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Formal normal form of A_k slow–fast systems

Forme normale formelle des systèmes lents–rapides de type A_k

Hildeberto Jardón-Kojakhmetov

Johann Bernoulli Institute for Mathematics and Computer Science, University of Groningen, P.O. Box 407, 9700 AK, Groningen, The Netherlands

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ABSTRACT

An A_k slow–fast system is a particular type of singularly perturbed ODE. The corresponding slow manifold is defined by the critical points of a universal unfolding of an A_k singularity. In this note we propose a formal normal form of A_k slow–fast systems.

RÉSUMÉ

Un système lent–rapide de type A_k est une équation différentielle ordinaire singulièrement perturbée avec une structure particulière. La varieté lente correspondante est définie par les points critiques d'un déploiement universel d'une singularité de type A_k. Dans cette note, nous proposons une forme normale formelle des systèmes lents–rapides de type A_k.

1. Introduction

A slow–fast system (SFS) is a singularly perturbed ODE usually written as

$$\begin{aligned} \dot{x} &= f(x,z,\varepsilon) \\ \varepsilon\dot{z} &= g(x,z,\varepsilon) \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^m$, $z \in \mathbb{R}^n$ and $0 < \varepsilon \ll 1$ is a small parameter, and where the over-dot denotes the derivative with respect to a time parameter t. Slow–fast systems are often used as mathematical models of phenomena that occur in two time scales. A couple of classical examples of real life phenomena that were modeled by an SFS are Zeeman's heartbeat and nerve-impulse models [17]. For $\varepsilon \neq 0$, we can define a new time parameter τ by $t = \varepsilon\tau$. With this new time τ, we can write (1) as

$$\begin{aligned} x' &= \varepsilon f(x,z,\varepsilon) \\ z' &= g(x,z,\varepsilon), \end{aligned} \tag{2}$$

where the prime denotes the derivative with respect to τ. An important geometric object in the study of SFSs is the *slow manifold*, which is defined by

E-mail address: h.jardon.kojakhmetov@gmail.com.

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$$S = \left\{ (x, z) \in \mathbb{R}^m \times \mathbb{R}^n \mid g(x, z, 0) = 0 \right\}. \tag{3}$$

When $\varepsilon = 0$, the manifold S serves as the phase space of (1) and as the set of equilibrium points of (2). In the rest of the document, we prefer to work with an SFS written as (2). Furthermore, to avoid working with an ε-parameter family of vector fields as in (2), we extend (2) by adding the trivial equation $\varepsilon' = 0$. To be more precise, we treat a C^∞-smooth vector field defined as follows.

Definition 1.1 *(A_k slow–fast system).* Let $k \in \mathbb{N}$ with $k \geq 2$. An A_k slow–fast system (for short A_k-SFS) is a vector field X of the form

$$X = \varepsilon(1 + \varepsilon f_1)\frac{\partial}{\partial x_1} + \sum_{i=2}^{k-1} \varepsilon^2 f_i \frac{\partial}{\partial x_i} - (G_k - \varepsilon f_k)\frac{\partial}{\partial z} + 0\frac{\partial}{\partial \varepsilon}, \tag{4}$$

where $G_k = z^k + \sum_{i=1}^{k-1} x_i z^{i-1}$ and where each $f_i = f_i(x_1, \ldots, x_{k-1}, z, \varepsilon)$ is a C^∞-smooth function vanishing at the origin.

Multi-scale models described by an A_k-SFS are of interest as they exhibit local, fast transitions between stable states, e.g., [8,13,14,17].

Remark 1.1. The slow manifold associated with an A_k-SFS is defined by

$$S = \left\{ (x, z) \in \mathbb{R}^k \mid z^k + \sum_{i=1}^{k-1} x_i z^{i-1} = 0 \right\}. \tag{5}$$

The manifold S can be regarded as the critical set of the universal unfolding of a smooth function with an A_k singularity at the origin [1,3]. Hence the name A_k-SFS.

Observe that the origin is a non-hyperbolic equilibrium point of X and thus, it is not possible to study its local dynamics with the classical Geometric Singular Perturbation Theory [5]. In this case, the blow-up technique [4,9] can be applied to desingularize the SFS. This methodology has been successfully used in many cases, e.g., [2,7,10,11,15,16], where many of these deal with an A_k-SFS with fixed $k = 2$ or $k = 3$. Briefly speaking, the blow-up technique consists in an appropriate change of coordinates under which the induced vector field is regular or has simpler singularities (hyperbolic or partially hyperbolic).

In this paper we propose a normal form of A_k-SFSs given by Definition 1.1. In this normalization, the unknown functions f_i of (4) are eliminated. As it is shown below, the structure of the A_k-SFS plays an important role in the normalization process. Moreover, this normalization greatly simplifies the local analysis of systems given by (4), as shown in [6,7].

2. Formal normal form of an A_k slow–fast system

We regard the vector field X of Definition 1.1 as $X = F + P$, where F and P are smooth vector fields called "the principal part" and "the perturbation" respectively. That is:

$$F = \varepsilon\frac{\partial}{\partial x_1} + \sum_{i=2}^{k-1} 0\frac{\partial}{\partial x_i} - G_k\frac{\partial}{\partial z} + 0\frac{\partial}{\partial \varepsilon}, \qquad P = \sum_{i=1}^{k-1} \varepsilon^2 f_i \frac{\partial}{\partial x_i} + \varepsilon f_k \frac{\partial}{\partial z} + 0\frac{\partial}{\partial \varepsilon}. \tag{6}$$

The idea of the rest of the document is motivated by [12]. In short, we want to formally simplify the expression of X by eliminating the perturbation P. The terminology used below is that of [12].

The vector field F is quasihomogeneous of type $r = (k, k-1, \ldots, 1, 2k-1)$ and quasidegree $k-1$ [1,12]. From now on, we fix the type of quasihomogeneity r. A quasihomogeneous object of type r will be called r-quasihomogeneous.

Definition 2.1 *(Good perturbation).* Let F be an r-quasihomogeneous vector field of quasidegree $k-1$. A good perturbation X of F is a smooth vector field $X = F + P$, where $P = P(x_1, \ldots, x_{k-1}, z, \varepsilon)$ satisfies the following conditions:

- P is a smooth vector field of quasiorder greater than $k-1$,
- $P = \sum_{i=1}^{k-1} P_i \frac{\partial}{\partial x_i} + P_k \frac{\partial}{\partial z} + 0\frac{\partial}{\partial \varepsilon}$, with $P|_{\varepsilon=0} = 0$.

Notation By $\mathcal{P}_\delta$ we denote the space of r-quasihomogeneous polynomials (in $k+1$ variables) of quasidegree δ. By $\mathcal{H}_\gamma$ we denote the space of r-quasihomogeneous vector fields (in $\mathbb{R}^{k+1}$) of quasidegree γ and such that for all $U \in \mathcal{H}_\gamma$ we have $U = \sum_{i=1}^{k} U_k \frac{\partial}{\partial x_i} + 0\frac{\partial}{\partial x_{k+1}}$. The formal series expansion of a function f is denoted by $\hat{f}$.

Definition 2.2 *(The inner product $\langle\cdot,\cdot\rangle_{r,\delta}$ [12]).* Let $x=(x_1,\dots,x_n)$, and $s,q\in\mathbb{N}^n$. Let $f,g\in\mathcal{P}_\delta$, that is $f=\sum_{(r,s)=\delta} f_s x^s$, where $f_s\in\mathbb{R}$, $x^s=x_1^{s_1}\cdots x_n^{s_n}$; and similarly for g. Then the inner product $\langle\cdot,\cdot\rangle_{r,\delta}$ is defined as

$$\langle f,g\rangle_{r,\delta}=\sum_{(r,s)=\delta} f_s g_s\frac{(s!)^r}{\delta!}, \tag{7}$$

where $(s!)^r=(s_1!)^{r_1}\cdots(s_n!)^{r_n}$, and where (r,s) denotes the dot product $r\cdot s$. So for monomials one has

$$\langle x^s,x^q\rangle_{r,\delta}=\begin{cases}\frac{(s_1!)^{r_1}\cdots(s_n!)^{r_n}}{\delta!} & \text{if}\quad s=q\quad\text{with}\quad (s,r)=\delta,\\ 0 & \text{otherwise.}\end{cases} \tag{8}$$

Accordingly, for vector fields: let $X=\sum_{i=1}^n X_i\frac{\partial}{\partial x_i}\in\mathcal{H}_\delta$, and $Y=\sum_{i=1}^n Y_i\frac{\partial}{\partial x_i}\in\mathcal{H}_\delta$. Then

$$\langle X,Y\rangle_{r,\delta}=\sum_{i=1}^n\langle X_i,Y_i\rangle_{r,\delta+r_i}. \tag{9}$$

Definition 2.3 *(The operators d, d^* and $\Box$ [12]).* The operator $d:\mathcal{H}_\gamma\to\mathcal{H}_{\gamma+k-1}$ (associated with F) is defined by $d(U)=[F,U]$ for any $U\in\mathcal{H}_\gamma$, where $[\cdot,\cdot]$ denotes the Lie bracket. The operator d^* is the adjoint operator of d with respect to the inner product of Definition 2.2. This is, given $U\in\mathcal{H}_\gamma$, $V\in\mathcal{H}_{\gamma+k-1}$ we have

$$\langle d(U),V\rangle_{r,\gamma+k-1}=\langle U,d^*(V)\rangle_{r,\gamma}. \tag{10}$$

For any quasidegree $\beta>k-1$, the self adjoint operator $\Box_\beta:\mathcal{H}_\beta\to\mathcal{H}_\beta$ is defined by $\Box_\beta(U)=dd^*(U)$ for all $U\in\mathcal{H}_\beta$.

Definition 2.4 *(Resonant vector field [12]).*

- We say that a vector field $U\in\mathcal{H}_\beta$ is resonant if $U\in\ker\Box_\beta$.
- A formal vector field is called resonant if all its quasihomogeneous components are resonant.

Definition 2.5 *(Normal form [12]).* A good perturbation $X=F+R$ of F is a *normal form with respect to* F if R is resonant.

It is important to note the following.

Lemma 2.1. $\ker\Box_\beta=\ker d^*|_{\mathcal{H}_\beta}$.

Proof. Let $\alpha=k-1$, then $d:\mathcal{H}_\gamma\to\mathcal{H}_{\gamma+\alpha}$ and $d^*:\mathcal{H}_{\gamma+\alpha}\to\mathcal{H}_\gamma$. Due to the fact that d^* is the adjoint of d, we have the decomposition $\mathcal{H}_\gamma=\operatorname{Im}d^*|_{\mathcal{H}_{\gamma+\alpha}}\oplus\ker d|_{\mathcal{H}_\gamma}$. Now let $U\in\mathcal{H}_{\gamma+\alpha}=\mathcal{H}_\beta$, then $\Box_\beta(U)=dd^*(U)=0$ if and only if $d^*U\in\ker d$. Furthermore, $d^*U\in\operatorname{Im}d^*$. That is $d^*U\in\operatorname{Im}d^*\cap\ker d$. However $\operatorname{Im}d^*$ and $\ker d$ are orthogonal. Then $\Box_\beta(U)=0$ if and only if $d^*U=0$. □

We now recall a result of [12] (Proposition 4.4); we only adapt it for the present context.

Theorem 2.1 *(Formal normal form [12]). Let $X=F+P$ be a good perturbation of F as in Definition 2.1. Then there exists a formal diffeomorphism $\hat\Phi$ such that $\hat\Phi$ conjugates $\hat X$ to a vector field $F+R$, where R is a resonant formal vector field in the sense of Definition 2.4.*

Finally, we present our result: we prove that the resonant vector field R in Theorem 2.1 associated with F given by (6) is $R=0$.

Theorem 2.2. *Let $X=F+P$ be a good perturbation of the vector field*

$$F=\varepsilon\frac{\partial}{\partial x_1}+\sum_{i=2}^{k-1}0\frac{\partial}{\partial x_i}-\left(z^k+\sum_{j=1}^{k-1}x_jz^{j-1}\right)\frac{\partial}{\partial z}+0\frac{\partial}{\partial\varepsilon}. \tag{11}$$

Then, there exists a formal diffeomorphism $\hat\Phi$ that conjugates $\hat X$ with F, this is $\hat\Phi_*\hat X=F$.

Proof. From Theorem 2.1 and Lemma 2.1 we will show that if $P\in\ker d^*|_{\mathcal{H}_{\geq k}}$ then $P=0$. To simplify the notation, let $\alpha\geq k$, $P\in\mathcal{H}_\alpha$, $\beta=\alpha-k+1$, and $Q\in\mathcal{H}_\beta$; and let $x=(x_1,\dots,x_{k-1},z,\varepsilon)=(x_1,\dots,x_{k-1},x_k,x_{k+1})$. If D is an operator, its adjoint with respect to the inner product Definition 2.2 is always denoted as D^*. We start with the inner product (Definition 2.2)

$$\langle d(Q), P\rangle_{r,\alpha} = \langle Q, d^*(P)\rangle_{r,\beta}. \tag{12}$$

We can write $d(Q) = \sum_{i=1}^{k+1} F(Q_i) - Q(F_i)$, where $F(Q_i) = \sum_{j=1}^{k+1} F_j \frac{\partial Q_i}{\partial x_j}$ and similarly for $Q(F_i)$, then

$$\begin{aligned}
\langle d(Q), P\rangle_{r,\alpha} &= \sum_{i=1}^{k+1} \langle F(Q_i) - Q(F_i), P_i\rangle_{r,\beta} = \sum_{i=1}^{k+1} \langle F(Q_i), P_i\rangle_{r,\alpha+r_i} - \langle Q(F_i), P_i\rangle_{r,\alpha+r_i} \\
&= \sum_{i=1}^{k+1} \langle Q_i, F^*(P_i)\rangle_{r,\beta+r_i} - \langle Q(F_i), P_i\rangle_{\alpha+r_i} = \sum_{i=1}^{k+1} \langle Q_i, F^*(P_i)\rangle_{r,\beta+r_i} - \sum_{j=1}^{k+1} \langle Q_j, \left(\frac{\partial F_i}{\partial x_j}\right)^* (P_i)\rangle_{\beta+r_j} \\
&= \sum_{i=1}^{k+1} \langle Q_i, F^*(P_i) - \sum_{j=1}^{k+1} \left(\frac{\partial F_j}{\partial x_i}\right)^* (P_j)\rangle_{\beta+r_i}
\end{aligned} \tag{13}$$

Comparing (13) to $\langle Q, d^*(P)\rangle_{r,\beta}$ we can write

$$d^*(P) = \begin{bmatrix} F^* - \left(\frac{\partial F_1}{\partial x_1}\right)^* & -\left(\frac{\partial F_2}{\partial x_1}\right)^* & \cdots & -\left(\frac{\partial F_{k+1}}{\partial x_1}\right)^* \\ -\left(\frac{\partial F_1}{\partial x_2}\right)^* & F^* - \left(\frac{\partial F_2}{\partial x_2}\right)^* & \cdots & -\left(\frac{\partial F_{k+1}}{\partial x_2}\right)^* \\ \vdots & \vdots & \ddots & \vdots \\ -\left(\frac{\partial F_1}{\partial x_{k+1}}\right)^* & -\left(\frac{\partial F_2}{\partial x_{k+1}}\right)^* & \cdots & F^* - \left(\frac{\partial F_{k+1}}{\partial x_{k+1}}\right)^* \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_{k+1} \end{bmatrix}. \tag{14}$$

Plugging in the expressions of F and P into (14) we get

$$d^*(P) = \begin{bmatrix} F^* & 0 & \cdots & 0 & 1 & 0 \\ 0 & F^* & \cdots & 0 & z^* & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \\ 0 & 0 & \cdots & F^* & \left(z^{k-1}\right)^* & 0 \\ 0 & 0 & \cdots & 0 & F^* + Z^* & 0 \\ -1 & 0 & \cdots & 0 & 0 & F^* \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_{k-1} \\ P_k \\ 0 \end{bmatrix} = 0, \tag{15}$$

where $Z^* = \left(kz^{k-1} + \sum_{i=2}^{k-1}(i-1)x_i z^{i-2}\right)^*$. Note that (15) implies $P_1 = P_k = 0$ and $F^*(P_j) = 0$ for all $j = 2, \dots, k-1$.

Remark 2.1. For $k = 2$ the result is trivial: we have $F = \varepsilon \frac{\partial}{\partial x_1} - (z^2 + x_1)\frac{\partial}{\partial z} + 0\frac{\partial}{\partial \varepsilon}$. Therefore $d^*(P) = 0$ is written as

$$d^*(P) = \begin{bmatrix} F^* & 1 & 0 \\ 0 & F^* + 2z^* & 0 \\ -1 & 0 & F^* \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ 0 \end{bmatrix} = 0, \tag{16}$$

which immediately implies $P_1 = P_2 = 0$.

Now, we study $F^*(P_j) = 0$. Recall that $P = P(x_1, \dots, x_{k-1}, z, \varepsilon)$ is not any vector field, but it has the property that $P(x_1, \dots, x_{k-1}, z, 0) = 0$. That is, we can write

$$P = \sum_{i=1}^{k-1} \varepsilon \bar{P}_i \frac{\partial}{\partial x_i} + \varepsilon \bar{P}_k \frac{\partial}{\partial z} + 0\frac{\partial}{\partial \varepsilon}, \tag{17}$$

where $\bar{P}_j \in \mathcal{P}_{\alpha+r_j-2k+1}$. This is because the (quasihomogeneous) weight of ε is $2k-1$. Now, since it is complicated to work with the adjoint, we first rewrite the problem $F^*(\varepsilon \bar{P}_j) = 0$. We then prove that $F^*(\varepsilon \bar{P}_j) = 0$ implies that $\bar{P}_j = 0$.

Note that $F^*(\varepsilon \bar{P}_j) = 0$ is equivalent to $\langle Q, F^*(\varepsilon \bar{P}_j)\rangle_{\beta+r_j} = 0$ for all $Q \in \mathcal{P}_{\beta+r_j}$. Next, we use the definition of F^* that is

$$\langle Q, F^*(\varepsilon \bar{P}_j)\rangle_{r,\beta+r_j} = \langle F(Q), \varepsilon \bar{P}_j\rangle_{r,\alpha+r_j} = 0. \tag{18}$$

We will now show that if $\langle F(Q), \varepsilon \bar{P}_j\rangle_{r,\alpha+r_j} = 0$ for all $Q \in \mathcal{P}_{\beta+r_j}$, then $\bar{P}_j = 0$. Note that by (18), this is the same as proving that $F^*(\varepsilon \bar{P}_j) = 0$ implies $\bar{P}_j = 0$. First, we choose an element x^q of the basis of $\mathcal{P}_{\beta+r_j}$; this is

$$x^q = x_1^{q_1} \cdots x_{k-1}^{q_{k-1}} z^{q_k} \varepsilon^{q_{k+1}}, \qquad (r, q) = \beta + r_j, \tag{19}$$

then, it follows that

$$F(x^q) = q_1 x_1^{q_1-1} \cdots x_{k-1}^{q_{k-1}} z^{q_k} \varepsilon^{q_{k+1}+1} - \left(z^k + \sum_{i=1}^{k-1} x_i z^{i-1} \right) q_k x_1^{q_1} \cdots x_{k-1}^{q_{k-1}} z^{q_k-1} \varepsilon^{q_{k+1}}. \tag{20}$$

Let us write $\varepsilon \bar{P}_j \in \mathcal{P}_{\alpha+r_j}$ as

$$\varepsilon \bar{P}_j = \varepsilon \sum_{(r,p)=\alpha+r_j-2k+1} a_p x_1^{p_1} \cdots x_{k-1}^{p_{k-1}} z^{p_k} \varepsilon^{p_{k+1}}, \tag{21}$$

where $a_p \in \mathbb{R}$. We now proceed by recursion on the exponent of ε. Let $q_{k+1} = 0$, then the inner product $\langle F(Q), \varepsilon \bar{P}_j \rangle_{\alpha+r_j}$ has only one term, since $F(Q)$ has only one monomial containing ε. That is

$$\langle F(Q), \varepsilon \bar{P}_j \rangle_{\alpha+r_j}|_{q_{k+1}=0} = \langle q_1 x_1^{q_1-1} \cdots x_{k-1}^{q_{k-1}} z^{q_k} \varepsilon, \varepsilon a_p x_1^{p_1} \cdots x_{k-1}^{p_{k-1}} z^{p_k} \rangle_{r,\alpha+r_j} = 0. \tag{22}$$

We naturally consider $q_1 > 0$. If $q_1 = 0$, then the equality is automatically satisfied. Recalling Definition 2.2 of the inner product, the equality (22) means that

$$\langle q_1 x_1^{q_1-1} \cdots x_{k-1}^{q_{k-1}} z^{q_k} \varepsilon, \varepsilon a_p x_1^{p_1} \cdots x_{k-1}^{p_{k-1}} z^{p_k} \rangle_{r,\alpha+r_j} = q_1 a_p \frac{(q!)^r}{(\alpha + r_j)!} = 0, \tag{23}$$

and therefore from (22) we have

$$a_p = a_{q_1-1, p_2, \ldots, p_k, 1} = 0, \tag{24}$$

for all $q_1 > 0$, $p_2, \ldots, p_k \geq 0$ (naturally, also satisfying the degree condition $(r, p) = \alpha + r_j$). Next, let $q_{k+1} = 1$. Then

$$F(x^q) = q_1 x_1^{q_1-1} \cdots x_{k-1}^{q_{k-1}} z^{q_k} \varepsilon^2 - \left(z^k + \sum_{i=1}^{k-1} x_i z^{i-1} \right) q_k x_1^{q_1} \cdots x_{k-1}^{q_{k-1}} z^{q_k-1} \varepsilon. \tag{25}$$

Once again, the inner product $\langle F(Q), \varepsilon \bar{P}_j \rangle_{r,\alpha+r_j}$ has only one term, now this is due to the fact that all coefficients a_p of monomials containing ε are zero due to (24). Then

$$\langle F(Q), \varepsilon \bar{P}_j \rangle_{\alpha+r_j}|_{q_{k+1}=1} = \langle q_1 x_1^{q_1-1} \cdots x_{k-1}^{q_{k-1}} z^{q_k} \varepsilon^2, \varepsilon a_p x_1^{p_1} \cdots x_{k-1}^{p_{k-1}} z^{p_k} \varepsilon \rangle_{r,\alpha+r_j} = 0. \tag{26}$$

Therefore, similarly as above, we have the condition

$$a_p = a_{q_1-1, p_2, \ldots, p_k, 2} = 0, \tag{27}$$

for all $q_1 > 0$, $p_2, \ldots, p_k \geq 0$ (naturally, also satisfying the degree condition $(r, p) = \alpha + r_j$). By recursion arguments, assume $q_{k+1} = n$ and that all the coefficients

$$a_p = a_{p_1, p_2, \ldots, p_k, m} = 0, \qquad \forall m \leq n. \tag{28}$$

Then, again, the inner product $\langle F(Q), \varepsilon \bar{P}_j \rangle_{r,\alpha+r_j}$ has only one term, namely

$$\langle F(Q), \varepsilon \bar{P}_j \rangle_{\alpha+r_j}|_{q_{k+1}=n} = \langle q_1 x_1^{q_1-1} \cdots x_{k-1}^{q_{k-1}} z^{q_k} \varepsilon^{n+1}, \varepsilon a_p x_1^{p_1} \cdots x_{k-1}^{p_{k-1}} z^{p_k} \varepsilon^n \rangle_{r,\alpha+r_j} = 0. \tag{29}$$

The latter implies:

$$a_p = a_{q_1-1, p_2, \ldots, p_k, n+1} = 0. \tag{30}$$

This finishes the proof of $\langle F(Q), \varepsilon \bar{P}_j \rangle_{r,\alpha+r_j} = 0$, which implies $\bar{P}_j = 0$ for all $j = 2, \ldots, k-1$. □

Remark 2.2. Theorem 2.2, together with Borel's lemma [3], implies that an A_k-SFS $X = F + P$ is *smoothly* conjugate to a smooth vector field $Y = F + H$ where H is flat at the origin. The benefits of this normal form are exploited in [6,7].

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