u\rangle_H = 0, \qquad k\ge 1.$$

We have

$$E\left[\exp\left(\mathrm{i}\int_0^T u_t\,\mathrm{d}B_t\right)\,\middle|\,\int_0^T |u_t|^2\,\mathrm{d}t\right] = \exp\left(-\frac{1}{2}\int_0^T |u_t|^2\,\mathrm{d}t\right),$$

provided that $\frac{1}{2}\int_0^T |u_t|^2\,\mathrm{d}t$ is exponentially integrable.

In the particular case where $u_t = R_t h$, $t\in[0,T]$, $h\in H$, where R is a random, adapted (or quasi-nilpotent) isometry of H, we find that $\int_0^T |u_t|^2\,\mathrm{d}t = \int_0^T |h(t)|^2\,\mathrm{d}t$ is deterministic, hence

$$\langle u, (Du)^k u\rangle_H = \frac{1}{2}\langle (Du)^{k-1}u, D\langle u,u\rangle_H\rangle_H = 0, \qquad k\ge 1,$$

and Theorem 2 shows that $\int_0^T (R_t h)\,\mathrm{d}B_t$ has a centered Gaussian distribution with variance $\int_0^T |h(t)|^2\,\mathrm{d}t$, as in Theorem 2.1(b) of [5].

Theorems 1 and 2 also apply when $\int_0^T |u_t|^2 \, dt$ is random, for example when $(u_t)_{t\in[0,T]}$ takes the form $u_t = g(B_t)$, $t \in [0,T]$, where $g \in \mathcal{C}_b^1(\mathbb{R}^d;\mathbb{R}^d)$ satisfies the condition $\langle g(x), ((\nabla g(x))^\dagger)^k g(x)\rangle_{\mathbb{R}^d} = 0$, $x \in \mathbb{R}^d$, $k \geq 1$. Next, we check that this condition is satisfied on concrete examples based on [6], when g is a linear mapping of the form $g(x) = Ax$, $x \in \mathbb{R}^d$.

2.1. Vanishing of $A^\dagger A^2$

Applying Theorem 1 to the adapted process $(u_t)_{t\in[0,T]} := (AB_t)_{t\in[0,T]}$ under Yor's [6] condition $A^\dagger A^2 = 0$, by the relation $D_t B_s = \mathbb{1}_{[0,s]}(t) I_{\mathbb{R}^d}$ we obtain the vanishing

$$\begin{aligned}\langle u_t, (Du)^k u_t\rangle_{\mathbb{R}^d} &= \int_0^T \cdots \int_0^T \langle u_t, (D_{t_k} u_t)^\dagger (D_{t_{k-1}} u_{t_k})^\dagger \cdots (D_{t_1} u_{t_2})^\dagger u_{t_1}\rangle_{\mathbb{R}^d} \, dt_1 \cdots dt_k \\ &= \int_0^t \int_0^{t_k} \cdots \int_0^{t_2} \langle AB_t, (A^\dagger)^k AB_t\rangle_{\mathbb{R}^d} \, dt_1 \cdots dt_k \\ &= 0, \qquad t \in [0,T], \quad k \geq 1.\end{aligned}$$

This yields the next corollary of Theorem 1, in which the condition $A^\dagger A^2 = 0$ includes 2-nilpotent matrices as a particular case.

Corollary 3. *Assume that $A^\dagger A^2 = 0$. We have*

$$E\left[\exp\left(i \int_0^T AB_t \, dB_t\right) \middle| (|AB_t|)_{t\in[0,T]}\right] = \exp\left(-\frac{1}{2}\int_0^T |AB_t|^2 \, dt\right). \tag{6}$$

Note that the filtration of $(|AB_t|)_{t\in[0,T]}$ coincides with the filtration $(\mathcal{F}_t^k)_{t\in[0,T]}$ generated by k independent Brownian motions where k is the number of nonzero eigenvalues of $A^\dagger A$, cf. Corollary 2 of [6].

3. Skew-symmetric $A^\dagger A^2$

When $A^\dagger A$ has only one nonzero eigenvalue, i.e. $A^\dagger A$ is proportional to a projection, the condition $A^\dagger A^2 = 0$ can be relaxed using stochastic calculus, by only assuming that $A^\dagger A^2$ is skew-symmetric. We start with the following variation of Corollary 2 of [6].

Lemma 4. *Assume that $A^\dagger A^2$ is skew-symmetric and $A^\dagger A$ has a unique nonzero eigenvalue λ_1. Then the processes*

$$Y_t^1 := \frac{1}{\sqrt{\lambda_1}} \int_0^t \frac{AB_s}{|AB_s|} dAB_s, \quad \textit{and} \quad Y_t^2 := \int_0^t \frac{AB_s}{|AB_s|} dB_s, \quad t \in [0,T], \tag{7}$$

are independent standard Brownian motions.

Proof. Since $A^\dagger A$ is symmetric it can be written as $A^\dagger A = R^\dagger C R$, where R is orthogonal and C is diagonal, therefore since $(RB_t)_{t\in[0,T]}$ is also a standard Brownian motion we can assume that $A^\dagger A$ has the form $A^\dagger A = (\lambda_k \mathbb{1}_{\{1\leq k=l\leq r\}})_{1\leq k,l\leq d}$ with $\lambda_i > 0$, $1 \leq i \leq r$. Clearly $(Y_t^2)_{t\in[0,T]}$ is a standard Brownian motion, and

$$d\langle Y^1, Y^2\rangle_t = \frac{\langle A^\dagger A^2 B_t, B_t\rangle}{|AB_t|^2 \sqrt{\lambda_1}} dt = 0.$$

In addition, we have $dY_t^1 = \frac{\lambda_1^{-1/2}}{|AB_t|} \sum_{i=1}^r \lambda_i B_t^i \, dB_t^i$ and

$$d\langle Y^1, Y^1\rangle_t = \frac{(\lambda_1 B_t^1)^2 + \cdots + (\lambda_r B_t^r)^2}{\lambda_1 (\lambda_1 (B_t^1)^2 + \cdots + \lambda_r (B_t^r)^2)} dt,$$

hence $(Y_t^1)_{t\in[0,T]}$ is also a standard Brownian motion when $\lambda_1 = \cdots = \lambda_r$. □

The following result relaxes the vanishing hypothesis of Corollary 3.

Corollary 5. *Assume that $A^\dagger A^2$ is skew-symmetric and $A^\dagger A$ has a unique nonzero eigenvalue λ_1. Then we have*

$$E\left[\exp\left(\mathrm{i}\int_0^T AB_t\,\mathrm{d}B_t\right)\middle|(|AB_t|)_{t\in[0,T]}\right]=\exp\left(-\frac{1}{2}\int_0^T |AB_t|^2\,\mathrm{d}t\right). \tag{8}$$

Proof. We let $S_t := |AB_t|^2$, $t\in[0,T]$, and note that by Corollary 2 of [6], the filtration generated by $(|AB_t|)_{t\in[0,T]}$ coincides with the filtration $(\mathcal{F}_t^1)_{t\in[0,T]}$ of $(Y_t^1)_{t\in[0,T]}$. Next, Itô's formula shows that

$$S_t = 2\int_0^t AB_s\,\mathrm{d}AB_s + \mathrm{Tr}\left(A^\dagger A\right)t = 2\int_0^t \sqrt{\lambda_1 S_s}\,\mathrm{d}Y_s^1 + r\lambda_1 t, \qquad t\in[0,T],$$

hence $(|AB_t|)_{t\in[0,T]}$ is $(\mathcal{F}_t^1)_{t\in[0,T]}$-adapted and therefore independent of $(Y^2)_{t\in[0,T]}$, hence

$$\int_0^T AB_t\,\mathrm{d}B_t = \int_0^T |AB_t|\,\mathrm{d}Y_t^2$$

is centered Gaussian with variance $\int_0^T |AB_t|^2\,\mathrm{d}t$ given $\mathcal{F}_T^1$, which yields (8). □

3.1. Commutation with orthogonal matrices

Under the assumptions of Corollaries 3 or 5 it follows that

$$E\left[\exp\left(\mathrm{i}\int_0^T AB_t\,\mathrm{d}B_t\right)\middle||AB_t|\right]=E\left[\exp\left(-\frac{1}{2}\int_0^T |AB_t|^2\,\mathrm{d}t\right)\middle||AB_t|\right], \tag{9}$$

since $(|AB_t|)_{t\in[0,T]}$ and $(Y_t^1)_{t\in[0,T]}$ generate the same filtration on $(\mathcal{F}_t^1)_{t\in[0,T]}$.

Corollary 6. *Assume that either $A^\dagger A^2=0$, or $A^\dagger A^2$ is skew-symmetric and $A^\dagger A$ has a unique nonzero eigenvalue. If in addition A commutes with orthogonal matrices, then we have*

$$E\left[\exp\left(\mathrm{i}\int_0^T AB_s\mathrm{d}B_s\right)\middle|AB_t\right]=E\left[\exp\left(-\frac{1}{2}\int_0^T |AB_s|^2\,\mathrm{d}s\right)\middle|AB_t\right], \tag{10}$$

$0\le t\le T$.

Proof. We check that for any $d\times d$ orthogonal matrix R we have

$$E\left[\exp\left(\mathrm{i}\int_0^T AB_t\,\mathrm{d}B_t\right)\middle|AB_t=Rx\right]=E\left[\exp\left(\mathrm{i}\int_0^T AB_t\,\mathrm{d}B_t\right)\middle|AB_t=x\right],$$

$x\in\mathbb{R}^d$, which shows that

$$E\left[\exp\left(\mathrm{i}\int_0^T AB_t\,\mathrm{d}B_t\right)\middle|AB_t\right]=E\left[\exp\left(\mathrm{i}\int_0^T AB_t\,\mathrm{d}B_t\right)\middle||AB_t|\right]$$

and similarly for the right-hand side, and we conclude by (9). □

3.2. Skew-symmetric orthogonal A

We note that when A is skew-symmetric and orthogonal, the condition $A^\dagger A^2$ skew-symmetric is satisfied as in this case we have $(A^\dagger A^2)^\dagger = A^\dagger A^\dagger A = A^\dagger = -A = -A^\dagger A^2$, and (10) can be written as

$$E\left[\exp\left(\mathrm{i}\int_0^T AB_s\,\mathrm{d}B_s\right)\middle|B_t\right]=E\left[\exp\left(-\frac{1}{2}\int_0^T |AB_s|^2\,\mathrm{d}s\right)\middle|B_t\right], \tag{11}$$

$0 \leq t \leq T$. This holds in particular when $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, in which case $A^\dagger A = I_{\mathbb{R}^2}$ has the unique eigenvalue $\lambda_1 = 1$ and $A^\dagger A^2 = A$ is skew-symmetric, in which case we recover the result of [7] that has been used to show that (11) holds when A is skew-symmetric and not necessarily orthogonal in Theorem 2.1 of [1].

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