C. R. Acad. Sci. Paris, Ser. I 353 (2015) 1123–1127

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

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Algebraic geometry

Twisted cubic curves in the Segre variety

Courbes rationnelles dans la variété de Segre

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A R T I C L E I N F O

Article history:
Received 30 April 2015
Accepted after revision 11 September 2015
Available online 23 October 2015

Presented by Claire Voisin

Keywords:
Rational curves
Stable maps
Stable sheaves

A B S T R A C T

Let $X = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ be the Segre variety. Let $\mathbf{S}$ be the space of twisted cubic curves in X with tri-degree $(1, 1, 1)$. In this note, we prove that $\mathbf{S}$ is a rational, smooth variety of dimension 6. Also, we compute the Poincaré polynomial of $\mathbf{S}$ by stratifying the space into projective space fibration over some base spaces.

R É S U M É

Soit $X = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ la variété de Segre. Soit $\mathbf{S}$ l'espace des courbes cubiques rationnelles de tridegré $(1, 1, 1)$ dans X. Dans cet article, nous prouvons que $\mathbf{S}$ est une variété rationnelle, lisse, de dimension 6. Nous calculons également le polynôme de Poincaré de $\mathbf{S}$ à l'aide d'une stratification dont les strates sont des fibrés projectifs.

1. Introduction and results

Rational curves in a projective variety X have been studied by many algebraic geometers from various viewpoints: curve counting theory; minimal model program; construction of new varieties. One of the key issues in the research is to compactify the space $\mathbf{R}$ of smooth rational curves in different ways [3,4]. But, in general, the space $\mathbf{R}$ may not be irreducible. When the target space is a homogeneous variety, then it is shown that there exists a unique irreducible component consisting of smooth rational curves in each compactification for a fixed curve class β [6]. Let X be the projective variety of the product $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{P}^7$ embedded by the complete linear system $|\mathcal{O}(1, 1, 1)|$. Let us fix the curve class of the type $\beta = (1, 1, 1) \in H_2(X) = \mathbb{Z}^{\oplus 3}$. In this paper, we consider the compactification of $\mathbf{R}_\beta$ in the stable maps space $\mathbf{M}$, stable sheaves space $\mathbf{S}$, and Hilbert scheme $\mathbf{H}$, respectively. In [2], the authors studied the geometry of the spaces $\mathbf{S}$ and $\mathbf{H}$. Concerning the similar situation, the main results of this paper are the following ones.

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http://dx.doi.org/10.1016/j.crma.2015.09.008
1631-073X/

Theorem 1.1.

(1) *The spaces are isomorphic to each other:*

$$\mathbf{M} \cong \mathbf{S} \cong \mathbf{H}.$$

(2) *The space* **S** *is a smooth, irreducible and rational variety of dimension* 6.

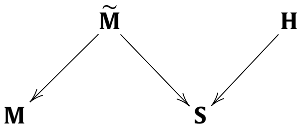
Also we compute the Poincaré polynomial of **S** by using the proof of Theorem 1.1 (for detail, see Proposition 3.2).

Remark 1.2. The results of [2, Proposition 4.8] have been strengthened through Theorem 1.1.

2. Proof of Theorem 1.1

2.1. Application of the results in [4]

Since the variety $X = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ clearly satisfies all conditions stated in [4, Lemma 2.1], we can apply the main result in [4]. That is, there exist blow-up/down diagrams among **M**, **S** and **H**:



where the blow-up centers are loci of a multiple cover of lines (or the locus of plane curves) (for detail, see [4]).

First step of the proof of Theorem 1.1. Let $f(C)$ be the image curve of the stable map $[f : C \to X] \in \mathbf{M}$. By definition, $[f(C)] = (1, 1, 1) \in H_2(X)$ and thus the map is not a multiple cover onto its image. This implies that the blow-up centers in **M** are empty. Thus the first isomorphism in item (1) holds by [4, Theorem 1.7]. On the other hand, the blow-up centers in **S** are empty because X does not contain any planes and no plane cubic curve. Therefore, the second part of item (1) is proved by [4, Theorem 4.16]. The smoothness of item (2) is exactly Proposition 4.13 in [4]. The irreducibility of **S** comes from that of **M** [6]. □

Remark 2.1. In [2], the authors proved that **S** is smooth in the complement $\mathbf{S} \setminus D$ such that $D \cong X$ parameterizes the union of three lines meeting at a single point $x \in X$. One checks that **S** is smooth everywhere with the help of the computer program Macaulay2 [5] as follow. Without loss of generality, let us assume that

$$I_X = \langle x_4x_7 - x_5x_6, x_2x_7 - x_3x_6, x_2x_5 - x_3x_4, x_1x_7 - x_3x_5,$$
$$x_1x_6 - x_2x_5, x_0x_7 - x_2x_5, x_0x_6 - x_2x_4, x_0x_5 - x_1x_4, x_0x_3 - x_1x_2\rangle,$$

where $x_0, x_, \cdots, x_7$ are the homogeneous coordinates of $\mathbb{P}^7$. Also, let us define a union of three lines C by

$$I_C = \langle x_3, x_5, x_6, x_7, x_1x_2, x_1x_4, x_2x_4\rangle$$

such that the three lines meet at the point $x = [1 : 0] \times [1 : 0] \times [1 : 0]$. Then the tangent space of **S** at $[\mathcal{O}_C]$ is isomorphic to

$$\mathrm{Ext}^1_X(\mathcal{O}_C, \mathcal{O}_C) \cong \mathbb{C}^6$$

and thus **S** is smooth at $[\mathcal{O}_C]$. This holds for every point in D because X is homogeneous.

2.2. Rationality of the space **S**

By using the notion of the relative extension, we construct a $\mathbb{P}^3$-bundle over an affine space that is birational with the space **S**. Let us start with the following observation. Let $C \subset X$ be a general twisted cubic curve with the degree $\beta = (1, 1, 1)$. Then the projection $C_0 = \pi_{12}(C)$ is a smooth conic where $\pi_{12} : \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$ is the projection map into the first two components. Let us consider the surface $S = C_0 \times \mathbb{P}^1$. Then clearly $C \subset S$. Hence there exists a structure sequence

$$0 \to I_{S,X} \to I_{C,X} \to I_{C,S} \to 0.$$

Remark that $I_{S,X} = \mathcal{O}_X(-1, -1, 0)$ and $I_{C,S} \cong I_{C',S} = \mathcal{O}_S(-1, -1)$ for any twisted cubic curves C and C' in X such that $\pi_{12}(C) = \pi_{12}(C') = C_0$. Conversely, let us fix a conic C_0 and thus the ruled surface S. Then there exists a one-to-one correspondence between $I_{C,X}$'s and the points

$$P = \mathbb{P}(\mathrm{Ext}^1_X(I_{C,S}, I_{S,X})).$$

On the other hand,

$$\begin{aligned}\mathrm{Ext}^1_X(I_{C,S},\mathcal{O}_X(-1,-1,0)) &= \mathrm{Ext}^2_X(O_X(-1,-1,0), I_{C,S}(-2,-2,-2))^* \\ &= H^2(I_{C,S}(-1,-1,-2))^* = H^2(\mathcal{O}_S(-1,-1)\otimes\mathcal{O}_S(-2,-2))^* \\ &= H^2(\mathcal{O}_S(-3,-3))^* = H^0(\mathcal{O}_S(1,1)) \cong \mathbb{C}^4.\end{aligned}$$

Hence $P = \mathbb{P}^3$. Let us relativize this situation.

End of the proof of Theorem 1.1. Let $Z \subset Gr(3,4)\times\mathbb{P}^1\times\mathbb{P}^1 \to Gr(3,4)$ be the universal family of conics. Let us consider the family of $\mathbb{P}^1\times\mathbb{P}^1$ in X, which is provided by the direct product

$$p : \mathcal{S} := Z \times \mathbb{P}^1 \subset Gr(3,4)\times X \to Gr(3,4).$$

Let q be the projection $\mathcal{S} \to X$. Let

$$\mathcal{E} := \mathcal{E}xt^1_p(\mathcal{O}_{\mathcal{S}}(-1,-1), q^*\mathcal{O}_X(-1,-1,0))$$

be the relative extension sheaf on the space $Gr(3,4)$. We claim that there exists a birational map

$$\Psi : \mathbb{P}(\mathcal{E}) \longrightarrow \mathbf{S}$$

provided by the tautological family $\mathcal{K}$ on $\mathbb{P}(\mathcal{E})$:

$$0 \to q^*\mathcal{O}_X(-1,-1,0)\otimes\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \to \mathcal{K} \to \mathcal{O}_{\mathcal{S}}(-1,-1) \to 0.$$

Let $\pi : \mathbb{P}(\mathcal{E}) \to Gr(3,4)$ be the structure morphism. Then we claim that the map Ψ is well-defined and injective on the locus $Gr(3,4)\setminus\Delta$ of the smooth conics. Let $(C_0,\kappa)\in\mathbb{P}(\mathcal{E})$ for a smooth conic $C_0 \in Gr(3,4)$ and $\kappa(\neq 0) \in \mathrm{Ext}^1(\mathcal{O}_S(-1,-1),\mathcal{O}_X(-1,-1,0)) \cong \mathrm{Hom}(\mathcal{O}_S(-1,-1),\mathcal{O}_S)$, where $S = C_0 \times \mathbb{P}^1$. By the definition of the pulling-back and the injection $\kappa : \mathcal{O}_S(-1,-1) \hookrightarrow \mathcal{O}_S$, we have a commutative diagram

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \uparrow & & \uparrow & & \\ & & & & \mathcal{O}_C & = & \mathcal{O}_C & & \\ & & & & \uparrow & & \uparrow & & \\ 0 & \to & \mathcal{O}_X(-1,-1,0) & \to & \mathcal{O}_X & \to & \mathcal{O}_S & \to & 0 \\ & & \| & & \uparrow & & \uparrow{\scriptstyle\kappa} & & \\ 0 & \to & \mathcal{O}_X(-1,-1,0) & \dashrightarrow & \mathcal{K}_{(C_0,\kappa)} & \dashrightarrow & \mathcal{O}_S(-1,-1) & \to & 0 \\ & & & & \uparrow & & \uparrow & & \\ & & & & 0 & & 0 & & \end{array}$$

such that C is a rational cubic curve in X with tri-degree $(1,1,1)$. This implies that $\mathcal{K}_{(C_0,\kappa)} \cong I_{C,X}$. The map Ψ restricted on the fiber $\pi^{-1}(C_0) = \mathbb{P}^3$ is injective because κ parameterizes the conics in S. Also, let $C_0, C_1 \in Gr(3,4)$ be two different smooth conics. Then one can see that the intersection of C_0 and C_1 consists of two points. This observation with the structures of S_0 and S_1 in X enable us to conclude that the intersection of S_0 and S_1 is the union of two lines. Hence there does not exist any cubic curves lying on the intersection part $S_0 \cap S_1$. Finally, the map Ψ is injective on the $\mathbb{P}^3$-bundle over $Gr(3,4)\setminus\Delta$. Since $\dim\mathbb{P}(\mathcal{E}) = \dim\mathbf{S} = 6$, the map Ψ is generically embedding. Thus we proved the claim. □

3. Poincaré polynomial of S

This section is devoted to compute the Poincaré polynomial of **S**. The *virtual Poincaré polynomial* of X is defined by

$$P(X) = \sum (-1)^{i+j} \dim_{\mathbb{Q}} \mathrm{gr}^j_W H^i_c(X,\mathbb{Q}) q^{i/2},$$

where $\mathrm{gr}^j_W H^i_c(X,\mathbb{Q})$ is the j-th weight-graded piece of the mixed Hodge structure on the i-th cohomology of X with compact supports. Since odd cohomology groups of moduli spaces of our interest always vanish, their virtual Poincaré polynomial is a *polynomial* indeed. Let $e(X) := \sum_i (-1)^i \dim H^i(X)$ be the *virtual* Euler number of the variety X. The virtual Poincaré polynomial has the well-known *motivic* properties:

Proposition 3.1.

(1) *$P(X) = P(X - Z) + P(Z)$ for a closed subvariety Z of X.*
(2) *Let X and Y be quasi-projective varieties. Let $\pi : X \to Y$ be a Zariski locally trivial fibration with fiber F. Then $P(X) = P(Y) \cdot P(F)$.*
(3) *Let $f : X \to Y$ be a bijective morphism. Then $P(X) = P(Y)$.*
(4) *If X is a smooth and projective variety, then the virtual Poincaré polynomial is the usual one.*

In (2), if the fiber is $F = Gr(k, n)$, the same conclusion holds even though π is an analytic fibration [1, Lemma 3.1].

Proposition 3.2. *The Poincaré polynomial of* **S** *is given by*

$$1 + 3q + 7q^2 + 10q^3 + 7q^4 + 3q^5 + q^6.$$

Proof. Let us consider the map

$$\Psi : \mathbf{M} = \mathbf{M}(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1, (1,1,1)) \longrightarrow \mathbf{M}(\mathbb{P}^1 \times \mathbb{P}^1, (1,1)) \cong \mathbb{P}^3$$

induced by the projection $\pi_{12} : \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$ into the first two components. As we have seen in the proof of Theorem 1.1, the map Ψ is a $\mathbb{P}^3$-fibration over the complement of the locus $\Delta \cong \mathbb{P}^1 \times \mathbb{P}^1$ of degenerated conics. The inverse image $\Psi^{-1}(\Delta)$ consists of two irreducible components

$$\Psi^{-1}(\Delta) = \Delta_1 \cup \Delta_2$$

such that Δ_1 (resp. Δ_2) consists of the cubic curves $C = L_1 \cup Q$ (resp $C = L_2 \cup Q$) where $[L_1] = (1,0,0) \in H_2(X)$ and $[Q] = (0,1,1) \in H_2(X)$ (resp. $[L_2] = (0,1,0) \in H_2(X)$ and $[Q] = (1,0,1) \in H_2(X)$). Note that $\Delta_1 \cong \Delta_2$ by switching the lines L_1 and L_2. Also $\Delta_1 \cap \Delta_2$ consists of the reducible cubic curves $L = L_1 \cup L_2 \cup L_3$. We show that

$$\Delta_1 = A \cup B,$$

where the locus A parameterizes cubic curves $C = L \cup Q$ such that $[L] = (1,0,0) \in H_2(X)$ and $[Q] = (0,1,1) \in H_2(X)$. Note that $L \cap Q = \{\text{pt}\}$ because $\chi(\mathcal{O}_{LQ}) = 1$. Hence one can easily see that A is a $\mathbb{P}^2$-bundle over $(\mathbb{P}^1)^3$. The second component B parameterizes the closure of the locus of cubic curves $C = L \cup L_2 \cup L_3$ such that $L_2 \cap L_3 = \emptyset$. Therefore, the locus B is a $\mathbb{P}^1$-bundle over $(\mathbb{P}^1)^3$. Note that the intersection $A \cap B$ is isomorphic to a $(\mathbb{P}^1)^3$. Summarizing, we obtain

$$P(\Delta_1) = P(\mathbb{P}^2) \cdot P((\mathbb{P}^1)^3) + P(\mathbb{P}^1) \cdot P((\mathbb{P}^1)^3) - P((\mathbb{P}^1)^3). \tag{1}$$

On the other hand, the intersection $\Delta_1 \cap \Delta_2$ is a union of two irreducible components

$$\Delta_1 \cap \Delta_2 = D \cup E$$

such that D (resp. E) is the locus of cubic curves $C = L_1 \cup L_2 \cup L_3$ such $L_1 \cap L_2 \neq \emptyset$ (resp. $= \emptyset$). Then, by the similar argument as before, we obtain

$$\begin{aligned} P(\Delta_1 \cap \Delta_2) &= P(D) + P(E) - P(D \cap E) \\ &= 2 \cdot P(\mathbb{P}^1) \cdot (P(\mathbb{P}^1))^3 + P(\mathbb{P}^1) \cdot P(\mathbb{P}^1)^3 - 2 \cdot P(\mathbb{P}^1)^3. \end{aligned} \tag{2}$$

By equations (1), (2) and Proposition 3.1, we have

$$\begin{aligned} P(\mathbf{M}) &= P(\mathbb{P}^3) \cdot P(\mathbb{P}^3 - \Delta) + P(\Delta_1) + P(\Delta_2) - P(\Delta_1 \cap \Delta_2) \\ &= P(\mathbb{P}^3) \cdot P(\mathbb{P}^3 - \Delta) + 2 \cdot [P(\mathbb{P}^2) \cdot P((\mathbb{P}^1)^3) + P(\mathbb{P}^1) \cdot P((\mathbb{P}^1)^3) - P((\mathbb{P}^1)^3)] - P(\Delta_1 \cap \Delta_2) \\ &= 1 + 3q + 7q^2 + 10q^3 + 7q^4 + 3q^5 + q^6. \end{aligned}$$

Since $\mathbf{S} \cong \mathbf{M}$ is smooth, we proved the claim. □

Remark 3.3. In particular, the Euler number of **M** is $e(\mathbf{M}) = 32$; this is obtained using the torus localization technique.

Acknowledgement

The first named author was supported by Korea NRF grant 2013R1A1A2006037. The second named author was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (2014R1A1A4A01008404).

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