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Beurling's theorem for the Bessel–Struve transform

Théorème de Beurling pour la transformée de Bessel–Struve

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ABSTRACT

The Bessel–Struve transform satisfies some uncertainty principles in a similar way to the Euclidean Fourier transform. Beurling's theorem is obtained for the Bessel–Struve transform $\mathcal{F}^{\alpha}_{B,S}$.

RÉSUMÉ

La transformé de Bessel–Struve satisfait quelques principes d'incertitude de manière similaire au cas de la transformée de Fourier euclidienne. Le théorème de Beurling est obtenu pour la transformée de Bessel–Struve.

1. Introduction

There are many known theorems that state that a function and its classical Fourier transform on $\mathbb{R}$ cannot both be sharply localized. That it is impossible for a non-zero function and its Fourier transform to be simultaneously small. Hardy [5], Miyachi [9], Cowling and Price [3], and Beurling [1] for example interpreted the smallness as sharp pointwise estimates for integrable decay of functions. Beurling's theorem, which was found by Beurling and his proof was published much later by Hörmander, says that for any non-trivial function $f \in L^2(\mathbb{R})$, the product $f(x)\widehat{f}(y)$ is never integrable on $\mathbb{R}^2$ with respect to the measure $\mathrm{e}^{|x||y|}\mathrm{d}x\mathrm{d}y$, where $\widehat{f}$ stands for the Fourier transform of f. A far-reaching generalization of this result has been recently proved by Bonami, Demange and Jaming [2]. They proved that

Theorem 1.1. *If $f \in L^2(\mathbb{R})$ satisfies for an integer N*

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|f(x)||\widehat{f}(y)|}{(1+|x|+|y|)^N}\mathrm{e}^{|x||y|}\mathrm{d}x\mathrm{d}y < \infty,$$

then f is of the form $f(x) = P(x)\mathrm{e}^{-bx^2}$, where P is a polynomial of degree strictly lower than $\frac{N-1}{2}$ and b is a positive constant.

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Many authors have established the analogous of Beurling's theorem in other various setting of harmonic analysis (see for instance [7,8]). The purpose of this paper is to establish an analogous of Beurling's theorem for the Bessel–Struve transform. The outline of the content of this paper is as follows.

Section 2 is dedicated to some properties and results concerning the Bessel–Struve transform.

Section 3 is devoted to the Beurling's theorem for the Bessel–Struve transform.

2. Bessel–Struve transform

We consider the Bessel–Struve operator l_α, $\alpha > -\frac{1}{2}$, defined on $\mathcal{C}^\infty(\mathbb{R})$ by

$$l_\alpha u(x) = \frac{\mathrm{d}^2 u}{\mathrm{d}x^2}(x) + \frac{2\alpha+1}{x}\left[\frac{\mathrm{d}u}{\mathrm{d}x}(x) - \frac{\mathrm{d}u}{\mathrm{d}x}(0)\right].$$

For $\lambda \in \mathbb{C}$, the differential equation:

$$\begin{cases} l_\alpha u(x) = \lambda^2 u(x) \\ u(0) = 1,\ u'(0) = \frac{\lambda\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{3}{2})} \end{cases} \tag{1}$$

possesses a unique solution denoted $\Phi_\alpha(\lambda)$. This eigenfunction, called the Bessel–Struve kernel, is given by:

$$\Phi_\alpha(\lambda x) = j_\alpha(\mathrm{i}\lambda x) - \mathrm{i}h_\alpha(\mathrm{i}\lambda x), \quad x \in \mathbb{R}.$$

j_α and h_α are respectively the normalized Bessel and Struve functions of index α. These kernels are given as follows:

$$j_\alpha(z) = \Gamma(\alpha+1)\sum_{k=0}^{+\infty}\frac{(-1)^k\left(\frac{z}{2}\right)^{2k}}{k!\Gamma(k+\alpha+1)}$$

and

$$h_\alpha(z) = \Gamma(\alpha+1)\sum_{k=0}^{+\infty}\frac{(-1)^k\left(\frac{z}{2}\right)^{2k+1}}{\Gamma\left(k+\frac{3}{2}\right)\Gamma\left(k+\alpha+\frac{3}{2}\right)}.$$

Let $p \in [1, +\infty]$, we denote by $L^p_\alpha(\mathbb{R})$ the space of real-valued functions f measurable on $\mathbb{R}$ such that

$$\|f\|_{p,\alpha} = \left(\int_{\mathbb{R}} |f(x)|^p \,\mathrm{d}\mu_\alpha(x)\right)^{\frac{1}{p}} < +\infty \quad \text{if } p < +\infty$$

where

$$\mathrm{d}\mu_\alpha(x) = A(x)\,\mathrm{d}x \ \text{ and } A(x) = |x|^{2\alpha+1},$$

$$\|f\|_{\infty,\alpha} = ess\sup_{x\in\mathbb{R}}|f(x)| < +\infty \ \text{ if } p = \infty.$$

The kernel Φ_α possesses the following integral representation:

$$\Phi_\alpha(\lambda x) = \frac{2\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}\int_0^1 (1-t^2)^{\alpha-\frac{1}{2}}\mathrm{e}^{\lambda x t}\mathrm{d}t, \quad \forall x \in \mathbb{R}, \quad \forall \lambda \in \mathbb{C}.$$

The Bessel–Struve kernel Φ_α is related to the exponential function by

$$\forall x \in \mathbb{R}, \quad \forall \lambda \in \mathbb{C}, \quad \Phi_\alpha(\lambda x) = \mathcal{X}_\alpha(\mathrm{e}^{\lambda \cdot})(x),$$

where $\mathcal{X}_\alpha$ is the Bessel–Struve intertwining operator (see [4]).

Definition 1. The Bessel–Struve transform is defined on $L^1_\alpha(\mathbb{R})$ by

$$\forall \lambda \in \mathbb{R}, \quad \mathcal{F}^\alpha_{B,S}(f)(\lambda) = \int_{\mathbb{R}} f(x)\Phi_\alpha(-\mathrm{i}\lambda x)\mathrm{d}\mu_\alpha(x).$$

Definition 2. For $f \in L^1_\alpha(\mathbb{R})$ with bounded support, the integral transform W_α, given by

$$W_\alpha(f(x)) = \frac{2\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}|x|^{2\alpha+1}\int_{|x|}^{+\infty}(y^2-x^2)^{\alpha-\frac{1}{2}}yf(\mathrm{sgn}(x)y)\mathrm{d}y, \quad x \in \mathbb{R}\backslash\{0\}$$

is called Weyl integral transform associated with the Bessel–Struve operator.

Proposition 1. *(See [4].) W_α is a bounded operator from $L^1_\alpha(\mathbb{R})$ to $L^1(\mathbb{R})$, where $L^1(\mathbb{R})$ is the space of Lebesgue-integrable functions.*

Proposition 2. *(See [4].) We have $\forall f \in L^1_\alpha(\mathbb{R})$, $\mathcal{F}^\alpha_{B,S} = \mathcal{F} \circ W_\alpha(f)$ where $\mathcal{F}$ is the classical Fourier transform defined on $L^1(\mathbb{R})$ by*

$$\mathcal{F}(g)(\lambda) = \int_{\mathbb{R}} g(x)\mathrm{e}^{-\mathrm{i}\lambda x}\mathrm{d}x.$$

Proposition 3. *(See [4].) Let $a > 0$, the Weyl integral transform verifies*

$$W_\alpha(e_{-a}) = Ce_{-a}$$

where $C = \frac{\Gamma(\alpha+1)}{2\sqrt{\pi}a^{\alpha+\frac{1}{2}}}$.

Proposition 4. *(See [4].) Let $a > 0$ and let f be a continuous function on $\mathbb{R}$ such that*

$$\forall x \in \mathbb{R}, \quad |f(x)| \le C\mathrm{e}^{-ax^2} \tag{$*$}$$

Then $W_\alpha(f)$ is of class C^1 on $\mathbb{R}\backslash\{0\}$ and verifies

$$\forall x \in \mathbb{R}\backslash\{0\}, \quad [W_{\frac{1}{2}}f]'(x) = -xf(x)$$

and

$$\forall \alpha > \frac{1}{2}, \quad \forall x \in \mathbb{R}\backslash\{0\}, \quad [W_\alpha f]'(x) = -2\alpha x W_{\alpha-1}f(x).$$

Proposition 5. *(See [4].) For $\alpha = k + \frac{1}{2}$, $k \in \mathbb{N}$, let f be a continuous function on $\mathbb{R}$ verifying $(*)$. Then W_α is of class C^{k+1} on $\mathbb{R}\backslash\{0\}$ and we have $V_\alpha \circ W_\alpha(f) = f$ where*

$$V_\alpha f(x) = (-1)^{k+1}\frac{2^{2k+1}k!}{(2k+1)!}(\frac{\mathrm{d}}{\mathrm{d}x^2})^{k+1}(f(x)), \quad x \in \mathbb{R}^* \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}x^2} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}.$$

Lemma 1. *(See [6].) Let $g \in \mathcal{E}(\mathbb{R}\backslash\{0\})$, m and k are two integers nonnegative, we have*

$$\forall x \in \mathbb{R}^*, \quad (\frac{\mathrm{d}}{\mathrm{d}x^2})^k(x^m g(x)) = \sum_{i=0}^{k} b^k_i x^{m-2k+i}g^{(i)}(x)$$

where $\mathcal{E}(\mathbb{R})$ designates the space of infinitely differentiable functions on $\mathbb{R}$ and b^k_i are constants depending on i, k and m.

Lemma 2. *(See [6].) Let f be in $\mathcal{D}(\mathbb{R})$. We have $V_\alpha \circ (W_\alpha(f)) = f$.*

Theorem 2.1. *(See [4].) Let f be a measurable function on $\mathbb{R}$ such that*

$$\|e_a f\|_{p,\alpha} < +\infty \quad \text{and} \quad \|e_b \mathcal{F}^\alpha_{B,S}(f)\|_{q,\alpha} < +\infty \tag{2}$$

for some constants $a > 0$, $b > 0$, $1 \le p,\ q \le +\infty$ and at least one of p and q is finite. We have

1. *If $ab \ge \frac{1}{4}$, then $f = 0$ a.e.*
2. *If $ab < \frac{1}{4}$, then for all $\delta \in]a, \frac{1}{4b}[$, the functions having the form $f(x) = P(x)\mathrm{e}^{-\delta x^2}$ where P is an even polynomial on $\mathbb{R}$ satisfy relation (2).*

3. Beurling's theorem for the Bessel–Struve transform

In this section, we will prove our result main.

Theorem 3.1. *Let* $N,\ k \in \mathbb{N}$, $\alpha = k + \frac{1}{2}$ *and* $f \in L^2_\alpha(\mathbb{R})$. *Then*

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|f(x)||\mathcal{F}^\alpha_{B,S}(f)(y)|}{(1+|x|+|y|)^N} e^{|x||y|} |x|^{2\alpha+1} \mathrm{d}x\mathrm{d}y < \infty \tag{3}$$

implies $f(x) = P(x)e^{-rx^2}$, *where* $r > 0$ *and* P *is an even polynomial of degree strictly lower than* $\frac{N-1}{2}$.

Proof. We start with the following lemma.

Lemma 3. *We suppose that* $f \in L^2_\alpha(\mathbb{R})$ *satisfies (3), then* $f \in L^1_\alpha(\mathbb{R})$.

Proof. We may suppose that $f \neq 0$ in $L^2_\alpha(\mathbb{R})$. (3) and the Fubini theorem imply that for almost every $y \in \mathbb{R}$,

$$\frac{|\mathcal{F}^\alpha_{B,S}(f)(y)|}{(1+|y|)^N} \int_{\mathbb{R}} \frac{|f(x)|}{(1+|x|)^N} e^{|x||y|} |x|^{2\alpha+1} \mathrm{d}x < \infty.$$

Since $\mathcal{F}^\alpha_{B,S}(f) \neq 0$, there exists $y_0 \in \mathbb{R}$, $y_0 \neq 0$ such that $\mathcal{F}^\alpha_{B,S}(f)(y_0) \neq 0$.
Therefore

$$\int_{\mathbb{R}} \frac{|f(x)|}{(1+|x|)^N} e^{|x||y_0|} |x|^{2\alpha+1} \mathrm{d}x < \infty.$$

Since $\dfrac{e^{|x||y_0|}}{(1+|x|)^N} \geqslant 1$ for large $|x|$, it follows that $\displaystyle\int_{\mathbb{R}} |f(x)||x|^{2\alpha+1}\mathrm{d}x < \infty$. □

This lemma and Proposition 1 imply that $W_\alpha(f)$ is well defined almost everywhere on $\mathbb{R}$. By Proposition 1 we can find a constant $C > 0$ such that

$$\int_{\mathbb{R}} |W_\alpha(f)(x)|\mathrm{d}x \leq C \int_{\mathbb{R}} |f(x)||x|^{2\alpha+1}\mathrm{d}x,$$

thus

$$\begin{aligned}\int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|W_\alpha(f(x))||\mathcal{F}^\alpha_{B,S}(f)(y)|}{(1+|x|+|y|)^N} e^{|x||y|}\mathrm{d}x\mathrm{d}y &\leq C \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|f(x)||\mathcal{F}^\alpha_{B,S}(f)(y)|}{(1+|x|+|y|)^N} e^{|x||y|}|x|^{2\alpha+1}\mathrm{d}x\mathrm{d}y \\ &< \infty.\end{aligned}$$

It follows from Proposition 2 that

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|W_\alpha(f(x))||\mathcal{F} \circ W_\alpha(f)(y)|}{(1+|x|+|y|)^N} e^{|x||y|}\mathrm{d}x\mathrm{d}y < \infty.$$

According to Theorem 1.1, we can deduce that for all $x \in \mathbb{R}$, $W_\alpha(f)(x) = P(x)e^{-rx^2}$, where $r > 0$ and P is a polynomial of degree strictly lower than $\frac{N-1}{2}$.
Hence applying Lemma 1, we can find constants b_s such that

$$f(x) = V_\alpha \circ W_\alpha(f)(x) = \sum_{|s|<\frac{N-1}{2}} b_s x^s e^{-rx^2}.$$

Then it follows from Lemma 3 that $\displaystyle\int_{\mathbb{R}} |\sum_{|s|<\frac{N-1}{2}} b_s x^s|e^{-rx^2}|x|^{2\alpha+1}\mathrm{d}x < \infty$. Then for some constants $a \in]0, r[$, we have $\|e_a f\|_{1,\alpha} < +\infty$. On the other hand,

$$\mathcal{F}^\alpha_{B,S}(f)(y) = \mathcal{F} \circ W_\alpha(f)(y) = \mathcal{F}(P(x)e^{-rx^2}) = R(y)e^{\frac{-y^2}{4r}}$$

where R is a polynomial of degree $\deg P$. Then for some constants $b \in]0, \frac{1}{4}r[$, we have $\|e_b \mathcal{F}^\alpha_{B,S}(f)\|_{2,\alpha} < +\infty$. According to Theorem 2.1, we can deduce that $f(x) = P(x)e^{-rx^2}$, where P is an even polynomial of degree strictly lower than $\frac{N-1}{2}$. □

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