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Sharp estimates of integral functionals on classes of functions with small mean oscillation



Estimations précises de certaines fonctionnelles intégrales sur des classes de fonctions avec une petite oscillation moyenne

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ABSTRACT

We unify several Bellman function problems treated in [1,2,4–6,9–12,14–16,18–25]. For that purpose, we define a class of functions that have, in a sense, small mean oscillation (this class depends on two convex sets in \mathbb{R}^2). We show how the unit ball in the BMO space, or a Muckenhoupt class, or a Gehring class can be described in such a fashion. Finally, we consider a Bellman function problem on these classes, discuss its solution and related questions.

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RÉSUMÉ

Nous unifions plusieurs problèmes concernant la fonction de Bellman traités dans [1, 2,4–6,9–12,14–16,18–25]. Dans ce but, nous introduisons une classe de fonctions dont l'oscillation moyenne est petite dans un certain sens (cette classe dépend de deux sous-ensembles convexes de \mathbb{R}^2). Nous démontrons que la boule unité de l'espace BMO, ou de la classe de Muckenhoupt, ou de la classe de Gehring, peut être décrite de cette façon. Finalement, nous considérons un problème de fonction de Bellman sur chacune de ces classes et discutons sa résolution ainsi que des questions voisines.

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Since Slavin [12] and Vasyunin [18] proved the sharp form of the John–Nirenberg inequality (see [15]), there have been many papers where similar principles were used to prove sharp estimates of this kind. However, there is no theory or even a unifying approach; moreover, the class of problems to which the method can be applied has not been described yet.

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There is a portion of heuristics in the folklore that is each time applied to a new problem in a very similar manner. The first attempt to build a theory (at least for BMO) was made in [16], then the theory was developed in the paper [4] (see the short report [5] also). We would also like to draw the reader's attention to the preprint [6], which can be considered as a description of the theory for the BMO space in a sufficient generality. Problems of this kind were considered not only in BMO, but in Muckenhoupt classes, Gehring classes, etc. (see [1,2,11,13,19,20]). In this short note, we define a class of functions and an extremal problem on it that includes all the problems discussed above.

1. Setting

Let Ω_0 be a non-empty open strictly convex subset of \mathbb{R}^2 and let Ω_1 be an open strictly convex subset of Ω_0 . We define the domain Ω as $cl(\Omega_0 \setminus \Omega_1)$ (the word “domain” comes from “domain of a function”; the symbol cl denotes the closure) and the class \mathbf{A}_Ω of summable \mathbb{R}^2 -valued functions on an interval $I \subset \mathbb{R}$ as follows:

$$\mathbf{A}_\Omega = \left\{ \varphi \in L^1(I, \mathbb{R}^2) \mid \varphi(I) \subset \partial\Omega_0 \text{ and } \forall \text{ subinterval } J \subset I \quad \langle \varphi \rangle_J \notin \Omega_1 \right\}. \quad (1)$$

Here $\langle \varphi \rangle_J = \frac{1}{|J|} \int_J \varphi(s) ds$ is the average of φ over J . In Section 2 we show how the unit ball in BMO as well as the “unit balls” in Muckenhoupt and Gehring classes can be represented in the form (1). Let $f: \partial\Omega_0 \rightarrow \mathbb{R}$ be a bounded from below Borel measurable locally bounded function. We are interested in sharp bounds for the expressions of the form $\langle f(\varphi) \rangle_I$, where $\varphi \in \mathbf{A}_\Omega$.

Again, in Section 2 we explain how the John–Nirenberg inequality or other inequalities of harmonic analysis can be rewritten as estimations of such an expression. The said estimates are delivered by the corresponding Bellman function

$$\mathbf{B}_{\Omega,f}(x) = \sup \left\{ \langle f(\varphi) \rangle_I \mid \langle \varphi \rangle_I = x, \varphi \in \mathbf{A}_\Omega \right\}. \quad (2)$$

Problem 1.1. Given a domain Ω and a function f , calculate the function $\mathbf{B}_{\Omega,f}$.

The particular cases of this problem were treated in the papers [1,2,4–6,9–12,14–16,18–25] (see Section 2 for a detailed explanation). The main reason for Problem 1.1 to be solvable (and it has been heavily used in all the preceding work) is that the function \mathbf{B} enjoys good properties.

Definition 1.2. Let ω be a subset of \mathbb{R}^d . We call a function $G: \omega \rightarrow \mathbb{R} \cup \{+\infty\}$ locally concave on ω if for every segment $\ell \subset \omega$ the restriction $G|_\ell$ is concave.

Define the class of functions on Ω :

$$\Lambda_{\Omega,f} = \left\{ G: \Omega \rightarrow \mathbb{R} \cup \{+\infty\} \mid G \text{ is locally concave on } \Omega, \forall x \in \partial\Omega_0 \quad G(x) \geq f(x) \right\}. \quad (3)$$

The function $\mathfrak{B}_{\Omega,f}$ is given as follows:

$$\mathfrak{B}_{\Omega,f}(x) = \inf_{G \in \Lambda_{\Omega,f}} G(x). \quad (4)$$

Conjecture 1.3. $\mathbf{B}_{\Omega,f} = \mathfrak{B}_{\Omega,f}$.

In particular, the conjecture states that the Bellman function is locally concave (because the function $\mathfrak{B}_{\Omega,f}$ is).

Problem 1.4. Prove Conjecture 1.3 in adequate generality.

Though it may seem that one should solve Problem 1.4 before turning to Problem 1.1, it is not really the case. All the preceding papers used Conjecture 1.3 as an assumption that allowed the authors to guess \mathbf{B} , then to prove that this function was the Bellman function indeed, and only then verify Conjecture 1.3 for Ω and f chosen. However, to treat Problem 1.4 in itself, one has to invent a different approach, see Section 3.

2. Examples

From now on, we follow the agreement: if $g: \mathbb{R} \rightarrow \mathbb{R}^2$ is some fixed parameterization of $\partial\Omega_0$, then the function $f(g): \mathbb{R} \rightarrow \mathbb{R}$ is denoted by \tilde{f} .

The BMO space. We consider the BMO space with the quadratic seminorm. Let ε be a positive number. Set $\Omega_0 = \{x \in \mathbb{R}^2 \mid x_1^2 < x_2\}$ and $\Omega_1 = \{x \in \mathbb{R}^2 \mid x_1^2 + \varepsilon^2 < x_2\}$. A function

$$\varphi = (\varphi_1, \varphi_2): I \rightarrow \partial\Omega_0$$

belongs to the class \mathbf{A}_Ω if and only if its first coordinate φ_1 belongs to BMO_ε (the ball of radius ε in BMO). Indeed, for any $t \in I$, we have $\varphi_2(t) = \varphi_1^2(t)$; therefore, the condition $\langle \varphi \rangle_J \notin \Omega_1$ can be rewritten as

$$\langle \varphi_1^2 \rangle_J \leq \langle \varphi_1 \rangle_J^2 + \varepsilon^2,$$

which is the same as

$$\langle (\varphi_1 - \langle \varphi_1 \rangle_J)^2 \rangle_J \leq \varepsilon^2. \tag{5}$$

Now we see that the class \mathbf{A}_Ω corresponds to BMO_ε . The Bellman function (2) estimates the functional $\langle \tilde{f}(\varphi_1) \rangle_J$. The solution to Problem 1.1 with $\tilde{f}(t) = e^{\lambda t}$ leads to the John–Nirenberg inequality in its integral form, the case $\tilde{f}(t) = \chi_{(-\infty, -\lambda] \cup [\lambda, \infty)}(t)$ corresponds to the weak form of the John–Nirenberg inequality, and the case $f(t) = |t|^p$ leads to equivalent definitions of BMO . We refer the reader to the paper [4] for a detailed discussion. This case is the subject of study for the papers [4,6,9,10,15,16,21,22,25].

Classes A_{p_1, p_2} . Let p_1 and p_2 , $p_1 > p_2$, be real numbers and let $Q \geq 1$. Suppose

$$\Omega_0 = \{x \in \mathbb{R}^2 \mid x_1, x_2 > 0, x_2^{\frac{1}{p_2}} < x_1^{\frac{1}{p_1}}\} \quad \text{and} \quad \Omega_1 = \{x \in \mathbb{R}^2 \mid x_1, x_2 > 0, Qx_2^{\frac{1}{p_2}} < x_1^{\frac{1}{p_1}}\}.$$

If a function φ belongs to the class \mathbf{A}_Ω , then its first coordinate φ_1 belongs to the so-called A_{p_1, p_2} class. The “norm” in this class is defined as

$$[\psi]_{A_{p_1, p_2}} = \sup_{J \subset I} \langle \psi^{p_1} \rangle_J^{\frac{1}{p_1}} \langle \psi^{p_2} \rangle_J^{-\frac{1}{p_2}}, \tag{6}$$

where the supremum is taken over all subintervals of I . These classes were introduced in [20]. If $p \in (1, \infty)$, then $A_{1, \frac{1}{p-1}} = A_p$, where A_p stands for the classical Muckenhoupt class. When $p_2 = 1$ and $p_1 > 1$, the class A_{p_1, p_2} coincides with the so-called Gehring class (see [7] or [8]). Estimates of integral functionals as provided by the Bellman function (2) lead to various sharp forms of the reverse Hölder inequality, see [20]. These cases were treated in the papers [1,2,11,19,20].

Reverse Jensen classes. These classes were introduced in [7]. Let $\Phi: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a convex function. Let $Q > 1$. Consider the class of functions $\psi: I \rightarrow \mathbb{R}_+$ such that

$$\forall J \subset I \quad \langle \Phi(\psi) \rangle_J \leq Q \Phi(\langle \psi \rangle_J).$$

Surely, both a Muckenhoupt class and a Gehring class can be described as certain Reverse Jensen classes. The corresponding domain is $\{x \in \mathbb{R}^2 \mid x_1, x_2 \geq 0, \Phi(x_1) \leq x_2 \leq Q \Phi(x_1)\}$. Consult a very recent paper [13], where the Bellman function on the domain $\{x \in \mathbb{R}^2 \mid e^{x_1} \leq x_2 \leq C e^{x_1}\}$, $C > 1$, provides sharp constants in the John–Nirenberg inequality for the BMO space equipped with the L^p -type seminorm.

3. Hints to solutions

First, we note that strict convexity of Ω_0 implies the fact that $\mathbf{B}(x) = f(x)$ for $x \in \partial\Omega_0$. Second, we need Ω to fulfill several assumptions that all the domains listed in Section 2 do satisfy.

1. The domains Ω_0 and Ω_1 are unbounded. (7)
2. The boundary of Ω_1 is C^2 -smooth. (8)
3. Every ray inside Ω_0 can be translated to belong to Ω_1 entirely. (9)

The first two conditions are technical in a sense, the third one is essential, since (under assumption (7)) it is equivalent to the fact that for any $x \in \Omega$ there exists a function $\varphi \in \mathbf{A}_\Omega$ such that $\langle \varphi \rangle_I = x$ (i.e. the supremum in formula (2) is taken over a non-empty set). Now we are ready to present a solution to Problem 1.4.

Theorem 3.1. *Let the domain Ω satisfy the conditions (7), (8), (9). If the function f is bounded from below, then $\mathfrak{B}_{\Omega, f} = \mathbf{B}_{\Omega, f}$.*

The condition that f is bounded from below is not necessary. However, we note that without this condition the extremal problem in formula (2) is not well posed (the integral of $f(\varphi)$ may be not well defined). In [17] the reader can find the proof of Theorem 3.1 for the case $c\Omega_1 \subset \Omega_0$ as well as its analog where f can be unbounded from below.

To solve Problem 1.1, we need to consider even more restrictive conditions, we introduce some notation for that purpose. Choose $g = (g_1, g_2): \mathbb{R} \rightarrow \mathbb{R}^2$ to be a continuous parameterization of $\partial\Omega_0$; let the domain Ω lie on the left of this oriented curve. For any number $u \in \mathbb{R}$ we draw two tangents from the point $g(u)$ to the set Ω_1 ; by a tangent we mean not a line, but a segment connecting $g(u)$ with the tangency point. We denote the lengths of the left and the right tangents by $\ell_L(u)$ and $\ell_R(u)$ correspondingly (the left tangent lies between the right one and g' , see [4] for explanations about this notation).

1. The boundaries $\partial\Omega_0$ and $\partial\Omega_1$ are C^3 -smooth curves, the function f is C^3 -smooth. (10)

2. The curve $\gamma(t) = (g_1(t), g_2(t), \tilde{f}(t))$ changes the sign of its torsion a finite number of times. (11)

3. The integrals $\int_{-\infty}^0 \frac{1}{\ell_R}$ and $\int_0^{+\infty} \frac{1}{\ell_L}$ diverge. (12)

In Condition (12), the integration is with respect to the natural parameterization of the curve $\partial\Omega_1$, where the functions ℓ_R and ℓ_L are considered as the functions of their tangency points lying on $\partial\Omega_1$. The last Condition (12) is more mysterious; we believe that our considerations may work without it.

We also need a summability assumption for the function f . Let $\alpha_R(u)$ denote the oriented angle between the right tangent at the point u and the vector $(1, 0)$, let $\alpha_L(u)$ denote the oriented angle between the left tangent at the point u and the vector $(1, 0)$. Then, the summability condition requires the bulky integral

$$\int_{-\infty}^t \exp\left(\int_{\tau}^t \frac{g'_1}{\ell_R \cos(\alpha_R)}\right) \frac{\tan(\alpha_R(\tau))g'_1(\tau) - g'_2(\tau)}{(g'_1(\tau)g'_2(\tau) - g'_2(\tau)g'_1(\tau))^2} \begin{vmatrix} \tilde{f}'(\tau) & \tilde{f}''(\tau) & \tilde{f}'''(\tau) \\ g'_1(\tau) & g''_1(\tau) & g'''_1(\tau) \\ g'_2(\tau) & g''_2(\tau) & g'''_2(\tau) \end{vmatrix} d\tau \tag{13}$$

to converge for any $t \in \mathbb{R}$ (and a similar condition with R replaced by L and with $-\infty$ replaced by $+\infty$).

Claim: under Conditions (7), (9), (10), (11), (12), and the mentioned convergence conditions for the integrals (13), we can solve Problem 1.1.

As in [4], by “solution” we mean an expression for the function B , which may include roots of implicit equations, differentiations, and integrations. Though at the first sight, the benefit of such a “solution” may seem questionable, it occurs to be useful if one has a specific domain Ω and a function f at hand, see examples in the papers [4,6], the whole paper [22] that treats the cases of functions f extremely difficult from an algebraic point of view, and other papers on the subject.

It appears that to solve Problem 1.1, one has to reformulate reasonings from [4] and [6] in geometric terms and observe that in such terms they work for a more general setting of the problem considered. For example, the integral (13) plays the role of the force function coming from $-\infty$ (see [4] for the definition in the case of BMO) in the general setting. However, the geometric essence of the matter is even more revealed in the example of the chordal domain (it has already been used in [3]). We remind the reader that a chordal domain is a type of foliation (see [4] for the definition) that consists of chords, i.e. segments that connect two points of $\partial\Omega_0$. In the case of the parabolic strip $g(t) = (t, t^2)$, the chordal domain could match B_f if and only if it satisfied the cup equation

$$\frac{\tilde{f}(b) - \tilde{f}(a)}{b - a} = \frac{\tilde{f}'(b) + \tilde{f}'(a)}{2}; \quad (a, a^2) \text{ and } (b, b^2) \text{ are the endpoints of a chord,}$$

and two special differential inequalities (“inequalities for the differentials”) for each of its chord. In the general setting of Problem 1.1, the cup equation turns into

$$\begin{vmatrix} g'_1(a) & g'_2(a) & \tilde{f}'(a) \\ g'_1(b) & g'_2(b) & \tilde{f}'(b) \\ g_1(b) - g_1(a) & g_2(b) - g_2(a) & \tilde{f}(b) - \tilde{f}(a) \end{vmatrix} = 0; \quad g(a) \text{ and } g(b) \text{ are the endpoints of a chord,}$$

which has the following geometrical meaning: the tangent vectors to the curve $\gamma(t) = (g_1(t), g_2(t), \tilde{f}(t))$ at the points a and b lie in one two-dimensional plane with the vector $\gamma(a) - \gamma(b)$. The special differential inequalities (the so-called inequalities for the differentials) can also be re-stated in purely geometric terms (the triple product of $\gamma'(a)$, $\gamma(b) - \gamma(a)$, and the normal to γ at the point a should be negative; the same should be fulfilled with a and b interchanged) and then generalized to fit Problem 1.1.

In [4] the roots of \tilde{f}''' played the main role. Indeed, the cups sit on the points where \tilde{f}''' changes its sign from $+$ to $-$. In the general case, the function \tilde{f}''' should be replaced by the torsion of the curve γ . One can see the traces of the torsion in formula (13). Moreover, now we see that Condition (11) is a straightforward generalization of the regularity condition from [4].

We recall that in [4] the problem was treated not in the full generality (we assumed that the roots of \tilde{f}''' were well separated). This narrowed the list of local types of foliations. However, without such an assumption, the collection of figures is wider, see the preprint [6] for the general theory, and the example [22], where almost all figures from the general case appear. The latter paper also highlights the notation that becomes very important when there are lots of different figures (it appeared that a foliation corresponds to a special weighted graph). We only mention that all the figures are transferred to the general setting of Problem 1.1, as well as all the monotonicity lemmas for forces and tails (see [4] for definitions).

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References

- [1] O. Beznosova, A. Reznikov, Sharp estimates involving A_∞ and $L \log L$ constants, and their applications to PDE, *Algebra Anal.* 26 (1) (2014) 40–67.
- [2] M. Dindoš, T. Wall, The sharp A_p constant for weights in a reverse-Hölder class, *Rev. Mat. Iberoam.* 25 (2) (2009) 559–594.
- [3] P. Ivanishvili, D.M. Stolyarov, P.B. Zatitskiy, Bellman VS Beurling: sharp estimates of uniform convexity for L^p spaces, *Algebra Anal.* 27 (2) (2015) 218–231 (in Russian); to be translated in *St. Petersburg Math. J.*, <http://arxiv.org/abs/1405.6229>.
- [4] P. Ivanishvili, D.M. Stolyarov, N.N. Osipov, V.I. Vasyunin, P.B. Zatitskiy, Bellman function for extremal problems in BMO, *Trans. Amer. Math. Soc.* (2015), in press, <http://arxiv.org/abs/1205.7018v3>, <http://www.ams.org/journals/tran/0000-000-00/S0002-9947-2015-06460-5/home.html>, <http://dx.doi.org/10.1090/tran/6460#sthash.iIlsWYIQW.dpuf>.
- [5] P. Ivanishvili, N.N. Osipov, D.M. Stolyarov, V.I. Vasyunin, P.B. Zatitskiy, On Bellman function for extremal problems in BMO, *C. R. Acad. Sci. Paris, Ser. I* 350 (11) (2012) 561–564.
- [6] P. Ivanishvili, D.M. Stolyarov, V.I. Vasyunin, P.B. Zatitskiy, Bellman function for extremal problems on BMO II: evolution, 2015, <http://arxiv.org/abs/1510.01010>.
- [7] A.A. Korenovskii, The exact continuation of a reverse Hölder inequality and Muckenhoupt's conditions, *Mat. Zametki* 52 (6) (1992) 32–44 (in Russian); translated in *Math. Notes* 52 (6) (1992) 1192–1201.
- [8] A.A. Korenovskii, A.M. Stokolos, Some applications of equimeasurable rearrangements, in: *Recent Advances in Harmonic Analysis and Applications*, in: *Proc. in Math. & Stat.*, vol. 25, Springer, 2013, pp. 181–196.
- [9] A.A. Logunov, L. Slavin, D.M. Stolyarov, V. Vasyunin, P.B. Zatitskiy, Weak integral conditions for BMO, *Proc. Amer. Math. Soc.* 143 (2015) 2913–2926, <http://arxiv.org/abs/1309.6780>.
- [10] A. Osękowski, Sharp inequalities for BMO functions, *Chin. Ann. Math., Ser. B* 36 (2) (March 2015) 225–236.
- [11] A. Reznikov, Sharp weak type estimates for weights in the class A_{p_1, p_2} , *Rev. Mat. Iberoam.* 29 (2) (2013) 433–478.
- [12] L. Slavin, Bellman function and BMO, Ph.D. thesis, Michigan State University, 2004.
- [13] L. Slavin, The John–Nirenberg constant of BMO^p , $1 \leq p \leq 2$, 2015, <http://arxiv.org/abs/1506.04969>.
- [14] L. Slavin, A. Stokolos, V. Vasyunin, Monge–Ampère equations and Bellman functions: the dyadic maximal operator, *C. R. Acad. Sci. Paris, Ser. I* 346 (1) (2008) 585–588.
- [15] L. Slavin, V. Vasyunin, Sharp results in the integral form John–Nirenberg inequality, *Trans. Amer. Math. Soc.* 363 (8) (2011) 4135–4169.
- [16] L. Slavin, V. Vasyunin, Sharp L^p estimates on BMO, *Indiana Univ. Math. J.* 61 (3) (2012) 1051–1110.
- [17] D.M. Stolyarov, P.B. Zatitskiy, Theory of locally concave functions and its applications to sharp estimates of integral functionals, 2015, <http://arxiv.org/abs/1412.5350>.
- [18] V. Vasyunin, The sharp constant in the John–Nirenberg inequality, *POMI Prepr.* 20 (2003).
- [19] V.I. Vasyunin, The sharp constant in the reverse Hölder inequality for Muckenhoupt weights, *Algebra Anal.* 15 (1) (2003) 73–117 (in Russian); translated in *St. Petersburg Math. J.* 15 (1) (2004) 49–79.
- [20] V.I. Vasyunin, Mutual estimates of L^p -norms and the Bellman function, *Zap. Nauč. Semin. POMI* 355 (2008) 81–138 (in Russian); translated in *J. Math. Sci.* 156 (5) (2009) 766–798.
- [21] V. Vasyunin, Sharp constants in the classical weak form of the John–Nirenberg inequality, *POMI Prepr.* 10 (2011) 1–9.
- [22] V.I. Vasyunin, An example of constructing Bellman function for extremal problems in BMO, *Zap. Nauč. Semin. POMI* 424 (2014) 33–125 (in Russian).
- [23] V. Vasyunin, A. Volberg, The Bellman function for a certain two-weight inequality: a case study, *Algebra Anal.* 18 (2) (2006) 24–56 (in Russian); translated in *St. Petersburg Math. J.* 18 (2) (2007) 201–222.
- [24] V. Vasyunin, A. Volberg, Monge–Ampère equation and Bellman optimization of Carleson embedding theorems, *Amer. Math. Soc. Transl., Ser. 2* 226 (2009) 195–238.
- [25] V. Vasyunin, A. Volberg, Sharp constants in the classical weak form of the John–Nirenberg inequality, *Proc. Lond. Math. Soc.* 108 (6) (2014) 1417–1434.