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Complex analysis

Generalizations of starlike harmonic functions

Généralisations des fonctions harmoniques étoilées

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ABSTRACT

In this paper we investigate some generalizations of classes of harmonic functions. By using the extreme points theory we obtain coefficients estimates distortion theorems and integral mean inequalities in these classes of functions.

RÉSUMÉ

Dans cette Note, nous étudions des généralisations des classes de fonctions harmoniques liées aux fonctions de Janowski. En utilisant la théorie des points extrémaux, nous obtenons des estimations de coefficients, des théorèmes de distorsion et des inégalités de moyenne intégrale dans ces classes de fonctions.

1. Introduction

Harmonic functions are famous for their use in the study of minimal surfaces and also play important roles in a variety of problems in applied mathematics. Harmonic functions have been studied by differential geometers such as Choquet [1], Kneser [8], Lewy [9], and Rado [11]. Recent interest in harmonic complex functions has been triggered by geometric function theorists Clunie and Sheil-Small [2]. Let $\mathcal{H}$ denote the family of continuous complex-valued functions that are harmonic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ and let $\mathcal{A}$ denote the class of functions that are analytic in $\mathbb{U}$. By $\mathcal{S}_{\mathcal{H}}$ we denote the family of functions $f \in \mathcal{H}$ of the form

$$f(z) = h + \overline{g},\ h(z) = z + \sum_{k=2}^{\infty} a_k z^k,\ \ g(z) = \sum_{k=2}^{\infty} b_k z^k \quad (z \in \mathbb{U})\,, \tag{1}$$

which are univalent and sense-preserving in $\mathbb{U}$. The class $\mathcal{S}_{\mathcal{H}}$ was studied by Clunie and Sheil-Small [2] and Sheil-Small [13] together with some geometric subclasses of $\mathcal{S}_{\mathcal{H}}$. In particular, they investigated harmonic starlike functions and harmonic

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convex functions, which are defined as follows. We say that $f \in \mathcal{S}_{\mathcal{H}}$ is a harmonic starlike function if $f(\mathbb{U})$ is a starlike domain with respect to the origin. Likewise $f \in \mathcal{S}_{\mathcal{H}}$ is said to be a harmonic convex function if $f(\mathbb{U})$ is a convex domain.

In [12] Ruscheweyh introduced an operator $\mathcal{D}^n : \mathcal{A} \to \mathcal{A}$, $n \in \mathbb{N}_0 := \{0, 1, 2, \ldots\}$, defined by

$$\mathcal{D}^n f(z) = \frac{z\left(z^{n-1} f(z)\right)^{(n)}}{n!} \quad (z \in \mathbb{U}).$$

Motivated by the Ruscheweyh derivative $\mathcal{D}^n$, we consider the linear operator $\mathcal{D}^n_{\mathcal{H}} : \mathcal{H} \to \mathcal{H}$ defined for a function $f = h + \overline{g} \in \mathcal{H}$ by $\mathcal{D}^n_{\mathcal{H}} f := \mathcal{D}^n h + (-1)^n \overline{\mathcal{D}^n g}$. For a function $f \in \mathcal{H}$ of the form (1), we have

$$\mathcal{D}^n_{\mathcal{H}} f(z) = z + \sum_{k=2}^{\infty} \lambda_{k,n} a_k z^k + (-1)^n \sum_{k=2}^{\infty} \lambda_{k,n} \overline{b}_k \overline{z}^k \quad (z \in \mathbb{U}),$$

where

$$\lambda_{k,0} := 1, \; \lambda_{k,n} := \frac{k \cdot \ldots \cdot (k+n-1)}{n!} \quad (n \in \mathbb{N} := \{1, 2, \ldots\}). \tag{2}$$

We say that a function $f \in \mathcal{H}$ is *subordinate* to a function $F \in \mathcal{H}$, and write $f(z) \prec F(z)$ (or simply $f \prec F$) if there exists a complex-valued function ω that maps $\mathbb{U}$ into oneself with $\omega(0) = 0$ such that $f(z) = F(\omega(z))$ $(z \in \mathbb{U})$.

Let n be nonnegative integer and let $-B \le A < B \le 1$, $0 \le \alpha < 1$. We denote by $\mathcal{S}^n_{\mathcal{H}}(A, B)$ the class of functions $f \in \mathcal{S}_{\mathcal{H}}$ such that

$$\frac{\mathcal{D}^{n+1}_{\mathcal{H}} f(z)}{\mathcal{D}^n_{\mathcal{H}} f(z)} \prec \frac{1 + Az}{1 + Bz}. \tag{3}$$

The classes $\mathcal{S}^*_{\mathcal{H}}(\alpha) := \mathcal{S}^0_{\mathcal{H}}(2\alpha - 1, 1)$ and $\mathcal{S}^c_{\mathcal{H}}(\alpha) := \mathcal{S}^1_{\mathcal{H}}(2\alpha - 1, 1)$ were investigated by Jahangiri [6,7]. Moreover, we should notice that each function from the class $\mathcal{S}^*_{\mathcal{H}} := \mathcal{S}^*_{\mathcal{H}}(0)$ is a harmonic starlike function, and each function from the class $\mathcal{S}^c_{\mathcal{H}} := \mathcal{S}^c_{\mathcal{H}}(0)$ is a harmonic convex function.

2. Coefficients conditions

Theorem 1. *A function $f \in \mathcal{H}$ of the form (1) belongs to the class $\mathcal{S}^n_{\mathcal{H}}(A, B)$ if it satisfies the condition*

$$\sum_{k=2}^{\infty} (\alpha_k |a_k| + \beta_k |b_k|) \le B - A, \tag{4}$$

where

$$\alpha_k = \lambda_{k,n}\left((1+B)k - (1+A)\right), \quad \beta_k = \lambda_{k,n}\left((1+B)k + (1+A)\right). \tag{5}$$

Proof. It is clear that the theorem is true for the function $f(z) \equiv z$. Let $f \in \mathcal{H}$ be a function of the form (1) and let us assume that there exists $m \in \{2, 3, \ldots\}$ such that $a_m \ne 0$ or $b_m \ne 0$. Since $\frac{\alpha_k}{B-A} \ge k$, $\frac{\beta_k}{B-A} \ge k$ $(k = 2, 3, \ldots)$, then by (4) we have

$$\sum_{k=2}^{\infty} (k |a_k| + k |b_k|) \le 1 \tag{6}$$

and

$$\begin{aligned} \left|h'(z)\right| - \left|g'(z)\right| &\ge 1 - \sum_{k=2}^{\infty} k |a_k| |z|^k - \sum_{k=2}^{\infty} k |b_k| |z|^k \ge 1 - |z| \sum_{k=2}^{\infty} (k |a_k| + k |b_k|) \\ &\ge 1 - \frac{|z|}{B-A} \sum_{k=2}^{\infty} (\alpha_k |a_k| + \beta_k |b_k|) \ge 1 - |z| > 0 \quad (z \in \mathbb{U}). \end{aligned}$$

Thus, the function f is locally univalent and sense-preserving in $\mathbb{U}$. Moreover, if $z_1, z_2 \in \mathbb{U}$, $z_1 \ne z_2$, then $\left|\frac{z_1^k - z_2^k}{z_1 - z_2}\right| = \left|\sum_{l=1}^{k} z_1^{l-1} z_2^{k-l}\right| \le \sum_{l=1}^{k} |z_1|^{l-1} |z_2|^{k-l} < k$ $(k = 2, 3, \ldots)$. Hence, by (6), we have

$$\begin{aligned}|f(z_1)-f(z_2)| &\geq |h(z_1)-h(z_2)|-|g(z_1)-g(z_2)|\\ &\geq \left|z_1-z_2-\sum_{k=2}^{\infty}a_k\left(z_1^k-z_2^k\right)\right|-\left|\sum_{k=2}^{\infty}\overline{b_k\left(z_1^k-z_2^k\right)}\right|\\ &\geq |z_1-z_2|\left(1-\sum_{k=2}^{\infty}|a_k|\left|\frac{z_1^k-z_2^k}{z_1-z_2}\right|-\sum_{k=2}^{\infty}|b_k|\left|\frac{z_1^k-z_2^k}{z_1-z_2}\right|\right)\\ &> |z_1-z_2|\left(1-\sum_{k=2}^{\infty}k|a_k|-\sum_{k=2}^{\infty}k|b_k|\right)\geq 0.\end{aligned}$$

This leads to the univalence of f *i.e.* $f\in\mathcal{S}_{\mathcal{H}}$. Therefore, $f\in\mathcal{S}_{\mathcal{H}}^n(A,B)$ if and only if there exists a complex-valued function ω, $\omega(0)=0$, $|\omega(z)|<1$ $(z\in\mathbb{U})$ such that $\frac{\mathcal{D}_{\mathcal{H}}^{n+1}f(z)}{\mathcal{D}_{\mathcal{H}}^{n}f(z)}=\frac{1+A\omega(z)}{1+B\omega(z)}$ $(z\in\mathbb{U})$, or equivalently

$$\left|\frac{\mathcal{D}_{\mathcal{H}}^{n+1}f(z)-\mathcal{D}_{\mathcal{H}}^{n}f(z)}{B\mathcal{D}_{\mathcal{H}}^{n+1}f(z)-A\mathcal{D}_{\mathcal{H}}^{n}f(z)}\right|<1\quad(z\in\mathbb{U}). \tag{7}$$

Thus, it is sufficient to prove that $\left|\mathcal{D}_{\mathcal{H}}^{n+1}f(z)-\mathcal{D}_{\mathcal{H}}^{n}f(z)\right|-\left|B\mathcal{D}_{\mathcal{H}}^{n+1}f(z)-A\mathcal{D}_{\mathcal{H}}^{n}f(z)\right|<0$ $(z\in\mathbb{U}\setminus\{0\})$. Indeed, letting $|z|=r$ $(0<r<1)$ we have

$$\begin{aligned}&\left|\mathcal{D}_{\mathcal{H}}^{n+1}f(z)-\mathcal{D}_{\mathcal{H}}^{n}f(z)\right|-\left|B\mathcal{D}_{\mathcal{H}}^{n+1}f(z)-A\mathcal{D}_{\mathcal{H}}^{n}f(z)\right|\\ &\leq\sum_{k=2}^{\infty}\lambda_{k,n}(k-1)|a_k|r^k+\sum_{k=2}^{\infty}\lambda_{k,n}(k+1)|b_k|r^k-(B-A)r\\ &\quad+\sum_{k=2}^{\infty}\lambda_{k,n}(Bn-A)|a_k|r^k+\sum_{k=2}^{\infty}\lambda_{k,n}(Bn+A)|b_k|r^k\\ &\leq r\left\{\sum_{k=2}^{\infty}(\alpha_k|a_k|+\beta_k|b_k|)r^{k-1}-(B-A)\right\}<0\end{aligned}$$

whence $f\in\mathcal{S}_{\mathcal{H}}^n(A,B)$. □

Motivated by Silverman [14], we denote by $\mathcal{S}_{\mathcal{T}}^n(A,B)$ the class of functions $f\in\mathcal{S}_{\mathcal{H}}^n(A,B)$ of the form (1) such that $a_k=-|a_k|$, $b_k=(-1)^n|b_k|$ $(k=2,3,\ldots)$ *i.e.*

$$f=h+\overline{g},\quad h(z)=z-\sum_{k=2}^{\infty}|a_k|z^k,\quad g(z)=(-1)^n\sum_{k=2}^{\infty}|b_k|\overline{z}^k\quad(z\in\mathbb{U}). \tag{8}$$

Now, we show that the condition (4) is also the sufficient condition for a function to be in the class $\mathcal{S}_{\mathcal{T}}^n(A,B)$.

Theorem 2. *Let $f\in\mathcal{H}$ be a function of the form (8). Then $f\in\mathcal{S}_{\mathcal{T}}^n(A,B)$ if and only if the condition (4) holds true.*

Proof. In view of Theorem 1, we need only show that each function $f\in\mathcal{S}_{\mathcal{T}}^n(A,B)$ satisfies the coefficient inequality (4). If $f\in\mathcal{S}_{\mathcal{T}}^n(A,B)$, then it satisfies (7) or equivalently

$$\left|\frac{\sum_{k=2}^{\infty}\lambda_{k,n}\left\{(k-1)|a_k|z^k+(k+1)|b_k|\overline{z}^k\right\}}{(B-A)z-\sum_{k=2}^{\infty}\lambda_{k,n}\left\{(Bn-A)|a_k|z^k+(Bn+A)|b_k|\overline{z}^k\right\}}\right|<1\quad(z\in\mathbb{U}).$$

Therefore, putting $z=r$ $(0\leq r<1)$ we obtain:

$$\frac{\sum_{k=2}^{\infty}\lambda_{k,n}\{(k-1)|a_k|+(k+1)|b_k|\}r^{k-1}}{(B-A)-\sum_{k=2}^{\infty}\lambda_{k,n}\{(Bn-A)|a_k|+(Bn+A)|b_k|\}r^{k-1}}<1. \tag{9}$$

It is clear that the denominator of the left-hand side can not vanish for $r \in \langle 0, 1)$. Moreover, it is positive for $r = 0$, and in consequence for $r \in \langle 0, 1)$. Thus, by (9) we have $\sum_{k=2}^{\infty} (\alpha_k |a_k| + \beta_k |b_k|) r^{k-1} < B - A$, which, upon letting $r \to 1^-$, readily yields assertion (4). □

3. Topological properties

We consider the usual topology on $\mathcal{H}$ defined by a metric in which a sequence $\{f_k\}$ in $\mathcal{H}$ converges to f if and only if it converges to f uniformly on each compact subset of $\mathbb{U}$. It follows from the theorems of Weierstrass and Montel that this topological space is complete.

Let $\mathcal{F}$ be a subclass of class $\mathcal{H}$. A function $f \in \mathcal{F}$ is called *an extreme point of* $\mathcal{F}$ if the condition $f = \gamma f_1 + (1-\gamma) f_2$ $(f_1, f_2 \in \mathcal{F},\ 0 < \gamma < 1)$ implies $f_1 = f_2 = f$. We shall use the notation $E\mathcal{F}$ to denote the set of all extreme points of $\mathcal{F}$. It is clear that $E\mathcal{F} \subset \mathcal{F}$.

A real-valued functional $\mathcal{J} : \mathcal{H} \to \mathbb{R}$ is called *convex* on a convex class $\mathcal{F} \subset \mathcal{H}$ if

$$\mathcal{J}(\gamma f + (1-\gamma) g) \le \gamma \mathcal{J}(f) + (1-\gamma) \mathcal{J}(g) \quad (f, g \in \mathcal{F},\ 0 \le \gamma \le 1).$$

The Krein–Milman theorem implies the following important result (see [5, p. 45]).

Lemma 1. *Let $\mathcal{F}$ be a non-empty compact subclass of the class $\mathcal{H}$ and $\mathcal{J} : \mathcal{H} \to \mathbb{R}$ be a real-valued, continuous and convex functional on $\mathcal{F}$. Then*

$$\max\{\mathcal{J}(f) : f \in \mathcal{F}\} = \max\{\mathcal{J}(f) : f \in E\mathcal{F}\}.$$

Since $\mathcal{H}$ is a complete metric space, Montel's theorem (see [10]) implies the following lemma.

Lemma 2. *A class $\mathcal{F} \subset \mathcal{H}$ is compact if and only if $\mathcal{F}$ is closed and locally uniformly bounded.*

Theorem 3. *The class $\mathcal{S}_{\mathcal{T}}^{n}(A, B)$ is a convex and compact subset of $\mathcal{H}$.*

Proof. Let $0 \le \gamma \le 1$ and let $f_l \in \mathcal{S}_{\mathcal{T}}^{n}(A, B)$ be functions of the form

$$f_l(z) = \sum_{k=0}^{\infty} a_{l,k} z^k + \sum_{k=1}^{\infty} \overline{b_{l,k}} \overline{z}^k \quad (z \in \mathbb{U},\ l \in \mathbb{N}). \tag{10}$$

Since

$$\gamma f_1(z) + (1-\gamma) f_2(z) = z + \sum_{k=2}^{\infty} \left\{ \left(\gamma a_{1,k} + (1-\gamma) a_{2,k}\right) z^k + \overline{\left(\gamma b_{1,k} + (1-\gamma) b_{2,k}\right)} \overline{z}^k \right\},$$

and by Theorem 2, we have

$$\begin{aligned}
&\sum_{k=2}^{\infty} \left\{ \alpha_k \left|\gamma a_{1,k} + (1-\gamma) a_{2,k}\right| + \beta_k \left|\gamma b_{1,k} + (1-\gamma) b_{2,k} z^k\right| \right\} \\
&\quad \le \gamma \sum_{k=2}^{\infty} \left\{ \alpha_k \left|a_{1,k}\right| + \beta_k \left|b_{1,k}\right| \right\} + (1-\gamma) \sum_{k=2}^{\infty} \alpha_k \left|a_{2,k}\right| + \beta_k \left|b_{2,k}\right| \\
&\quad \le \gamma (B - A) + (1-\gamma)(B - A) = B - A,
\end{aligned}$$

the function $\phi = \gamma f_1 + (1-\gamma) f_2$ belongs to the class $\mathcal{S}_{\mathcal{T}}^{n}(A, B)$. Hence, the class is convex. Furthermore, for $f \in \mathcal{S}_{\mathcal{T}}^{n}(A, B)$, $|z| \le r$, $0 < r < 1$, we have

$$|f(z)| \le r + \sum_{k=2}^{\infty} (|a_k| + |b_k|) r^k \le r + \sum_{k=2}^{\infty} (\alpha_k |a_k| + \beta_k |b_k|) \le r + (B - A). \tag{11}$$

Thus, we conclude that the class $\mathcal{S}_{\mathcal{T}}^{n}(A, B)$ is locally uniformly bounded. By Lemma 2, we only need to show that it is closed, *i.e.* if $f_l \in \mathcal{S}_{\mathcal{T}}^{n}(A, B)$ $(l \in \mathbb{N})$ and $f_l \to f$, then $f \in \mathcal{S}_{\mathcal{T}}^{n}(A, B)$. Let f_l and f be given by (10) and (1), respectively. Using Theorem 2, we have $\sum_{k=2}^{\infty} \left(\alpha_k \left|a_{l,k}\right| + \beta_k \left|b_{l,k}\right|\right) \le B - A$ $(l \in \mathbb{N})$. Since $f_l \to f$, we conclude that $\left|a_{l,k}\right| \to |a_k|$ and $\left|b_{l,k}\right| \to |b_k|$ as $l \to \infty$ $(k \in \mathbb{N})$. This gives condition (4), and, in consequence, $f \in \mathcal{S}_{\mathcal{T}}^{n}(A, B)$, which completes the proof. □

Theorem 4.

$$E\mathcal{S}_{\mathcal{T}}^{n}(A,B)=\{h_k: k\in\mathbb{N}\}\cup\{g_k: k\in\{2,3,\dots\}\},$$

where

$$h_1(z)=z,\ h_k(z)=z-\frac{B-A}{\alpha_k}z^k,\ g_k(z)=z+(-1)^n\frac{B-A}{\beta_k}\overline{z}^k\ \ (z\in\mathbb{U}). \tag{12}$$

Proof. By using (4), we easily verify that the functions of the form (12) are the extreme points of the class $\mathcal{S}_{\mathcal{T}}^{n}(A,B)$. Now, suppose that $f\in E\mathcal{S}_{\mathcal{T}}^{n}(A,B)$ and f is not of the form (12). Then there exists $m\in\{2,3,\dots\}$ such that $0<|a_m|<\frac{B-A}{\alpha_m}$ or $0<|b_m|<\frac{B-A}{\beta_m}$. If $0<|a_m|<\frac{B-A}{\alpha_m}$, then putting $\gamma=\frac{|a_m|\alpha_m}{B-A}$, $\varphi=\frac{1}{1-\gamma}(f-\gamma h_m)$, we have $0<\gamma<1$, $h_m,\varphi\in\mathcal{S}_{\mathcal{T}}^{*}(A,B)$, $h_m\neq\varphi$ and $f=\gamma h_m+(1-\gamma)\varphi$. Thus, $f\notin E\mathcal{S}_{\mathcal{T}}^{n}(A,B)$. Similarly, we get $f\notin E\mathcal{S}_{\mathcal{T}}^{n}(A,B)$ if $0<|b_m|<\frac{B-A}{\beta_m}$, and the proof is completed. □

4. Applications

It is clear that if the class $\mathcal{F}=\{f_k\in\mathcal{H}: k\in\mathbb{N}\}$ is locally uniformly bounded, then its closed convex hull is given by

$$\overline{co}\mathcal{F}=\left\{\sum_{k=1}^{\infty}\gamma_k f_k:\ \sum_{k=1}^{\infty}\gamma_k=1,\ \gamma_k\geq 0\ (k\in\mathbb{N})\right\}. \tag{13}$$

Thus, by Theorem 4 we have the following corollary.

Corollary 1. *Let h_k, g_k be defined by (12). Then*

$$\mathcal{S}_{\mathcal{T}}^{n}(A,B)=\left\{\sum_{k=1}^{\infty}(\gamma_k h_k+\delta_k g_k): \sum_{k=1}^{\infty}(\gamma_k+\delta_k)=1,\ \delta_1=0,\gamma_k,\delta_k\geq 0\ (k\in\mathbb{N})\right\}.$$

For each fixed value of $k\in\mathbb{N}$, $z\in\mathbb{U}$, the following real-valued functionals are continuous and convex on $\mathcal{H}$:

$$\mathcal{J}(f)=|a_k|,\ \mathcal{J}(f)=|b_k|,\ \mathcal{J}(f)=|f(z)|,\ \mathcal{J}(f)=\left|\mathcal{D}_{\mathcal{H}}^{k}f(z)\right|\quad(f\in\mathcal{H}), \tag{14}$$

$$\mathcal{J}(f)=\left(\frac{1}{2\pi}\int_0^{2\pi}\left|f\left(re^{i\theta}\right)\right|^{\gamma}\mathrm{d}\theta\right)^{1/\gamma}\quad(f\in\mathcal{H},\ \gamma\geq 1,\ 0<r<1). \tag{15}$$

Therefore, by Lemma 1 and Theorem 4 we have the corollaries listed below.

Corollary 2. *Let $f\in\mathcal{S}_{\mathcal{T}}^{n}(A,B)$ be a function of the form (8). Then*

$$|a_k|\leq\frac{B-A}{\alpha_k},\ |b_k|\leq\frac{B-A}{\beta_k}\ \ (k=2,3,\dots), \tag{16}$$

where α_k,β_k are defined by (5). The result is sharp. The functions h_k, g_k of the form (12) are the extremal functions.

Corollary 3. *Let $f\in\mathcal{S}_{\mathcal{T}}^{n}(A,B)$, $|z|=r<1$. Then*

$$r-\frac{B-A}{(n+1)(1+2B-A)}r^2\leq|f(z)|\leq r+\frac{B-A}{(n+1)(1+2B-A)}r^2,$$
$$r-\frac{(B-A)\lambda_{2,m}}{(n+1)(1+2B-A)}r^2\leq\left|\mathcal{D}_{\mathcal{H}}^{m}f(z)\right|\leq r+\frac{(B-A)\lambda_{2,m}}{(n+1)(1+2B-A)}r^2\quad(m\in\mathbb{N}),$$

where $\lambda_{2,m}$ is defined by (2). The result is sharp. The function h_2 of the form (12) is the extremal function.

Corollary 4. *Let* $0 < r < 1$, $\gamma \geq 1$. *If* $f \in \mathcal{S}_{\mathcal{T}}^{n}(A, B)$, *then*

$$\frac{1}{2\pi}\int_{0}^{2\pi}\left|f(re^{i\theta})\right|^{\gamma} d\theta \leq \frac{1}{2\pi}\int_{0}^{2\pi}\left|h_2(re^{i\theta})\right|^{\gamma} d\theta,$$

where h_2 *is defined by* (12).

Remark 1. By choosing parameters A, B, n in the defined classes of functions, we can obtain new and also well-known results (see, for example, [3,4,6,7,14]).

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