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Simultaneous empirical interpolation and reduced basis method for non-linear problems

Une méthode EIM bases réduites simultanée pour les équations aux dérivées partielles non linéaires et non affines

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ABSTRACT

In this paper, we focus on the reduced basis methodology in the context of non-linear non-affinely parameterized partial differential equations in which affine decomposition necessary for the reduced basis methodology are not obtained [9,8]. To deal with this issue, it is now standard to apply the EIM methodology [1,3] before deploying the Reduced Basis (RB) methodology. However, the computational cost is generally huge as it requires many finite element solves, hence making it inefficient, to build the EIM approximation of the non-linear terms [3,2]. We propose a simultaneous EIM reduced basis algorithm, named SER, which provides a huge computational gain and requires as little as $N+1$ finite-element solves where N is the dimension of the RB approximation. The paper is organized as follows: we first review the EIM and RB methodologies applied to non-linear problems and identify the main issue, then we present SER and some variants and finally illustrates its performances in a benchmark proposed in [3].

RÉSUMÉ

Dans ce papier, nous nous intéressons à la méthodologie bases réduites (RB) dans le contexte d'équations aux dérivées partielles paramétrisées non linéaires et non affines, pour lesquelles la décomposition affine nécessaire à la méthodologie RB ne peut être obtenue [9,8]. Pour traiter ce problème, il est à présent standard d'appliquer la méthodologie EIM [1,3] avant de déployer la méthodologie RB. Cependant, le coût de calcul de cette approche est en général considérable, car il requiert de nombreuses évaluations éléments finis pour construire l'approximation EIM des termes non linéaires [3,2], ce qui la rend très peu compétitive. Nous proposons l'algorithme SER, qui construit simultanément l'approximation EIM et la méthodologie RB, fournissant ainsi un gain de calcul considérable, et qui requiert au minimum $N+1$ résolutions éléments finis, où N est la dimension de l'approximation RB. Le papier est organisé comme suit : tout d'abord, nous passons en revue les méthodes EIM et RB appliquées aux problèmes non linéaires et identifions la

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difficulté principale, puis nous présentons SER et quelques variantes ; finalement, nous illustrons ses performances sur un *benchmark* proposé par [3].

1. Introduction

Reduced-order modeling is more and more used in engineering problems due to efficient evaluation of quantities of interest. The Reduced Basis Method (see [5,8,9,4,6,7]) has been especially designed for real-time and many-query contexts, and cover a large range of problems, among which non-affinely parameterized Partial Differential Equations (PDE). A core enabler of this method is the so-called offline/online decomposition of the problem. This allows for computing costly parameter independent terms that depend solely on the finite element dimension. However such decomposition is not necessarily or readily available in particular for non-affine/non-linear problems. The Empirical Interpolation Method (EIM, see [1,3]) has been developed to recover this core ingredient and is used prior to the reduced basis methodology on industrial based applications, see e.g. [2]. However the EIM building step can be costly when the terms are non-linear and requires many non-linear finite element solves. It is a deterring trait for this methodology, which forbids its application to non-linear applications.

In this paper, we propose a Simultaneous EIM–RB (named SER) construction that requires only but a few finite-element solves and builds together the affine decomposition as well as the RB ingredient. To start we first give an overview of both Reduced Basis and Empirical Interpolation methods in non-affinely parameterized PDE context to recall the necessary notions. Based on these considerations, a second part makes an assessment of the proposed simultaneous approach, discussing the changes to be made in the EIM offline step. The last part illustrates our method with preliminary results obtained on a benchmark introduced in [3].

2. Preliminaries

Let $u(\boldsymbol{\mu})$ be a solution of a non-linear parameterized PDE, where $\boldsymbol{\mu}$ is the p-vector of inputs ($\boldsymbol{\mu} \in \mathcal{D} \subset \mathbb{R}^p$ the parameter space). Considering the Hilbert space $X \equiv H_0^1(\Omega)$, the variational formulation of the PDE consists in finding $u(\boldsymbol{\mu}) \in X$ as a root of a functional r such that

$$r(u(\boldsymbol{\mu}), v; \boldsymbol{\mu}) = 0 \; \forall v \in X. \tag{1}$$

We denote by $\mathcal{N}$ the dimension of the finite-element approximation space $X^{\mathcal{N}} \subset X$, and $u_{\mathcal{N}}(\boldsymbol{\mu})$ the associated finite-element approximation of $u(\boldsymbol{\mu})$ solution of (1).

In the following, we use a Newton algorithm to deal with the non-linearity. Denoting ${}^k u(\boldsymbol{\mu})$ the solution at kth Newton's iteration, and j the Jacobian associated with the functional r, the problem consists in finding $\delta^{k+1} u(\boldsymbol{\mu}) \equiv {}^{k+1}u(\boldsymbol{\mu}) - {}^k u(\boldsymbol{\mu}) \in X^{\mathcal{N}}$ and thus ${}^{k+1}u(\boldsymbol{\mu}) \in X^{\mathcal{N}}$ such that

$$j(u, v; \boldsymbol{\mu}; {}^k u(\boldsymbol{\mu}))\delta^{k+1} u(\boldsymbol{\mu}) = -r({}^k u(\boldsymbol{\mu}), v; \boldsymbol{\mu}) \; \forall v \in X^{\mathcal{N}} \tag{2}$$

at each iteration k.

If j and r depend affinely in $\boldsymbol{\mu}$, there exists positive integers Q_j and Q_r, $\boldsymbol{\mu}$-dependent functions $\theta_j^{q_j}$, $\theta_r^{q_r}$ and functions j^{q_j}, r^{q_r} ($1 \leqslant q_r \leqslant Q_r$, $1 \leqslant q_j \leqslant Q_j$) allowing to write such affine decompositions as

$$j(u, v; \boldsymbol{\mu}; {}^k u) = \sum_{q_j=1}^{Q_j} \theta_j^{q_j}({}^k u; \boldsymbol{\mu}) j^{q_j}(u, v) \quad r({}^k u, v; \boldsymbol{\mu}) = \sum_{q_r=1}^{Q_r} \theta_r^{q_r}({}^k u; \boldsymbol{\mu}) r^{q_r}(v) \quad \forall u, v \in X^{\mathcal{N}}, \; \forall \boldsymbol{\mu} \in \mathcal{D} \tag{3}$$

For non-affinely parameterized problems, j and/or r depend on at least one non-affine function $w(u(\boldsymbol{\mu}), \boldsymbol{x}; \boldsymbol{\mu})$

$$j(u, v; \boldsymbol{\mu}; {}^k u(\boldsymbol{\mu}); w({}^k u(\boldsymbol{\mu}), \boldsymbol{x}; \boldsymbol{\mu}))\delta^{k+1} u(\boldsymbol{\mu}) = -r({}^k u(\boldsymbol{\mu}), v; \boldsymbol{\mu}; w({}^k u(\boldsymbol{\mu}), \boldsymbol{x}; \boldsymbol{\mu})) \tag{4}$$

and the affine decomposition (3) does not exist due to the definition of $w(u(\boldsymbol{\mu}), \boldsymbol{x}; \boldsymbol{\mu})$. In order to recover an affine decomposition for (4), we build an affine approximation $w_M(u(\boldsymbol{\mu}), \boldsymbol{x}; \boldsymbol{\mu})$ of $w(u(\boldsymbol{\mu}), \boldsymbol{x}; \boldsymbol{\mu})$

$$w_M(u(\boldsymbol{\mu}), \boldsymbol{x}; \boldsymbol{\mu}) = \sum_{m=1}^{M} \beta_m^M(u(\boldsymbol{\mu}); \boldsymbol{\mu}) q_m(\boldsymbol{x}) \tag{5}$$

2.1. Empirical interpolation method

In order to build an affine approximation $w_M(u(\boldsymbol{\mu}), \boldsymbol{x}; \boldsymbol{\mu}) = \sum_{m=1}^{M} \beta_m^M(u(\boldsymbol{\mu}); \boldsymbol{\mu}) q_m(\boldsymbol{x})$ of a non-affine parameter-dependent function w, we first introduce a sample $\bar{S}_M = \{\bar{\boldsymbol{\mu}}_1, \dots, \bar{\boldsymbol{\mu}}_M\} \in \mathcal{D}^M$, and the associated function space $\bar{W}_M = \text{span}\{\bar{\boldsymbol{\xi}}_m \equiv w(u(\bar{\boldsymbol{\mu}}_m), \boldsymbol{x}; \bar{\boldsymbol{\mu}}_m), 1 \leqslant m \leqslant M\}$. These sets are built from a subset Ξ of $\mathcal{D}$, in which the first sample point $\bar{\boldsymbol{\mu}}_1$ is picked (assuming that $\bar{\boldsymbol{\xi}}_1 \neq 0$). The basis functions q_m are based on $\bar{\boldsymbol{\xi}}_m$, and establishing β_m^M coefficients requires, online, the solution of a $M \times M$ system, ensuring that the EIM approximation $w_M(u, \boldsymbol{x}; \boldsymbol{\mu})$ is exact on a set of interpolation points $\{t_i\}_i$:

$$\bar{S}_1 = \{\bar{\boldsymbol{\mu}}_1\}, \quad \bar{\boldsymbol{\xi}}_1 = w(u(\bar{\boldsymbol{\mu}}_1), \boldsymbol{x}; \bar{\boldsymbol{\mu}}_1), \quad \bar{W}_1 = \text{span}\{\bar{\boldsymbol{\xi}}_1\} \quad \boldsymbol{t}_1 = \arg\sup_{\boldsymbol{x}\in\Omega} |\bar{\boldsymbol{\xi}}_1(\boldsymbol{x})|, \quad q_1 = \frac{\bar{\boldsymbol{\xi}}_1(\boldsymbol{x})}{\bar{\boldsymbol{\xi}}_1(\boldsymbol{t}_1)} \tag{6}$$

For $M \geqslant 2$, the sample points $\bar{\boldsymbol{\mu}}_M$ are determined from a Greedy algorithm:

$$\begin{aligned} \bar{\boldsymbol{\mu}}_M &= \arg\max_{\boldsymbol{\mu}\in\Xi} \inf_{z\in W_{M-1}} ||w(.;.;\boldsymbol{\mu}) - z||_{L^\infty(\Omega)} \\ \bar{\boldsymbol{\xi}}_M &= w\left(u(\bar{\boldsymbol{\mu}}_M), \boldsymbol{x}; \bar{\boldsymbol{\mu}}_M\right), \quad \bar{S}_M = \bar{S}_{M-1} \cup \{\bar{\boldsymbol{\mu}}_M\}, \quad \bar{W}_M = \bar{W}_{M-1} \oplus \text{span}\{\bar{\boldsymbol{\xi}}_M\} \end{aligned} \tag{7}$$

The system ensuring the exactness of w_{M-1} at $\{t_i\}_{i=1}^{M-1}$, gives $w_{M-1}(u(\bar{\boldsymbol{\mu}}_M), \boldsymbol{x}; \bar{\boldsymbol{\mu}}_M)$, leading to the residual $\boldsymbol{r}_M$ defined as $\boldsymbol{r}_M(\boldsymbol{x}) = w(u(\bar{\boldsymbol{\mu}}_M), \boldsymbol{x}; \bar{\boldsymbol{\mu}}_M) - w_{M-1}(u(\bar{\boldsymbol{\mu}}_M), \boldsymbol{x}; \bar{\boldsymbol{\mu}}_M)$, on which interpolation points $\{\boldsymbol{t}_M\}_{M\geqslant 2}$ and basis functions $\{q_M\}_{M\geqslant 2}$ are computed. The next $\boldsymbol{t}_M$ and $\boldsymbol{q}_M$ are given by

$$\boldsymbol{t}_M = \arg\sup_{\boldsymbol{x}\in\Omega} |\boldsymbol{r}_M(\boldsymbol{x})|, \quad \boldsymbol{q}_M(\boldsymbol{x}) = \frac{\boldsymbol{r}_M(\boldsymbol{x})}{\boldsymbol{r}_M(\boldsymbol{t}_M)} \tag{8}$$

We are now ready to apply EIM to the Jacobian j; the affine decomposition reads

$$j(u, v; \boldsymbol{\mu}; {}^k u; w_M(u(\boldsymbol{\mu}), \boldsymbol{x}; \boldsymbol{\mu})) = \underbrace{\sum_{q=1}^{Q_j^{\text{eim}}} \sum_{m=1}^{M_q^j} \gamma_{j,m}^q({}^k u; \boldsymbol{\mu}) j_m^q(u, v)}_{\text{non-affine part of } j} + \underbrace{\sum_{l=1}^{Q_j^{\text{aff}}} \theta_j^l({}^k u; \boldsymbol{\mu}) j^l(u, v)}_{\text{affine part of } j} \quad \forall u, v \in X, \ \forall \boldsymbol{\mu} \in \mathcal{D}, \tag{9}$$

with Q_j^{eim} (resp. Q_j^{aff}) the number of non-affinely (resp. affinely) parameterized terms of j, and $\gamma_{j,m}^q$ (resp. j_m^q) obtained from the EIM approximations. Similar decomposition is obtained for the residual r.

2.2. Reduced-basis method

We now turn to the RB methodology. We introduce $S_N = \{\boldsymbol{\mu}_1, \cdots, \boldsymbol{\mu}_N\}$ with $N << \mathcal{N}$, and we define the set of solutions $S_N^u = \{u_{\mathcal{N}}(\boldsymbol{\mu}_i)\}_{i=1}^N$, computed from the affine decomposition (9) with

$$\begin{aligned} &j(u_{\mathcal{N}}(\boldsymbol{\mu}_i), v; \boldsymbol{\mu}_i; {}^k u_{\mathcal{N}}(\boldsymbol{\mu}_i); w_M({}^k u_{\mathcal{N}}(\boldsymbol{\mu}_i), \boldsymbol{x}; \boldsymbol{\mu}_i))\delta^{k+1} u_{\mathcal{N}}(\boldsymbol{\mu}_i) = -r({}^k u_{\mathcal{N}}(\boldsymbol{\mu}_i), v; \boldsymbol{\mu}_i; w_M({}^k u_{\mathcal{N}}(\boldsymbol{\mu}_i), \boldsymbol{x}; \boldsymbol{\mu}_i)) \\ &\quad \forall v \in X, \forall i \in [1, N] \end{aligned} \tag{10}$$

and orthonormalized with respect to the $<,>_X$ inner product to provide $W_N = \{\boldsymbol{\xi}_n\}_{i=1}^N$.

The reduced-basis approximation $u_N(\boldsymbol{\mu}) \in W_N$ is expressed as a linear combination of W_N elements

$$u_N(\boldsymbol{\mu}) = \sum_{i=1}^{N} u_{N,i}(\boldsymbol{\mu}) \boldsymbol{\xi}_i \ \text{ such that } \ r(u_N, v; \boldsymbol{\mu}; w(u_N, \boldsymbol{x}; \boldsymbol{\mu})) = 0 \ \forall v \in W_N \tag{11}$$

The offline/online strategy used to build efficiently the reduced basis approximation is based on the affine decomposition (3), (9). Choosing $\{\boldsymbol{\xi}_n\}_{i=1}^N$ as test functions and using (11), we have for $1 \leqslant l \leqslant N$:

$$\sum_{j=1}^{N} j(\boldsymbol{\xi}_j, \boldsymbol{\xi}_l; \boldsymbol{\mu}; {}^k u_N; w({}^k u_N, \boldsymbol{x}; \boldsymbol{\mu}))({}^{k+1} u_{N,j} - {}^k u_{N,j}) = -r(\boldsymbol{\xi}_l; \boldsymbol{\mu}; {}^k u_N; w({}^k u_N, \boldsymbol{x}; \boldsymbol{\mu})) \tag{12}$$

The offline step consists in the pre-computation of the terms $j_m^q(\boldsymbol{\xi}_j, \boldsymbol{\xi}_l)$, $j^l(\boldsymbol{\xi}_j, \boldsymbol{\xi}_l)$ and $r_m^q(\boldsymbol{\xi}_l), r^l(\boldsymbol{\xi}_l)$, and is done offline once thanks to the affine decomposition. To evaluate online ${}^{k+1} u_N(\boldsymbol{\mu})$ for any given $\boldsymbol{\mu}$, the coefficients $\gamma_{j,m}^q({}^k u_N, \boldsymbol{\mu})$, $\theta_j^l({}^k u_N, \boldsymbol{\mu})$ and $\gamma_{r,m}^q({}^k u_N, \boldsymbol{\mu}), \theta_r^l({}^k u_N, \boldsymbol{\mu})$ are computed in order to recover the $N \times N$ system (11) giving ${}^{k+1} u_{N,j}$ coefficients and consequently ${}^{k+1} u_N(\boldsymbol{\mu})$.

3. A simultaneous EIM–RB method

The standard methodology described previously briefly requires the use of EIM prior to the RB methodology for each affine-parameter-dependent function w (4). These EIM approximations are built upfront, allowing us to then write the affine decomposition (9). However, the greedy algorithm (7) of the EIM offline step requires the computation of the solution $u_{\mathcal{N}}(\boldsymbol{\mu})$ for all points of $\Xi \subset \mathcal{D}$, *i.e* in the step (7).

The SER methodology proposes to reduce the computational cost by simply using the readily available reduced basis approximation based on the previous EIM step and build simultaneously EIM and RB. In particular the expensive step (7) will now use solely RB approximations.

The initialization of EIM offline stage (6) doesn't change since no reduced approximation $u_N(\boldsymbol{\mu})$ is available yet. The resulting rough EIM approximations w_1 of each non-affine function w are used to compute a first affine decomposition from which $\boldsymbol{\xi}_1$ can be built, leading to a first reduced approximation $u_1(\boldsymbol{\mu})$. After this initialization step, EIM and RB approximation spaces are enriched alternatively. Each new EIM basis function q_M: *(i)* is built from a reduced-basis approximation $u_{M-1}(\boldsymbol{\mu})$ obtained at previous iteration $M-1$; *(ii)* completes the EIM approximation to then build the affine decomposition for $u_M(\boldsymbol{\mu})$. To summarize, the EIM steps modified by SER now read:

$$\bar{\boldsymbol{\mu}}_M = \arg\max_{\boldsymbol{\mu}\in\Xi} \inf_{z\in W_{M-1}} ||w(u_{M-1}(\boldsymbol{\mu}); .; \boldsymbol{\mu}) - z||_{L^\infty(\Omega)} \tag{13}$$

$$\bar{\boldsymbol{\xi}}_M = w\left(u_{M-1}(\bar{\boldsymbol{\mu}}_M); \boldsymbol{x}; \bar{\boldsymbol{\mu}}_M\right), \quad \bar{S}_M = \bar{S}_{M-1} \cup \{\bar{\boldsymbol{\mu}}_M\}, \quad \bar{W}_M = \bar{W}_{M-1} \oplus \operatorname{span}\{\bar{\boldsymbol{\xi}}_M\} \tag{14}$$

The residual $\boldsymbol{r}_M(\boldsymbol{x}) = w(u_{M-1}(\bar{\boldsymbol{\mu}}_M), \boldsymbol{x}; \bar{\boldsymbol{\mu}}_M) - w_{M-1}(u_{M-1}(\bar{\boldsymbol{\mu}}_M), \boldsymbol{x}; \bar{\boldsymbol{\mu}}_M)$ then gives $\boldsymbol{t}_M$ and q_M:

$$\boldsymbol{t}_M = \arg\sup_{\boldsymbol{x}\in\Omega} |\boldsymbol{r}_M(\boldsymbol{x})|, \quad q_M(\boldsymbol{x}) = \frac{\boldsymbol{r}_M(\boldsymbol{x})}{\boldsymbol{r}_M(\boldsymbol{t}_M)} \tag{15}$$

The affine decomposition is then updated with the new EIM approximations to compute the next reduced basis $\boldsymbol{\xi}_M$ thanks to (10), to then enrich the W_M space.

Remark 1 *(RB updates).* The EIM approximations are changing during the building of the W_N space. Then, the operator – and the solved problem – evolves at each step, which can mildly deteriorate the approximation. Recomputing W_N elements using finite-element solves based on the current EIM approximation may be considered. The offline precomputations need, however, to be updated for all elements of W_N for each update of the affine decomposition.

Remark 2 *(EIM updates).* In the SER methodology, only one finite element solve of (1) is required for the initialization; afterwards, the EIM and RB approximations are updated alternatively. We may in fact update the RB approximation every r EIM steps: if $r = M$, we recover the standard method, if $r = 1$ we recover the SER method and if $1 < r < M$, we have an intermediary method that requires finite-element solves for the intensive EIM step (7) before the first RB update. Other alternatives are readily available and some will be discussed in a future publication.

4. Preliminary results

We now turn to numerical experiments of the SER method compared to the standard one based on the benchmark problem introduced in [3]. It reads, find u such that

$$-\Delta u + \mu_1 \frac{e^{\mu_2 u} - 1}{\mu_2} = 100 \sin(2\pi x)\sin(2\pi y) \text{ in } \Omega =]0,1[^2 \text{ and } \boldsymbol{\mu} = (\mu_1, \mu_2) \in \mathcal{D} = [0.01, 10]^2 \tag{16}$$

and we are interested in the output s, the average of the solution u over Ω. Thanks to the function $g(u, x; \boldsymbol{\mu}) = \mu_1 \frac{e^{\mu_2 u}-1}{\mu_2}$, we are in the setting of the SER methodology for non-linear problems. Following the standard methodology, we develop $g_M = \sum_{i=1}^{M} \beta_m^M(u, \boldsymbol{\mu}) q_m(\boldsymbol{x})$ of g thanks to the Empirical Interpolation Method (see Section 2.1). The training set $\Xi \subset \mathcal{D}$ used for EIM is composed of 15 elements in each direction, which represents a subset of size 225. We shall use the absolute errors on the solution and the output, defined as follows:

$$\epsilon_{M,N}^{u,r} = \| u_{\mathcal{N}} - u_N^r \|_{L_2} \quad \epsilon_{M,N}^{s,r} = |s_{\mathcal{N}} - s_N^r|, \tag{17}$$

where $\cdot_{\mathcal{N}}$ is the finite element solution/output of the initial problem (2), $\cdot_N^r$ the reduced basis solution/output, while r is the frequency at which the EIM are updated, see Remark 2.

We first display the maximum of the absolute errors (17) obtained with a test set composed of 500 inputs – selected randomly in $[0.01, 10]^2$ – using the standard method $r = M$ in Table 1a, which reproduces the results in [3]. Tables 1c and 1d correspond to the SER methodology ($r = 1$) with $N = M$ and investigate the influence of W_N recomputation in the SER method, see Remark 1. Table 1b is an intermediary stage such that $1 < r < M$, where the EIM basis functions are built

Table 1
SER algorithm: Maximum absolute errors on solution u and on output s.

N	M	$\max(\epsilon^{u,M}_{M,N})$	$\max(\epsilon^{s,M}_{M,N})$
4	5	7.38e−3	5.75e−3
8	10	1.01e−3	2.34e−4
12	15	1.49e−4	3.09e−5
16	20	2.21e−5	1.25e−5
20	25	5.88e−6	2.82e−6

(a) $r = M$

N	M	$\max(\epsilon^{u,5}_{M,N})$	$\max(\epsilon^{s,5}_{M,N})$
4	5	8.21e−3	6.31e−3
8	10	4.48e−3	6.18e−3
12	15	2.69e−4	2.36e−4
16	20	1.48e−4	9.31e−5
20	25	2.60e−5	1.46e−5

(b) $r = 5$

N	M	$\max(\epsilon^{u,1}_{M,N})$	$\max(\epsilon^{s,1}_{M,N})$
5	5	9.98e−3	7.77e−3
10	10	2.32e−3	1.86e−3
15	15	4.61e−4	3.75e−4
20	20	2.48e−4	2.02e−4
25	25	3.51e−5	2.33e−5

(c) $r = 1$
(W_N recomputed)

N	M	$\max(\epsilon^{u,1}_{M,N})$	$\max(\epsilon^{s,1}_{M,N})$
5	5	1.30e−2	1.02e−2
10	10	2.20e−3	1.50e−3
15	15	4.83e−4	4.05e−4
20	20	2.42e−4	1.98e−4
25	25	1.50e−5	1.24e−5

(d) $r = 1$
(W_N not recomputed)

by groups of r ($r = 5$). In this last case, since we do not have access to the RB yet, the first group of r EIM basis is built from finite-element approximations and reduced-basis approximations are used afterwards. This explains, for example, that $(N = 4, M = 5)$ is similar to the first one in Table 1a.

The maximum of the errors observed with the SER method ($r = 1$) is slightly higher than with the case $r = 5$, which itself displays results slightly higher than those obtained with the standard method $r = M$. This behavior was expected. The results show the pertinence of the SER method and evidence that we can expect good results within a reasonable computational budget, since the number of finite-element approximations can then be reduced to $N + 1$ (Table 1d).

5. Conclusion

We have now an algorithm and variants that allow an efficient use of EIM and RB in the context of non-linear non-affine partial differential equations. We have already deployed SER in real applications and the initial results are very promising, showing the same behavior as the benchmark problem of this paper. This will be reported as well as an analysis of SER in a subsequent paper with applications to non-linear multiphysic problems.

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