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On the regularity of solutions to Poisson's equation

Sur la régularité des solutions de l'équation de Poisson

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ABSTRACT

In this note, we announce new regularity results for some locally integrable distributional solutions to Poisson's equation. This includes, for example, the standard solutions obtained by convolution with the fundamental solution. In particular, our results show that there is no qualitative difference in the regularity of these solutions in the plane and in higher dimensions.

RÉSUMÉ

Dans cette note, nous annonçons de nouveaux résultats de régularité pour des solutions distributionelles localement intégrables à l'équation de Poisson. Cela comprend, par exemple, les solutions standard obtenues par convolution avec la solution fondamentale. En particulier, nos résultats montrent qu'il n'y a aucune différence qualitative de régularité entre ces solutions dans le plan et celles en dimensions supérieures.

1. Main results

If $f \in C_c^\infty(\mathbb{R}^N)$, a classical solution to Poisson's equation

$$-\Delta u = f \tag{1}$$

is given by convolution with the fundamental solution to Laplace's equation (see, for example, [6, Chapter 6, Theorem 6.21, p. 157]): When $N = 2$,

$$u(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \log \frac{1}{|x-y|} f(y)\, dy, \tag{2}$$

while for $N \geq 3$,

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$$u(x) = \frac{\Gamma(\frac{N}{2})}{2\pi^{\frac{N}{2}}(N-2)} \int_{\mathbb{R}^N} \frac{f(y)}{|y-x|^{N-2}}\,dy. \tag{3}$$

For $f \in L^p(\mathbb{R}^N)$, $1 < p < +\infty$, the existence of $L^1_{\text{loc}}(\mathbb{R}^N)$ distributional solutions to (1), i.e. $u \in L^1_{\text{loc}}(\mathbb{R}^N)$ which satisfy

$$-\int_{\mathbb{R}^N} u\Delta\varphi = \int_{\mathbb{R}^N} f\varphi, \tag{4}$$

for all $\varphi \in C_c^\infty(\mathbb{R}^N)$, follows from a priori estimates on potentials and density arguments. Moreover, if one considers only solutions that satisfy the growth condition

$$\frac{u(x)}{|x|} \to 0 \text{ as } |x| \to \infty, \tag{5}$$

then such a $u \in L^1_{\text{loc}}(\mathbb{R}^N)$ is unique up to a constant (for the convenience of the reader, we provide a short proof of this Liouville-type theorem below).

In the forthcoming work [4], we establish several new sharp estimates for potentials related to (2) and (3) mapping $f \in L^p(\mathbb{R}^N)$, $\frac{N}{2} < p \leq N$. In particular, for $N = 2$ we define the map (which is a modified form of the potentials considered by Strömberg and Wheeden [9, p. 294])

$$\tilde{T}_j h(x) := \frac{1}{2\pi} \int_{\mathbb{R}^2} \left[\frac{y_j - x_j}{|y-x|} - \frac{y_j}{|y|}\right] h(y)\,dy,$$

for which we show

Theorem 1.1. *Let $1 < p \leq 2$.*

i) *If $1 < p < 2$, then there exists $C = C(p)$ such that*

$$|\tilde{T}_j h(x) - \tilde{T}_j h(z)| \leq C|x-z|^{2-\frac{2}{p}} \|h\|_{L^p(\mathbb{R}^2)}$$

for all $h \in L^p(\mathbb{R}^2)$ and $j = 1, 2$.

ii) *If $p = 2$ and $1 \leq q < 2$, then there exists $C = C(q)$ such that*

$$|\tilde{T}_j h(x) - \tilde{T}_j h(z)| \leq C|x-z|\,(|\log|x-z|| + 1)^{\frac{1}{2}} \left(\|h\|_{L^2(\mathbb{R}^2)} + \|h\|_{L^q(\mathbb{R}^2)}\right)$$

for all $h \in L^2(\mathbb{R}^2) \cap L^q(\mathbb{R}^2)$ and $j = 1, 2$.

When $N \geq 3$, letting

$$\tilde{I}_2 f(x) := \frac{\Gamma(\frac{N}{2})}{2\pi^{\frac{N}{2}}(N-2)} \int_{\mathbb{R}^N} \left[\frac{1}{|y-x|^{N-2}} - \frac{1}{|y|^{N-2}}\right] f(y)\,dy$$

denote the modified Newtonian potential, we prove

Theorem 1.2. *Let $N \geq 3$. For any $1 \leq q < N$, there exists $C = C(q, N)$ such that*

$$|\tilde{I}_2 f(x) - \tilde{I}_2 f(z)| \leq C|x-z|\,(|\log|x-z|| + 1)^{\frac{1}{N'}} (\|f\|_{L^q(\mathbb{R}^N)} + \|f\|_{L^N(\mathbb{R}^N)})$$

for all $f \in L^N(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$, where N' denotes the Hölder conjugate of N.

Then, establishing

$$u := \tilde{T}_1 R_1 f + \tilde{T}_2 R_2 f \tag{6}$$

solves (4) for $N = 2$, where R_j is the standard j-th Riesz transform,

$$R_j f(x) := \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{x_j - y_j}{|x-y|^3} f(y)\,dy,$$

and $u = \tilde{I}_2 f$ solves (4) for $N \geq 3$, we are able to conclude the main result we announce in this note, the following theorem on the regularity of the unique solution to (4) satisfying (5).

Theorem 1.3. *Suppose that $N \geq 2$ and either $f \in L^p(\mathbb{R}^N)$ with $\frac{N}{2} < p < N$ or $f \in L^N(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$ for some $1 \leq q < N$. Then there exists $u \in L^1_{\text{loc}}(\mathbb{R}^N)$ which satisfies (4), for which one has the following regularity estimates.*

i) *If $\frac{N}{2} < p < N$, then*

$$|u(x) - u(z)| \leq C|x-z|^{2-\frac{N}{p}} \|f\|_{L^p(\mathbb{R}^N)}.$$

ii) *If $p = N$, then denoting by N' it's Hölder conjugate, one has*

$$|u(x) - u(z)| \leq C|x-z| \left(|\log|x-z|| + 1\right)^{\frac{1}{N'}} \left(\|f\|_{L^N(\mathbb{R}^N)} + \|f\|_{L^q(\mathbb{R}^N)}\right).$$

In particular, this solution satisfies (5), and hence it is unique up to a constant.

In the regime in which $\frac{N}{2} < p < N$, and when $N \geq 3$, sharp Hölder regularity follows easily from known potential estimates (see, for example, [8, Section 4.2, Theorem 2.2, p. 155]). Our result shows how this approach via potentials can be extended to $N = 2$, provided that one is willing to utilize some basic Fourier analysis. The experts will recognize that one could alternatively deduce Part i) of this result from the embeddings (on the Triebel–Lizorkin/Besov scale)

$$\dot{F}^2_{p2} \hookrightarrow \dot{F}^2_{pp} = \dot{B}^2_{pp} \hookrightarrow \dot{B}^{2-\frac{N}{p}}_{\infty\infty},$$

see [5, pp. 95–96]. The estimate for $p = N$ is new in this setting—here we recall that an analogous supercritical estimate was shown for functions in the Sobolev space $W^{2,N}(\mathbb{R}^N)$ by Brezis and Wainger [2, Corollary 5]. While one has the inclusion

$$W^{2,p}(\mathbb{R}^N) \subsetneq \{u \in L^1_{\text{loc}}(\mathbb{R}^N) : \Delta u \in L^p(\mathbb{R}^N)\},$$

it is, in general, strict. For example, taking $f = \chi_{B(0,1)}(x)$, one can verify that when $N = 2$ there is no $q \in [1, \infty]$ such that $u \in L^q(\mathbb{R}^2)$, while when $N = 3$ there is no $q \in [1, 3]$ such that $u \in L^q(\mathbb{R}^3)$. Therefore, while the known embeddings for $W^{2,p}(\mathbb{R}^N)$ cannot be applied, our result establishes that the solutions to Poisson's equation enjoy analogous regularity estimates (which are known to be sharp—see, for example, [7, Chapter 1, p. 62, Remark 1] and [2, Corollary 5]).

We now sketch a proof that the function defined by (6) solves (4). Here we take the convention

$$\widehat{\varphi}(\xi) = \int_{\mathbb{R}^2} \varphi(x)\, \mathrm{e}^{-2\pi \mathrm{i} x\cdot\xi}\, \mathrm{d}x$$

for the Fourier transform, and our computation should be interpreted in the sense of tempered distributions.

Proof 1. First, we remark that for $\frac{N}{2} < p < N$, one can show the estimate (whose precise proof can be found in [4])

$$\int_{\mathbb{R}^2} \left| \frac{y_j - x_j}{|y-x|} - \frac{y_j}{|y|} \right| |R_j f(y)|\, dy \leq C|x|^{2-\frac{N}{p}} \|f\|_{L^p(\mathbb{R}^2)},$$

for $j = 1, 2$. Therefore, Fubini's theorem implies

$$\begin{aligned} -\int_{\mathbb{R}^2} u\Delta\varphi &= -\frac{1}{2\pi} \sum_{j=1,2} \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} \left[\frac{y_j - x_j}{|y-x|} - \frac{y_j}{|y|} \right] R_j f(y)\, \mathrm{d}y \right) \Delta\varphi(x)\, \mathrm{d}x \\ &= -\frac{1}{2\pi} \sum_{j=1,2} \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} \left[\frac{y_j - x_j}{|y-x|} - \frac{y_j}{|y|} \right] \Delta\varphi(x)\, \mathrm{d}x \right) R_j f(y)\, \mathrm{d}y. \end{aligned}$$

Further, since the divergence theorem implies $\int_{\mathbb{R}^2} \Delta\varphi(x)\, \mathrm{d}x = 0$, we have that

$$\int_{\mathbb{R}^2} \left[\frac{y_j - x_j}{|y-x|} - \frac{y_j}{|y|} \right] \Delta\varphi(x)\, \mathrm{d}x = \int_{\mathbb{R}^2} \frac{y_j - x_j}{|y-x|} \Delta\varphi(x)\, \mathrm{d}x.$$

Now, we define

$$g_j(y) := -\frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{y_j - x_j}{|y-x|} \Delta\varphi(x)\, \mathrm{d}x.$$

If we can show that $g_j = R_j\varphi$ almost everywhere, then we would have

$$-\int_{\mathbb{R}^2} u\Delta\varphi = \sum_{j=1,2}\int_{\mathbb{R}^2} R_j\varphi R_j f$$
$$= \int_{\mathbb{R}^2} f\varphi,$$

which is the thesis. Notice that

$$g_j(y) = -\frac{y_j}{2\pi}\int_{\mathbb{R}^2}\frac{1}{|y-x|}\Delta\varphi(x)\,\mathrm{d}x + \frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{1}{|y-x|}x_j\Delta\varphi(x)\,\mathrm{d}x,$$

and therefore,

$$\begin{aligned}\widehat{g}_j(\xi) &= \frac{1}{2\pi\mathrm{i}}\frac{\partial}{\partial\xi_j}\left((2\pi|\xi|)^{-1}\widehat{\Delta\varphi}(\xi)\right) + (2\pi|\xi|)^{-1}\left(\widehat{x_j\Delta\varphi}(\xi)\right)\\ &= \mathrm{i}\left[\frac{\partial}{\partial\xi_j}(|\xi|\widehat{\varphi}(\xi)) - \frac{1}{|\xi|}\frac{\partial}{\partial\xi_j}(|\xi|^2\widehat{\varphi}(\xi)\right]\\ &= -\mathrm{i}\frac{\xi_j}{|\xi|}\widehat{\varphi}(\xi)\\ &= \widehat{R_j\varphi}.\end{aligned}$$

Thus, we have proved that $g_j = R_j\varphi$ as distributions, which implies almost everywhere equality as functions, and the result is demonstrated. □

An aspect of our result worth further mention is the new representation formula for the logarithmic potential given by (6). More generally, we show in [4] that such a factorization of the logarithmic potential holds in any number of dimensions, which is interestingly related to the $\mathcal{H}^1 - BMO$ duality. We recall that the abstract result of Fefferman and Stein [3] (or the later constructive proof of Uchiyama [10]) implies that any $u \in BMO(\mathbb{R}^N)$ has a representation

$$u = u_0 + \sum_{j=1}^{N} R_j u_j$$

for some $\{u_j\}_{j=0}^N \subset L^\infty(\mathbb{R}^N)$. More recently, Bourgain and Brezis [1] have shown that for certain *BMO* functions one can take $u_0 \equiv 0$. Our result shows that for the canonical example of a *BMO* function—$\log|x|$—one has explicitly $u_0 \equiv 0$ and $u_j = c\frac{x_j}{|x|}$.

Finally, we give a short proof of the Liouville-type theorem remarked in the beginning.

Proof 2. Given u_1, u_2 that satisfy (4) and (5), define $w := u_1 - u_2$. Then w is harmonic and also satisfies (5). Thus, first utilizing the mean value property for harmonic functions and then the growth condition (5), one has

$$\begin{aligned}|w(x) - w(y)| &\le \frac{C}{r^N}\int_{B(x,r)\Delta B(y,r)} |w(z)|\,\mathrm{d}z\\ &\le \frac{C\epsilon}{r^N}\int_{B(x,r)\Delta B(y,r)} |z|\,\mathrm{d}z,\end{aligned}$$

for all $\epsilon > 0$ and all r sufficiently large (depending on $\epsilon, |x|, |y|$). As $r \to \infty$ the right-hand side stays bounded for x, y fixed, after which one can send $\epsilon \to 0$. □

As mentioned earlier, the proofs of Theorems 1.1 and 1.2 will appear in the forthcoming work [4], where we also address regularity properties of more general cases of Riesz and Riesz-type potentials, as well as the application of these results to deduce the embedding theorem of Brezis and Wainger [2, Corollary 5] in the supercritical case.

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