C. R. Acad. Sci. Paris, Ser. I 353 (2015) 937–942

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Functional analysis/Dynamical systems

Disjoint mixing linear fractional composition operators in the unit ball ☆

Mélange disjoint d'opérateurs de composition linéaires fractionnaires dans la boule unité

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ARTICLE INFO

Article history:
Received 27 January 2014
Accepted after revision 9 July 2015
Available online 12 August 2015

Presented by the Editorial Board

ABSTRACT

In the present paper, we investigate the disjoint mixing property of finitely many linear fractional composition operators acting on the space of holomorphic functions on the unit ball in $\mathbb{C}^N$, and generalize parts of the results obtained by Bès, Martin and Peris in 2011.

RÉSUMÉ

Dans la présente note, nous étudions la propriété de mélange disjoint pour un nombre fini d'opérateurs de composition linéaires fractionnaires agissant sur l'espace des fonctions holomorphes sur la boule unité de $\mathbb{C}^N$, et nous généralisons une partie des résultats obtenus par Bès, Martin et Peris en 2011.

1. Introduction

Let $\mathbb{C}^N$ denote the N-dimensional complex space, for any $z=(z_1,z_2,\cdots,z_N)$ and $w=(w_1,w_2,\cdots,w_N)$ in $\mathbb{C}^N$; the inner product is defined by $\langle z,w\rangle=\sum_{i=1}^{N} z_i\overline{w_i}$, and $\|z\|^2=\langle z,z\rangle$.

Let $\mathbb{B}_N$ be the unit ball of $\mathbb{C}^N$, with the boundary $\partial\mathbb{B}_N$. The class of all holomorphic functions on $\mathbb{B}_N$ will be denoted by $H(\mathbb{B}_N)$. Let $\varphi(z)$ be a holomorphic self-map of $\mathbb{B}_N$, the composition operator is defined by $C_\varphi(f)(z)=f(\varphi(z))$ for any $f\in H(\mathbb{B}_N)$ and $z\in\mathbb{B}_N$. And each $u\in H(\mathbb{B}_N)$ induces an operator M_u on $H(\mathbb{B}_N)$ of pointwise multiplication by the weight symbol u. The weighted composition operator $uC_\varphi := M_uC_\varphi : H(\mathbb{B}_N)\to H(\mathbb{B}_N)$, is defined by

$$uC_\varphi(f)(z) := u(z)(f\circ\varphi)(z).$$

☆ The work was supported in part by the National Natural Science Foundation of China (Grant Nos. 11371276; 11401431) and by Tianjin City High School Science and Technology Fund Planning Project (Grant No. 20141002).

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http://dx.doi.org/10.1016/j.crma.2015.07.005
1631-073X/

For general references on the theory of composition operators, we refer the interested readers to the books [9,13].

Definition 1.1. An operator T on a topological vector space X is called supercyclic provided that there is some vector f in X whose projective orbit $\{\lambda T^n f : \lambda \in \mathbb{C}, n = 0, 1, \cdots\}$ is dense in X. Such f is called a supercyclic vector for T.

Definition 1.2. The supercyclic operators $T_1, \cdots, T_m$ $(m \geq 2)$ on a topological space X are said to be d-supercyclic provided there is some $f \in X$ for which the vector $(f, \cdots, f) \in X^m$ is supercyclic for the direct sum operator $\bigoplus_{s=1}^m T_s$ acting on the product space X^m, endowed with the product topology.

Definition 1.3. The operators $T_1, \cdots, T_m$ $(m \geq 2)$ on a topological space X are d-mixing provided for every open subsets $U_0, \cdots, U_m$ of X there exists $n_0 \in \mathbb{N}$ such that $\emptyset \neq U_0 \bigcap T_1^{-n}(U_1) \bigcap \cdots \bigcap T_m^{-n}(U_m)$ for each $n \geq n_0$.

Definition 1.4. A map φ will be called a linear fractional map if $\varphi(z) = (Az + B)(\langle z, C\rangle + D)^{-1}$ where A is an $N \times N$ matrix, B and C are (column) vectors in $\mathbb{C}^N$, and D is a complex number.

For $\varphi : \mathbb{B}_N \to \mathbb{B}_N$ is a linear fractional map, with no fixed points in $\mathbb{B}_N$. Then there exists a unique point $\tau \in \partial\mathbb{B}_N$ such that $\varphi(\tau) = \tau$ and $\langle d\varphi_\tau(\tau), \tau\rangle = \alpha(\varphi)$ with $0 < \alpha(\varphi) \leq 1$.

The point $\tau \in \partial\mathbb{B}_N$ is called the Denjoy–Wolff point of φ and $\alpha(\varphi)$ the boundary dilation coefficient of φ. We say that φ is hyperbolic if $\alpha(\varphi) < 1$ while we say it parabolic if $\alpha(\varphi) = 1$.

We recall that the Siegal upper half-plane $\mathbb{H}_N$ is defined by

$$\mathbb{H}_N = \{(w_1, \cdots, w_n) = (w_1, w') \in \mathbb{C}^N, \mathrm{Im}(w_1) > \|w'\|^2\}.$$

Let $e_1 = (1, 0, \cdots, 0) = (1, 0')$. The Cayley transform, defined by $\mathcal{C}(z) = \mathrm{i}(e_1 + z)/(1 - z_1)$ is a biholomorphic map of $\mathbb{B}_N$ onto $\mathbb{H}_N$.

We say that Φ is a generalized Heisenberg translation of $\mathbb{H}_N$ if it may be written as

$$\Phi(w_1, w') = (w_1 + 2\mathrm{i}\langle w', \gamma\rangle + b, w' + \gamma),$$

with $b \in \mathbb{C}, \gamma \in \mathbb{C}^{N-1}\backslash\{0\}$ and $\mathrm{Im}(b) \geq \|\gamma\|^2$. A generalized Heisenberg translation is an automorphism if and only if $\mathrm{Im}(b) = \|\gamma\|^2$. Next, we give the following definition, which is a kind of generalized hyperbolic linear fractional self-map of $\mathbb{B}_N$ as we usually met before. For brevity, we still call it as generalized hyperbolic linear fractional map.

Definition 1.5. Let $\varphi : \mathbb{B}_N \to \mathbb{B}_N$ be a linear fractional map. We say that φ is a generalized hyperbolic linear fractional self-map of $\mathbb{B}_N$, if it is conjugate to a self-map of $\mathbb{H}_N$ of the form

$$\Phi_0(w_1, w') = \frac{1}{\lambda}(w_1 + \frac{2}{\sqrt{\lambda}}\langle Uw', d\rangle + c, \sqrt{\lambda}Uw' + d),\ (w_1, w') \in \mathbb{H}_N,$$

where $\lambda < 1$ and $\lambda\,\mathrm{Im}(c) > |d|^2$, $U \in \mathbb{C}^{(N-1)\times(N-1)}$ is a unitary matrix.

As was shown in [11], up to conjugation with an automorphism of $\mathbb{B}_N$, we may assume that φ is conjugate with a map of the form

$$\Phi(w_1, w') = (\lambda w_1 + b, \sqrt{\lambda}Uw')\ (w_1, w') \in \mathbb{H}_N,$$

where $b \in \mathbb{C}$ with $\mathrm{Im}(b) > 0$, and $1/\lambda$ is the boundary dilation coefficient of φ, $\lambda > 1$. The map Φ has the attractive fixed point ∞ and an exterior fixed point.

Definition 1.6. Let $\varphi : \mathbb{B}_N \to \mathbb{B}_N$ be a linear fractional map. We say that φ is an unstable parabolic linear fractional self-map of $\mathbb{B}_N$, if:

(i) φ has a unique fixed point τ in $\overline{\mathbb{B}_N}$, which is located on the boundary $\partial\mathbb{B}_N$;
(ii) the boundary dilation coefficient at τ is 1;
(iii) φ does not fix as a set any non-trivial affine subset of $\mathbb{B}_N$.

Note that the above φ fixes only e_1 if and only if Φ is a linear fractional map of $\mathbb{H}_N$ which fixes only ∞.

In the past two decades, many authors focused on the dynamics of weighted composition operator, for example, see [5,4,2,3,7,8,11,10,14] and the related references therein. Recently, in [6], Bès, Martin and Peris showed the following theorem.

Theorem. *Let $\varphi_1, \cdots, \varphi_m$ be linear fractional maps of the unit disk $\mathbb{D}$, where $m \geq 2$. The following are equivalent:*

(a) *the operators $C_{\varphi_1}, \cdots, C_{\varphi_m}$ are d-supercyclic on $H(\mathbb{D})$;*
(b) *the operators $\mu_1 C_{\varphi_1}, \cdots, \mu_m C_{\varphi_m}$ are d-mixing on $H(\mathbb{D})$ for any non-zero scalars $\mu_1, \cdots, \mu_m$;*

(c) *the symbols* $\varphi_1, \cdots, \varphi_m$ *have no fixed points in* $\mathbb{D}$*, and satisfy that if any two* φ_l, φ_j *have the same attractive fixed point* α*, then the expressions* $\varphi_l'(\alpha) = \varphi_j'(\alpha) < 1$ *do not occur.*

For the higher dimensional case, things will be a little bit difference. Some properties are not easily managed, we need some new methods and calculation techniques. In this paper, we will discuss the higher dimensional case, and our theorems generalize parts of the results obtained in [6].

2. Some lemmas

In this section, we present some lemmas that will be used in the proofs of our main results in the next section. We first give the generalization of Runge theorem to the several variables.

Lemma 2.1. *(See [12, Theorem 4.24].) For a pseudoconvex open subset* Ω *of* $\mathbb{C}^N$*, a necessary and sufficient condition for the polynomial ring* $\mathbb{C}[z]$ *to be dense in* $H(\Omega)$ *is that, for every compact set* $K \subset \Omega$*, there exists a continuous plurisubharmonic exhaustion function* φ *defined on* $\mathbb{C}^N$ *such that* $K \subset \{z|\varphi(z) < 0\} \subset \Omega$*.*

From the above lemma, we have the following lemma.

Lemma 2.2. *The polynomial ring* $\mathbb{C}[z]$ *is dense in* $H(\mathbb{B}_N)$*.*

Proof. Fix any given compact set $K \subset \mathbb{B}_N$, let ρ denote the distance from K to $\partial\mathbb{B}_N$. The continuous plurisubharmonic exhaustion function φ can be defined by

$$\varphi(z) = \sum_{i=1}^{N} z_i\overline{z_i} - 1 + \rho/2.$$

Then the lemma follows by Lemma 2.1. □

Lemma 2.3. *(See [1, Theorem 3.1].) Let* φ *be an unstable parabolic linear fractional map of* $\mathbb{B}_N$*. Then* $\Phi := \mathcal{C} \circ \varphi \circ \mathcal{C}^{-1}$ *is a generalized Heisenberg translation.*

Lemma 2.4. *Assume that* $\varphi_j : \mathbb{B}_N \to \mathbb{B}_N$ $(j = 1, 2)$ *are univalent with no fixed points in* $\mathbb{B}_N$*, and have different attractive fixed points. Then for any compact subset* $K \subset \mathbb{B}_N$*, there exists some sufficiently large* n_0*, whenever* $n \ge n_0$*, then* $\varphi_j^{[n]}(K) \cap K = \emptyset$ *and* $\varphi_1^{[n]}(K) \cap \varphi_2^{[n]}(K) = \emptyset$*, where* $\varphi_j^{[n]}$ *denotes the n-fold composition of* φ_j *with itself.*

Proof. By Theorem 2.83 in [9], there exist different points ξ_j $(j = 1, 2)$ of norm 1 so that the iterates $\varphi_j^{[n]}$ of φ_j converge to ξ_j uniformly on compact subsets of $\mathbb{B}_N$. Given any compact set $K \subset \mathbb{B}_N$, let ρ_1 denote the distance from K to $\partial\mathbb{B}_N$, and ρ_2 denote the distance between ξ_1 and ξ_2. Set $\rho = \min\{\rho_1, \rho_2\}$ and $U_j := \{z \in \mathbb{B}_N : \|z - \xi_j\| < \rho/2\}$, then there is some n_j such that, whenever $n \ge n_j$, we have $\varphi_j^{[n]}(K) \subset U_j$. Hence, whenever $n \ge n_0 := \max\{n_1, n_2\}$, then $\varphi_j^{[n]}(K) \cap K = \emptyset$ and $\varphi_1^{[n]}(K) \cap \varphi_2^{[n]}(K) = \emptyset$. □

Lemma 2.5. *Assume that* φ_1, φ_2 *be generalized hyperbolic or unstable parabolic linear fractional maps of* $\mathbb{B}_N$ *and* $\varphi_1 \ne \varphi_2$*, let* β_1, β_2 *be the attractive fixed points of* φ_1^{-1} *and* φ_2^{-1}*, respectively. If* φ_1, φ_2 *have the same attractive fixed point* τ*, for* $1 \le j, l \le 2$*, then the following situations follows:*

(i) *if* $\alpha(\varphi_l) = \alpha(\varphi_j) = 1$*, then*

$$\varphi_l^{[-n]} \circ \varphi_j^{[n]} \to \tau \text{ locally uniformly on } \mathbb{B}_N;$$

(ii) *if* $\alpha(\varphi_l) < \alpha(\varphi_j) < 1$*, then*

$$\varphi_l^{[-n]} \circ \varphi_j^{[n]} \to \beta_l \text{ locally uniformly on } \mathbb{B}_N, \text{ and}$$

$$\varphi_j^{[-n]} \circ \varphi_l^{[n]} \to \tau \text{ locally uniformly on } \mathbb{B}_N;$$

(iii) *if* $\alpha(\varphi_l) < \alpha(\varphi_j) = 1$*, then*

$$\varphi_l^{[-n]} \circ \varphi_j^{[n]} \to \beta_l \text{ locally uniformly on } \mathbb{B}_N, \text{ and}$$

$$\varphi_j^{[-n]} \circ \varphi_l^{[n]} \to \tau \text{ locally uniformly on } \mathbb{B}_N.$$

Proof. Without loss of generality, we assume that $\tau = e_1$.

Case i: $\alpha(\varphi_l) = \alpha(\varphi_j) = 1$.

For $l = 1, 2$, upon conjugation with Cayley transform $\mathcal{C}(z) = \mathrm{i}(e_1 + z)/(1 - z_1)$, the maps $\Phi_l := \mathcal{C} \circ \varphi_l \circ \mathcal{C}^{-1}$ have $\mathcal{C}(e_1) = \infty$ as an attractive fixed point and, by Lemma 2.3, we have:

$$\Phi_l(w_1, w') = (w_1 + 2\mathrm{i}\langle w', \gamma_l\rangle + b_l, w' + \gamma_l).$$

Straight calculation shows that

$$\Phi_l^{-1}(w_1, w') = (w_1 - b_l + 2\mathrm{i}|\gamma_l|^2 - 2\mathrm{i}\langle w', \gamma_l\rangle, w' - \gamma_l);$$
$$\Phi_l^{[n]}(w_1, w') = (w_1 + nb_l + 2n\mathrm{i}\langle w', \gamma_l\rangle + n(n-1)\mathrm{i}|\gamma_l|^2, w' + n\gamma_l)$$

and

$$\Phi_l^{[-n]}(w_1, w') = (w_1 - nb_l - 2n\mathrm{i}\langle w', \gamma_l\rangle + n(n+1)\mathrm{i}|\gamma_l|^2, w' - n\gamma_l).$$

Thus

$$\Phi_l^{[-n]} \circ \Phi_j^{[n]}(w_1, w') = (w_1 + n(b_j - b_l) + 2n\mathrm{i}\langle w', \gamma_j - \gamma_l\rangle + n(n-1)\mathrm{i}|\gamma_j|^2 + n(n+1)\mathrm{i}|\gamma_l|^2 - 2n^2\mathrm{i}\langle r_j, r_l\rangle, w' + n(\gamma_j - \gamma_l)).$$

So $\Phi_l^{[-n]} \circ \Phi_j^{[n]}(w_1, w') \to \infty$ locally uniformly on $\mathbb{H}_N$, and thus

$$\varphi_l^{[-n]} \circ \varphi_j^{[n]} \to e_1 = \mathcal{C}^{-1}(\infty)$$

locally uniformly on $\mathbb{B}_N = \mathcal{C}^{-1}(\mathbb{H}_N)$.

Case ii: $1 < \lambda_1 = \frac{1}{\alpha(\varphi_1)} < \lambda_2 = \frac{1}{\alpha(\varphi_2)}$.

From Definition 1.5, without loss of generality, suppose that for $j = 1, 2$, φ_j fixes the point $\beta_j = (-r_j, 0')$ $(r_j > 1)$ outside $\overline{\mathbb{B}_N}$, with Denjoy–Wolff point $\tau = e_1 \in \partial\mathbb{B}_N$, then Φ_j fixes the point $\xi_j = (\mathrm{i}\frac{1-r_j}{1+r_j}, 0')$ and

$$\Phi_j(w_1, w') = \left(\lambda_j w_1 + (1 - \lambda_j)\mathrm{i}\frac{1 - r_j}{1 + r_j}, \sqrt{\lambda_j} U_j w'\right), \quad (w_1, w') \in \mathbb{H}_N.$$

By similar arguments as in [11] or Theorem 2.5 in [2], then the Φ_j is automorphism of the half-plane

$$\Omega_j = \left\{(w_1, w') \in \mathbb{C} \times \mathbb{C}^{N-1} : \mathrm{Im}(w_1) > \|w'\|^2 + \frac{1 - r_j}{1 + r_j}\right\}.$$

And the Cayley transform $\mathcal{C}$ is biholomorphic transform from the complex ellipsoid

$$\Delta_j = \left\{(z_1, z') \in \mathbb{C} \times \mathbb{C}^{N-1} : \frac{|z_1 - \frac{1-r_j}{2}|^2}{(\frac{1+r_j}{2})^2} + \frac{\|z'\|^2}{\frac{1+r_j}{2}} < 1\right\}$$

onto Ω_j, and $\mathbb{B}_N \subset \Delta_j$, $\mathbb{H}_N \subset \Omega_j$.

Now

$$\Phi_1^{[-n]} \circ \Phi_2^{[n]}(w_1, w') = \left(\lambda_1^{-n}(\lambda_2^n w_1 + (1 - \lambda_2^n)\mathrm{i}\frac{1 - r_2}{1 + r_2}) + (1 - \lambda_1^{-n})\mathrm{i}\frac{1 - r_1}{1 + r_1}, \lambda_1^{-n}\lambda_2^n (U_1^*)^n (U_2)^n w'\right).$$

So $\Phi_1^{[-n]} \circ \Phi_2^{[n]}(w_1, w') \to \infty$ locally uniformly on $\mathbb{H}_N$, and $\varphi_l^{[-n]} \circ \varphi_j^{[n]} \to e_1 = \mathcal{C}^{-1}(\infty)$ locally uniformly on $\mathbb{B}_N$. And

$$\Phi_2^{[-n]} \circ \Phi_1^{[n]}(w_1, w') = \left(\lambda_2^{-n}(\lambda_1^n w_1 + (1 - \lambda_1^n)\mathrm{i}\frac{1 - r_1}{1 + r_1}) + (1 - \lambda_2^{-n})\mathrm{i}\frac{1 - r_2}{1 + r_2}, \lambda_2^{-n}\lambda_1^n (U_1^*)^n (U_2)^n w'\right)$$

then

$$\Phi_2^{[-n]} \circ \Phi_1^{[n]}(w_1, w') \to \left(\mathrm{i}\frac{1 - r_2}{1 + r_2}, 0'\right)$$

locally uniformly on $\mathbb{H}_N$, thus $\varphi_2^{[-n]} \circ \varphi_1^{[n]} \to (-r_2, 0) = \mathcal{C}^{-1}((\mathrm{i}\frac{1-r_2}{1+r_2}, 0'))$ locally uniformly on $\mathbb{B}_N$.

Case iii: $1 = \lambda_1 < \lambda_2$.

Note that

$$\Phi_1^{[-n]}(w_1, w') = (w_1 - nb_1 - 2n\mathrm{i}\langle w', \gamma_1\rangle + n(n+1)\mathrm{i}|\gamma_1|^2, w' - n\gamma_1);$$
$$\Phi_2^{[n]}(w_1, w') = \left(\lambda_2^n w_1 + (1 - \lambda_2^n)\mathrm{i}\frac{1 - r_2}{1 + r_2}, \lambda_2^{\frac{n}{2}} (U_2)^n w'\right).$$

Direct calculation shows that $\Phi_1^{[-n]} \circ \Phi_2^{[n]}(w_1, w') \to \infty$ locally uniformly on $\mathbb{H}_N$, thus $\varphi_l^{[-n]} \circ \varphi_j^{[n]} \to e_1$ locally uniformly on $\mathbb{B}_N$. Now

$$\Phi_2^{[-n]}(w_1, w') = \left(\lambda_2^{-n} w_1 + (1-\lambda_2^{-n})\mathrm{i}\frac{1-r_2}{1+r_2}, \lambda_2^{-\frac{n}{2}}(U_2^*)^n w'\right),$$

and remember that

$$\Phi_1^{[n]}(w_1, w') = \left(w_1 + nb_1 + 2n\mathrm{i}\langle w', \gamma_1\rangle + n(n-1)\mathrm{i}|\gamma_1|^2, w' + n\gamma_1\right)$$

then

$$\Phi_2^{[-n]} \circ \Phi_1^{[n]}(w_1, w') \to \left(\mathrm{i}\frac{1-r_2}{1+r_2}, 0'\right)$$

locally uniformly on $\mathbb{H}_N$, so $\varphi_2^{[-n]} \circ \varphi_1^{[n]} \to (-r_2, 0)$ locally uniformly on $\mathbb{B}_N$. □

3. Main theorems

Proposition 3.1. *Let φ_1, φ_2 be generalized hyperbolic holomorphic linear fractional self-maps of $\mathbb{B}_N$ defined in Definition 1.5, with a common attractive fixed point τ, such that $\alpha(\varphi_1) = \alpha(\varphi_2)$, and different other fixed points outside $\overline{\mathbb{B}_N}$. Then $C_{\varphi_1}, C_{\varphi_2}$ are not d-supercyclic on $H(\mathbb{B}_N)$.*

Proof. As discussion in Lemma 2.5, set $\lambda = \frac{1}{\alpha(\varphi_1)} = \frac{1}{\alpha(\varphi_2)} > 1$, note that by our assumption we have $r_1 \neq r_2$. Set $r_1 < r_2$, and a straight calculation shows that

$$\Phi_j^{[n]}(w_1, w') = \left(\lambda^n w_1 + (1-\lambda^n)\mathrm{i}\frac{1-r_j}{1+r_j}, \lambda^{\frac{n}{2}}(U_j)^n w'\right)$$

and

$$\Phi_j^{[-n]}(w_1, w') = \left(\lambda^{-n} w_1 + (1-\lambda^{-n})\mathrm{i}\frac{1-r_j}{1+r_j}, \lambda^{-\frac{n}{2}}(U_j^*)^n w'\right),$$

where U^* denotes the adjoint of the unitary matrix U. Hence,

$$\Phi_1^{[-n]} \circ \Phi_2^{[n]}(w_1, w') = \left(w_1 + (\frac{1-r_2}{1+r_2} - \frac{1-r_1}{1+r_1})\mathrm{i}(\lambda^{-n} - 1), (U_1^*)^n (U_2)^n w'\right).$$

Now suppose that $C_{\varphi_1}, C_{\varphi_2}$ are d-supercyclic on $H(\mathbb{B}_N)$, and let $f \in H(\mathbb{B}_N)$ be a d-supercyclic vector for $C_{\varphi_1}, C_{\varphi_2}$. Therefore, for $g \in H(\mathbb{B}_N)$ with $g(z) = z_1$, there exist an increasing sequence (n_k) of positive integers and a sequence (μ_k) of non-zero scalars such that

$$\mu_k(f \circ \varphi_j^{[n_k]}) \to g, \ k \to \infty$$

in $H(\mathbb{B}_N)$ $(j = 1, 2)$. Set $\Psi^{[n]} := \varphi_1^{[-n]} \circ \varphi_2^{[n]} = \mathcal{C}^{-1} \circ \Phi_1^{[-n]} \circ \Phi_2^{[n]} \circ \mathcal{C}$, and let $z = (z_1, z') \in \mathbb{B}_N$ be fixed, $[\Psi]_1 := \lim_{n\to\infty} [\Psi^{[n]}]_1 = \frac{(1-z_1)\beta + 2\mathrm{i}z_1}{(1-z_1)\beta + 2\mathrm{i}}$, here $[\Psi^{[n]}]_1$ denotes the first component of $\Psi^{[n]}$ and $\beta = \mathrm{i}\left(\frac{1-r_1}{1+r_1} - \frac{1-r_2}{1+r_2}\right)$. Let $z \in \mathbb{B}_N$ be fixed, there exists $\epsilon > 0$ such that $K_\epsilon = \{(w_1, w') \in \mathbb{B}_N : |w_1 - [\Psi]_1(z)| \le \epsilon\}$. Since that $\Phi_1^{[-n]} \circ \Phi_2^{[n]}$ are parabolic with $\mathrm{Im}(\beta) > 0$, then map $\mathbb{H}_\mathbb{N}$ into $\mathbb{H}_\mathbb{N}$ when n is large enough, note that the Cayley transform is a biholomorphic map of $\mathbb{B}_\mathbb{N}$ onto $\mathbb{H}_\mathbb{N}$; we have $\Psi^{[n]}(z) \in K_\epsilon$ for n large enough. Notice that

$$\lim_{k\to\infty} \left|\mu_k(f \circ \varphi_1^{[n_k]} \circ \Psi^{[n_k]}) - g \circ \Psi^{[n_k]}\right|(z) \le \lim_{k\to\infty} \sup_{w \in K_\epsilon} \left|\mu_k(f \circ \varphi_1^{[n_k]}) - g\right|(w) = 0.$$

We obtain

$$\begin{aligned} z_1 = g(z) &= \lim_{k\to\infty} \mu_k(f \circ \varphi_2^{[n_k]})(z) \\ &= \lim_{k\to\infty} \mu_k(f \circ \varphi_1^{[n_k]} \circ \Psi^{[n_k]})(z) \\ &= \lim_{k\to\infty} (g \circ \Psi^{[n_k]})(z) = [\Psi]_1(z) \end{aligned}$$

for each $z \in \mathbb{B}_N$, which is a contradiction. □

Theorem 3.2. *Let $m \geq 2$, and assume that $\varphi_1, \cdots, \varphi_m$ be generalized hyperbolic or unstable parabolic linear fractional maps of $\mathbb{B}_N$, and satisfy that if any two φ_l, φ_j have the same attractive fixed point τ, then the expression $\alpha(\varphi_l) = \alpha(\varphi_j) < 1$ does not occur. And let $u_1, \cdots, u_m \in H(\mathbb{B}_N)$, with $u_i(z) \neq 0$ for every $z \in \mathbb{B}_N$ for each $1 \leq i \leq m$. Then the operators $u_1 C_{\varphi_1}, \cdots, u_m C_{\varphi_m}$ are d-mixing on $H(\mathbb{B}_N)$.*

Proof. Note that the compact-open topology on $H(\mathbb{B}_N)$ is independent of the chosen exhaustion. We set $K_n := \{z \in \mathbb{B}_N : \|z\| \leq 1 - 1/n\}, n \in \mathbb{N}$, which is an exhaustion of $\mathbb{B}_N$, then we endow $H(\mathbb{B}_N)$ with the topology induced by the seminorms $p_n(f) := \sup_{z \in K_n} |f(z)|$, $f \in H(\mathbb{B}_N)$. Let $U, V_1, \cdots, V_m$ be non-empty open subsets of $H(\mathbb{B}_N)$, and fix $f \in U, g_j \in V_j$, for $1 \leq j \leq m$. By the definition of the topology on $H(\mathbb{B}_N)$, there is a closed ball K centered on 0 and an $\epsilon > 0$ such that a holomorphic function h belongs to U (or to V_j) whenever $\sup_{z \in K} |f(z) - h(z)| < \epsilon$ (or $\sup_{z \in K} |g_j(z) - h(z)| < \epsilon$, respectively). Let $\widetilde{K}$ be a closed ball in $\mathbb{B}_N$ such that $K \subset \widetilde{K}^\circ \subset \widetilde{K}$. Since φ_j $(1 \leq j \leq m)$ are univalent and without fixed points in $\mathbb{B}_N$, we know by Lemmas 2.4 and 2.5 that there exists n_0 such that $\widetilde{K}, \varphi_1^{[n]}(\widetilde{K}), \cdots, \varphi_m^{[n]}(\widetilde{K})$ are pairwise disjoint, whenever $n \geq n_0$. Then the function f is holomorphic on some neighborhood of $\widetilde{K}$, and function $\frac{g_j \circ (\varphi^{[-n]})}{\prod_{k=1}^n (u_j \circ (\varphi^{[-k]}))}$ is holomorphic on some neighborhood of $\varphi_j^{[n]}(\widetilde{K})$. Since the compact set $\mathcal{K} := \widetilde{K} \bigcup \varphi_1^{[n]}(\widetilde{K}) \bigcup \cdots \bigcup \varphi_m^{[n]}(\widetilde{K})$ is a polynomial hull and has connected complement, there exists a polynomial h such that

$$\sup_{z \in \widetilde{K}} |f(z) - h(z)| < \epsilon \quad \text{and} \quad \sup_{y \in \varphi_j^{[n]}(\widetilde{K})} \left| \frac{g_j \circ (\varphi_j^{[-n]})}{\prod_{k=1}^n (u_j \circ (\varphi_j^{[-k]}))}(y) - h(y) \right| < \frac{\epsilon}{M_j},$$

where

$$M_j := \max_{y \in \varphi_j^{[n]}(\widetilde{K})} \left| \prod_{k=1}^n (u_j \circ (\varphi_j^{[-k]}))(y) \right|.$$

Hence for each $1 \leq j \leq m$ and $n \geq n_0$,

$$\sup_{z \in K} |f(z) - h(z)| < \epsilon$$

and

$$\sup_{z \in K} \left| g_j(z) - (u_j C_{\varphi_j})^n h(z) \right|$$
$$= \sup_{z \in K} \left| \prod_{k=1}^n (u_j \circ (\varphi_j^{[-k]}))(y) \left(\frac{g \circ (\varphi_j^{[-n]})}{\prod_{l=1}^n (u_j \circ (\varphi_j^{[-l]}))}(y) - h(y) \right) \right| < \epsilon,$$

where $y := \varphi^{[n]}(z)$. This shows that $h \in U$ and $(u_j C_{\varphi_j})^n h \in V_j$, that is, the operators $u_1 C_{\varphi_1}, \cdots, u_m C_{\varphi_m}$ are d-mixing on $H(\mathbb{B}_N)$. □

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