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Numerical analysis

A robust coarse space for optimized Schwarz methods: SORAS-GenEO-2

Un espace grossier robuste pour les méthodes de Schwarz optimisées : SORAS-GenEO-2

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A B S T R A C T

Optimized Schwarz methods (OSM) are very popular methods that were introduced in [11] for elliptic problems and in [3] for propagative wave phenomena. We build here a coarse space for which the convergence rate of the two-level method is guaranteed regardless of the regularity of the coefficients. We do this by introducing a symmetrized variant of the ORAS (Optimized Restricted Additive Schwarz) algorithm [17] and by identifying the problematic modes using two different generalized eigenvalue problems instead of only one as in [16,15] for the ASM (Additive Schwarz method), BDD (Balancing Domain Decomposition [12]) or FETI (Finite-Element Tearing and Interconnection [6]) methods.

R É S U M É

Les méthodes de Schwarz optimisées sont des méthodes très populaires, qui ont été introduites dans [11] pour des problèmes elliptiques et dans [3] pour des phénomènes de propagation d'ondes. Nous construisons ici un espace grossier pour lequel le taux de convergence de la méthode à deux niveaux peut être prescrit à l'avance sans hypothèse sur la régularité des coefficients. Ceci est rendu possible par l'introduction d'une version symétrisée de la méthode ORAS (*Optimized Restricted Additive Schwarz*) [17] ainsi que par l'identification des modes problématiques via deux problèmes aux valeurs propres généralisées au lieu d'un seul comme dans [16,15] pour les méthodes ASM (*Additive Schwarz method*), BDD (*Balancing Domain Decomposition* [12]) ou FETI (Finite-Element Tearing and Interconnection [6]).

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1. Introduction

Substructuring algorithms such as BNN or FETI are defined for non-overlapping domain decompositions, but not for overlapping subdomains. The Schwarz method [14] is defined only for overlapping subdomains. With the help of a coarse space correction, the two-level versions of both types of methods are weakly scalable, see [18] and references therein.

The domain decomposition method introduced by P.L. Lions [11] can be applied to both overlapping and non-overlapping subdomains. It is based on improving Schwarz methods by replacing the Dirichlet interface conditions by Robin interface conditions. This algorithm was extended to the Helmholtz problem by Després [4]. Robin interface conditions can be replaced by more general interface conditions that can be optimized (Optimized Schwarz methods, OSM) for a better convergence, see [8,7] and references therein. When the domain is decomposed into a large number of subdomains, these methods are, on a practical point of view, scalable if a second level is added to the algorithm via the introduction of a coarse space [10,5,2]. But there is no systematic procedure to build coarse spaces with a provable efficiency.

The purpose of this article is to define a general framework for building adaptive coarse space for OSM methods for decomposition into overlapping subdomains. We prove that we can achieve the same robustness as that of what was done for Schwarz [15] and FETI-BDD [16] domain decomposition methods with the so-called GenEO (Generalized Eigenvalue in the Overlap) coarse spaces. Compared to these previous works, we have to introduce a non-standard symmetric variant of the ORAS method as well as two generalized eigenvalue problems.

2. Symmetrized ORAS method

The problem to be solved is defined via a variational formulation on a domain $\Omega \subset \mathbb{R}^d$ for $d \in \mathbb{N}$.

$$\text{Find } u \in V \text{ such that} : a_\Omega(u, v) = l(v)\,, \quad \forall v \in V\,,$$

where V is a Hilbert space of functions from Ω with real values. The problem we consider is given through a symmetric positive definite bilinear form that is defined in terms of an integral over any open set $\omega \subset \Omega$. Typical examples are the Darcy equation ($\mathbf{K}$ is a diffusion tensor)

$$a_\omega(u, v) := \int_\omega \mathbf{K}\,\nabla u \cdot \nabla v \,\mathrm{d}x\,,$$

or the elasticity system ($\boldsymbol{C}$ is the fourth-order stiffness tensor and $\boldsymbol{\varepsilon}(\boldsymbol{u})$ is the strain tensor of a displacement field $\boldsymbol{u}$):

$$a_\omega(\boldsymbol{u},\, \boldsymbol{v}) := \int_\omega \boldsymbol{C} : \boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v})\,\mathrm{d}x\,.$$

The problem is discretized by a finite-element method. Let $\mathcal{N}$ denote the set of degrees of freedom and $(\phi_k)_{k\in\mathcal{N}}$ be a finite-element basis on a mesh $\mathcal{T}_h$. Let $A \in \mathbb{R}^{\#\mathcal{N}\times\#\mathcal{N}}$ be the associated finite-element matrix, $A_{kl} := a_\Omega(\phi_l, \phi_k)$, $k, l \in \mathcal{N}$. For some given right-hand side $\mathbf{F} \in \mathbb{R}^{\#\mathcal{N}}$, we have to solve a linear system in $\mathbf{U}$ of the form:

$$A\mathbf{U} = \mathbf{F}\,.$$

Domain Ω is decomposed into N overlapping subdomains $(\Omega_i)_{1\le i\le N}$ so that all subdomains are a union of cells of the mesh $\mathcal{T}_h$. This decomposition induces a natural decomposition of the set of indices $\mathcal{N}$ into N subsets of indices $(\mathcal{N}_i)_{1\le i\le N}$:

$$\mathcal{N}_i := \{k \in \mathcal{N} \mid \mathrm{meas}(\mathrm{supp}(\phi_k) \cap \Omega_i) > 0\}\,,\ 1 \le i \le N. \tag{1}$$

For all $1 \le i \le N$, let R_i be the restriction matrix from $\mathbb{R}^{\#\mathcal{N}}$ to the subset $R^{\#\mathcal{N}_i}$ and D_i be a diagonal matrix of size $\#\mathcal{N}_i \times \#\mathcal{N}_i$, so that we have a partition of unity at the algebraic level, $I_d = \sum_{i=1}^N R_i^{\mathrm{T}} D_i R_i$, where $I_d \in \mathbb{R}^{\#\mathcal{N}\times\#\mathcal{N}}$ is the identity matrix.

For all subdomains $1 \le i \le N$, let B_i be a SPD matrix of size $\#\mathcal{N}_i \times \#\mathcal{N}_i$, which comes typically from the discretization of boundary-value local problems using optimized transmission conditions, the ORAS preconditioner [17] is defined as

$$M^{-1}_{\mathrm{ORAS},1} := \sum_{i=1}^N R_i^{\mathrm{T}} D_i B_i^{-1} R_i\,. \tag{2}$$

Due to matrices D_i, this preconditioner is not symmetric. Note that in the special case where $B_i = R_i\, A\, R_i^{\mathrm{T}}$, the ORAS algorithm reduces to the RAS algorithm [1]. The symmetrized variant of RAS is the additive Schwarz method (ASM), $M^{-1}_{ASM,1} := \sum_{i=1}^N R_i^{\mathrm{T}} A_i^{-1} R_i$, which has been extensively studied and for which various coarse spaces have been analyzed. For the ORAS method considered in this note, it seems at first glance that we should accordingly study the following symmetrized variant:

$$M^{-1}_{\mathrm{OAS},1} := \sum_{i=1}^N R_i^{\mathrm{T}} B_i^{-1} R_i\,. \tag{3}$$

For reasons explained in Remark 1, we introduce another non-standard variant of the ORAS preconditioner (2), the symmetrized ORAS (SORAS) algorithm:

$$M^{-1}_{\mathrm{SORAS},1} := \sum_{i=1}^{N} R_i^{\mathrm{T}} D_i B_i^{-1} D_i R_i \,. \tag{4}$$

3. Two-level SORAS algorithm

In order to define the two-level SORAS algorithm, we introduce two generalized eigenvalue problems.

First, for all subdomains $1 \le i \le N$, we consider the following problem.

Definition 3.1.

$$\begin{aligned} &\text{Find } (\mathbf{U}_{ik}, \mu_{ik}) \in \mathbb{R}^{\#\mathcal{N}_i} \setminus \{0\} \times \mathbb{R} \text{ such that} \\ &D_i R_i A R_i^{\mathrm{T}} D_i \mathbf{U}_{ik} = \mu_{ik} B_i \mathbf{U}_{ik} \;\;. \end{aligned} \tag{5}$$

Let $\gamma > 0$ be a user-defined threshold, we define $Z^{\gamma}_{\mathrm{geneo}} \subset \mathbb{R}^{\#\mathcal{N}}$ as the vector space spanned by the family of vectors $(R_i^{\mathrm{T}} D_i \mathbf{U}_{ik})_{\mu_{ik} > \gamma\,, 1 \le i \le N}$ corresponding to eigenvalues larger than γ.

In order to define the second generalized eigenvalue problem, we introduce, for all subdomains $1 \le j \le N$, $\widetilde{A}^j$, the $\#\mathcal{N}_j \times \#\mathcal{N}_j$ matrix defined by

$$\mathbf{V}_j^{\mathrm{T}} \widetilde{A}^j \mathbf{U}_j := a_{\Omega_j}\left(\sum_{l \in \mathcal{N}_j} \mathbf{U}_{jl}\phi_l, \sum_{l \in \mathcal{N}_j} \mathbf{V}_{jl}\phi_l \right), \;\; \mathbf{U}_j, \mathbf{V}_j \in \mathbb{R}^{\mathcal{N}_j} \,. \tag{6}$$

When the bilinear form a results from the variational solve of a Laplace problem, the previous matrix corresponds to the discretization of local Neumann boundary value problems.

Definition 3.2. We introduce the generalized eigenvalue problem

$$\begin{aligned} &\text{Find } (\mathbf{V}_{jk}, \lambda_{jk}) \in \mathbb{R}^{\#\mathcal{N}_i} \setminus \{0\} \times \mathbb{R} \text{ such that} \\ &\widetilde{A}^i \mathbf{V}_{ik} = \lambda_{ik} B_i \mathbf{V}_{ik} \;. \end{aligned} \tag{7}$$

Let $\tau > 0$ be a user-defined threshold, we define $Z^{\tau}_{\mathrm{geneo}} \subset \mathbb{R}^{\#\mathcal{N}}$ as the vector space spanned by the family of vectors $(R_i^{\mathrm{T}} D_i \mathbf{V}_{ik})_{\lambda_{ik} < \tau\,, 1 \le i \le N}$ corresponding to eigenvalues smaller than τ.

We are now ready to define the SORAS two-level preconditioner.

Definition 3.3 *(The SORAS-GenEO-2 preconditioner).* Let P_0 denote the A-orthogonal projection on the coarse space

$$Z_{\mathrm{GenEO\text{-}2}} := Z^{\gamma}_{\mathrm{geneo}} \bigoplus Z^{\tau}_{\mathrm{geneo}} \,,$$

the two-level SORAS-GenEO-2 preconditioner is defined as follows:

$$M^{-1}_{\mathrm{SORAS},2} := P_0 A^{-1} + (I_d - P_0) \sum_{i=1}^{N} R_i^{\mathrm{T}} D_i B_i^{-1} D_i R_i (I_d - P_0^{\mathrm{T}}) \,. \tag{8}$$

Note that this definition is reminiscent of the balancing domain decomposition preconditioner [12] introduced for Schur's complement-based methods. Note that the coarse space is now defined by two generalized eigenvalue problems instead of one in [15,16] for ASM and FETI-BDD methods. We have the following theorem.

Theorem 3.1 *(Spectral estimate for the SORAS-GenEO-2 preconditioner). Let k_0 be the maximal multiplicity of the subdomain intersections, γ, $\tau > 0$ be arbitrary constants used in Definitions 3.2 and 3.3.*

Then the eigenvalues of the two-level preconditioned operator satisfy the following spectral estimate

$$\frac{1}{1 + \frac{k_0}{\tau}} \le \lambda(M^{-1}_{\mathrm{SORAS},2\;A}) \le \max(1, k_0\,\gamma)$$

where $\lambda(M^{-1}_{\mathrm{SORAS},2\,\mathrm{A}})$ is an eigenvalue of the preconditioned operator.

The proof is based on the fictitious space lemma [13].

Fig. 1. (Color online.) 2D Elasticity: coefficient distribution of steel and rubber.

Remark 1. An analysis of a two-level version of the preconditioner M_{OAS}^{-1} (3) following the same path yields the following two generalized eigenvalue problems:

$$\begin{aligned}&\text{find } (\mathbf{U}_{jk}, \mu_{jk}) \in \mathbb{R}^{\#\mathcal{N}_i} \setminus \{0\} \times \mathbb{R} \text{ such that}\\ &A^i \mathbf{U}_{ik} = \mu_{ik} B_i \mathbf{U}_{ik}\ ,\end{aligned}$$

and

$$\begin{aligned}&\text{find } (\mathbf{V}_{jk}, \lambda_{jk}) \in \mathbb{R}^{\#\mathcal{N}_i} \setminus \{0\} \times \mathbb{R} \text{ such that}\\ &\widetilde{A}^i \mathbf{V}_{ik} = \lambda_{ik} D_i B_i D_i \mathbf{V}_{ik}\ .\end{aligned}$$

In the general case for $1 \le i \le N$, matrices D_i may have zero entries for boundary degrees of freedom since they are related to a partition of unity. Moreover, very often matrices B_i and A_i differ only by the interface conditions, i.e. for entries corresponding to boundary degrees of freedom. Therefore, matrix $D_i B_i D_i$ on the right-hand side of the last generalized eigenvalue problem is not impacted by the choice of the interface conditions of the one-level optimized Schwarz method. This cannot lead to efficient adaptive coarse spaces.

4. Nearly incompressible elasticity

Although our theory does not apply in a straightforward manner to saddle-point problems, we use it for these difficult problems for which it is not possible to preserve both the symmetry and the positivity of the problem. Note that generalized eigenvalue problems (5) and (7) still make sense if A is the matrix of a saddle-point problem and matrices B_i and $\widetilde{A}_i$ are properly defined for each subdomain $1 \le i \le N$. The new coarse space was tested quite successfully on Stokes problems and nearly incompressible elasticity ones with a discretization based on saddle-point formulations in order to avoid locking phenomena. We first report 2D results for the latter case where comparisons are made with other methods in terms of iteration counts only. The mechanical properties of a solid can be characterized by its Young modulus E and Poisson ratio ν or alternatively by its Lamé coefficients λ and μ. These coefficients relate to each other by the following formulas:

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \quad \text{and} \quad \mu = \frac{E}{2(1+\nu)}\ . \tag{9}$$

The variational problem consists in finding $(\boldsymbol{u}_h, p_h) \in \mathcal{V}_h := \mathbb{P}_2^d \cap H_0^1(\Omega) \times \mathbb{P}_1$ such that for all $(\boldsymbol{v}_h, q_h) \in \mathcal{V}_h$

$$\begin{cases} \displaystyle\int_\Omega 2\mu \underline{\underline{\varepsilon}}(\boldsymbol{u}_h) : \underline{\underline{\varepsilon}}(\boldsymbol{v}_h)\,\mathrm{d}x - \int_\Omega p_h \mathrm{div}\,(\boldsymbol{v}_h)\mathrm{d}x = \int_\Omega \boldsymbol{f}\boldsymbol{v}_h\,\mathrm{d}x \\ \displaystyle -\int_\Omega \mathrm{div}\,(\boldsymbol{u}_h) q_h\,\mathrm{d}x - \int_\Omega \frac{1}{\lambda} p_h q_h = 0 \end{cases} \Longrightarrow A\mathbf{U} = \begin{bmatrix} H & B^T \\ B & C \end{bmatrix} \begin{bmatrix} \boldsymbol{u} \\ p \end{bmatrix} = \begin{bmatrix} \mathrm{f} \\ 0 \end{bmatrix} = \mathbf{F}. \tag{10}$$

Matrix $\widetilde{A}_i$ arises from the variational formulation (10) where the integration over domain Ω is replaced by the integration over subdomain Ω_i and finite-element space $\mathcal{V}_h$ is restricted to subdomain Ω_i. Matrix B_i corresponds to a Robin problem and is the sum of matrix $\widetilde{A}_i$ and of the matrix of the following variational formulation restricted to the same finite-element space:

$$\int_{\partial\Omega_i \setminus \partial\Omega} \frac{2\alpha\mu(2\mu+\lambda)}{\lambda+3\mu} \boldsymbol{u}_h \cdot \boldsymbol{v}_h \text{ with } \alpha = 10 \text{ in our test.}$$

We test various domain decomposition methods for a heterogeneous beam of eight layers of steel $(E_1, \nu_1) = (210 \cdot 10^9, 0.3)$ and rubber $(E_2, \nu_2) = (0.1 \cdot 10^9, 0.4999)$, see Fig. 1. The beam is clamped on its left and right sides. Iteration counts for a relative tolerance of 10^{-6} are given in Table 1. We compare the one-level Additive Schwarz (AS) and SORAS methods, the two level AS and SORAS methods with a coarse space consisting of rigid body motions that are zero-energy modes (ZEM) and finally AS with a GenEO coarse space and SORAS with the GenEO-2 coarse space defined in Definition 3.3 with $\tau = 0.4$ and $\gamma = 10^3$. Columns *dim* refer to the total size of the coarse space of a two-level method. Eigenvalue problem (7) accounts for roughly 90% of the GenEO-2 coarse space size. We performed large 3D simulations on 8192 cores of an IBM/Blue Gene Q machine with 1.6 GHz Power A2 processors for both elasticity (200 millions of d.o.f's in 200 s) and Stokes (200 millions of d.o.f's in 150 s) equations. Computing facilities were provided by an IDRIS-GENCI project.

Table 1
2D Elasticity. GMRES iteration counts for a solid made of steel and rubber. Simulations made with FreeFem++ [9].

Nb DOFs	Nb subdom	AS	SORAS	AS + CS(ZEM)		SORAS + CS(ZEM)		AS-GenEO		SORAS-GenEO-2	
		Iteration	Iteration	Iteration	*dim*	Iteration	*dim*	Iteration	*dim*	Iteration	*dim*
35 841	8	150	184	117	*24*	79	*24*	110	*184*	13	*145*
70 590	16	276	337	170	*48*	144	*48*	153	*400*	17	*303*
141 375	32	497	++1000	261	*96*	200	*96*	171	*800*	22	*561*
279 561	64	++1000	++1000	333	*192*	335	*192*	496	*1600*	24	*855*
561 531	128	++1000	++1000	329	*384*	400	*384*	++1000	*2304*	29	*1220*
1 077 141	256	++1000	++1000	369	*768*	++1000	*768*	++1000	*3840*	36	*1971*

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