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Internal null-controllability of the N-dimensional heat equation in cylindrical domains

Contrôlabilité à zéro interne de l'équation de la chaleur dans un domaine cylindrique

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ABSTRACT

In this Note, we consider the internal null-controllability of the N-dimensional heat equation on domains of the form $\Omega = \Omega_1 \times \Omega_2$, with $\Omega_1 = (0, 1)$ and Ω_2 a smooth domain of $\mathbb{R}^{N-1}$, $N > 1$. When the control is exerted on $\gamma = \{x_0\} \times \omega_2 \subset \Omega$, with x_0 an algebraic real number of order $d > 1$ and $\omega_2 \subsetneq \Omega_2$ a non-empty open subset, we show the null-controllability, for all time $T > 0$. This result is obtained through the Lebeau–Robbiano strategy and requires an upper bound of the cost of the one-dimensional null-control.

RÉSUMÉ

Dans cette note, on considère la contrôlabilité à zéro interne de l'équation de la chaleur N-dimensionnelle, sur des domaines de la forme $\Omega = \Omega_1 \times \Omega_2$, avec $\Omega_1 = (0, 1)$ et Ω_2 un domaine borné et régulier de $\mathbb{R}^{N-1}$, $N > 1$. Lorsque le contrôle est exercé sur $\gamma = \{x_0\} \times \omega_2 \subset \Omega$, avec x_0 un nombre réel algébrique de degré $d > 1$ et $\omega_2 \subsetneq \Omega_2$ un ouvert non vide, on montre la contrôlabilité à zéro, en tout temps $T > 0$. Ce résultat s'appuie sur la stratégie de Lebeau–Robbiano et exige une estimation du coût du contrôle monodimensionnel.

Version française abrégée

Ce travail s'appuie sur celui, relativement ancien, de Szymon Dolecki [2], qui donne une caractérisation de la contrôlabilité ponctuelle de l'équation de la chaleur monodimensionnelle (5). L'objectif est maintenant d'étendre ce résultat à la dimension supérieure. En notant $\varphi_k(x) = \sqrt{2}\sin(k\pi x)$, on montre plus précisément le résultat suivant.

Théorème 0.1. *Supposons que* $x_0 \in (0, 1) \setminus \mathbb{Q}$ *et soit*

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$$T_0(x_0) = -\liminf_{k\longrightarrow+\infty} \frac{\log|\varphi_k(x_0)|}{(k\pi)^2} \in [0;+\infty].$$

1. *Soit $\omega_2 = \Omega_2$. Pour tout $T > T_0(x_0)$, le système (1) est contrôlable à zéro au temps T. Pour tout $T < T_0(x_0)$, le système (1) n'est pas contrôlable à zéro au temps T.*
2. *Soit x_0 un nombre réel algébrique de degré $d > 1$ et $\omega_2 \subsetneq \Omega_2$. Le système (1) est contrôlable à zéro en tout temps $T > 0$ sur le domaine de contrôle $\gamma = \{x_0\} \times \omega_2$.*

Remarque 1. Par le théorème de Liouville (approximation diophantienne), on montre que, si x_0 est un nombre algébrique de degré $d > 1$, alors $T_0(x_0) = 0$; dans [2], l'auteur montre que, pour tout $\tau_0 \in [0, +\infty]$, il existe $x_0 \in (0, 1) \setminus \mathbb{Q}$ tel que $T_0(x_0) = \tau_0$.

1. Introduction

This work is based on that of Szymon Dolecki [2], which gives a characterization of the punctual controllability of the 1-dimensional heat equation. The aim is now to extend this result to the superior dimension. Let $\Omega = \Omega_1 \times \Omega_2$ with $\Omega_1 = (0, 1)$ and Ω_2 a smooth domain of $\mathbb{R}^{N-1}$, $N > 1$, and $\gamma = \{x_0\} \times \omega_2$ with $x_0 \in (0, 1)$ and $\omega_2 \subseteq \Omega_2$. Let us consider the following control problem:

$$\begin{cases} \partial_t y - \Delta y = \delta_{\{x_0\}\times\omega_2} v(x_2, t) & \text{in } \Omega \times (0, T), \\ y = 0 \text{ on } \partial\Omega \times (0, T), & y(\cdot, 0) = y_0 \text{ in } \Omega, \end{cases} \tag{1}$$

where $T > 0$ is the control time, y is the state, $y_0 \in L^2(\Omega)$ is the initial data, $v \in L^2\left(0, T; L^2(\omega_2)\right)$ is the control function and $\gamma = \{x_0\} \times \omega_2$ is the control domain. It is well known that System (1) is well posed.

We consider two different cases: $\omega_2 = \Omega_2$ and $\omega_2 \subsetneq \Omega_2$. In each case, we ask if for every $y_0 \in L^2(\Omega)$, there exists $v \in L^2\left(0, T; L^2(\omega_2)\right)$ such that the solution y of (1) satisfies $y(T; y_0, v) = 0$, or in an equivalent way, if there exists a constant $C = C(T) > 0$ such that

$$||z(T)||^2_{L^2(\Omega)} \le C_T^2 \int_0^T ||z(x_0, t)||^2_{L^2(\omega_2)} \mathrm{d}t, \quad \forall z_0 \in L^2(\Omega), \tag{2}$$

where z is the solution to the following adjoint system, associated with System (1):

$$\begin{cases} \partial_t z - \Delta z = 0 & \text{in } \Omega \times (0, T), \\ z = 0 \text{ on } \partial\Omega \times (0, T), & z(\cdot, 0) = z_0 \text{ in } \Omega. \end{cases} \tag{3}$$

Throughout this paper, we are going to consider the following two assumptions

Assumption (A1): The number $x_0 \in (0, 1) \setminus \mathbb{Q}$ and $\omega_2 = \Omega_2$,
Assumption (A2): The number x_0 is an algebraic real number of order $d > 1$ and $\omega_2 \subsetneq \Omega_2$.

The purpose of this note is to establish, by noting $\varphi_k(x) = \sqrt{2}\sin(k\pi x)$, the following result:

Theorem 1.1. *Assume that $x_0 \in (0, 1) \setminus \mathbb{Q}$ and consider*

$$T_0(x_0) = -\liminf_{k\longrightarrow+\infty} \frac{\log|\varphi_k(x_0)|}{(k\pi)^2} \in [0;+\infty]. \tag{4}$$

1. **Under the assumption (A1)**. *If $T > T_0(x_0)$, System (1) is null-controllable at time T. For any $T < T_0(x_0)$, System (1) is not null-controllable at time T.*
2. **Under the assumption (A2)**. *System (1) is null-controllable, for all time $T > 0$.*

Remark 1. By using Liouville's Theorem in Diophantine approximation, we show that if x_0 is an algebraic real number of order $d > 1$ then $T_0(x_0) = 0$. In [2], the author shows that, for all $\tau_0 \in [0, +\infty]$, there exists $x_0 \in (0, 1) \setminus \mathbb{Q}$ such that $T_0(x_0) = \tau_0$.

Remark 2.

1. The proof of the first assertion of Theorem 1.1 is much simpler and it does not need control cost estimate. On the other hand, it is essentially based on the results by S. Dolecki (see [2]), concerning the null-controllability of the one-dimensional heat equation.

2. The proof of the second assertion of Theorem 1.1 is based on the method of Lebeau–Robbiano (see [4]). This strategy requires an estimate of the cost of the 1-dimensional control. We establish, by using Liouville's Theorem in Diophantine approximation, that the cost of the one-dimensional null-control is bounded by $Ce^{C/T}$, for some $C > 0$, if x_0 is an algebraic real number of order $d > 1$.

2. Null-controllability of System (1) under the assumption (A1)

In one-dimensional space:

Szymon Dolecki in [2] proved, for all $x_0 \in (0,1) \setminus \mathbb{Q}$, that the following heat equation

$$\begin{cases} \partial_t y - \partial_{xx} y = \delta_{\{x_0\}} v(t) & \text{in } (0,1) \times (0,T), \\ y(0,\cdot) = y(1,\cdot) = 0 \text{ on } (0,T), & y(\cdot,0) = y_0 \text{ in } (0,1). \end{cases} \tag{5}$$

is null-controllable at time T, if $T > T_0(x_0)$ and for any $T < T_0(x_0)$, System (5) is not null-controllable at time T. He also showed the following observability inequality:

$$||\psi(\cdot,T)||_{L^2(0,1)} \le C_T ||\psi(x_0,\cdot)||_{L^2(0,T)}, \text{ where } C_T = \left[\sum_{k \ge 1} \frac{e^{-(k\pi)^2 T}}{d_k |\varphi_k(x_0)|} \right], \tag{6}$$

where $d_k(T)$ is the distance, in $L^2(0,T)$, between $e^{-(k\pi)^2 t}$ and the smallest closed subspace of $L^2[0,T]$ containing the functions $\{e^{-(n\pi)^2 t}\}_{n \neq k}$ and ψ being the solution to the following adjoint system of (5):

$$\begin{cases} \partial_t \psi - \partial_{xx} \psi = 0 & \text{in } (0,1) \times (0,T), \\ \psi(0,\cdot) = \psi(1,\cdot) = 0 \text{ on } (0,T), & \psi(\cdot,0) = \psi_0 \text{ in } (0,1), \end{cases} \tag{7}$$

where $\psi_0 \in L^2(0,T)$. The series in square parentheses (relation (6)) converges because of the estimation $\forall \varepsilon > 0$:

$$\frac{1}{d_k(T)} \le e^{2\varepsilon (k\pi)^2}, \text{ when } k \longrightarrow +\infty \text{ (see [5], Theorem 7.1)}.$$

Remark 3. The infimum of the constants C_T satisfying (6) is called the cost of the one-dimensional null-control on $(0,1)$, noted $C_T^{\Omega_1}$.

In *N*-dimensional space:

Let us consider the operator $A_{x_0,T} : L^2(0,T; L^2(\Omega_2)) \longrightarrow L^2(\Omega)$ defined by $A_{x_0,T}(z(x_0,\cdot,\cdot)) = z(\cdot,\cdot,T)$.

Let us show that $A_{x_0,T}$ is well defined and bounded for $T > T_0(x_0)$ and is not bounded for $T < T_0(x_0)$.

- Let $\lambda_j^{\Omega_2}$, $J \ge 1$ be the Dirichlet eigenvalues of the Laplacian on Ω_2, and let $\phi_j^{\Omega_2}$ be the corresponding normalized eigenfunction. Let us introduce the (closed) subspaces of $H_0^1(\Omega)$: $E_J = \left\{ \sum_{j=1}^J \left(u(x_1,\cdot), \phi_j^{\Omega_2} \right)_{L^2(\Omega_2)} \phi_j^{\Omega_2} \mid u \in H_0^1(\Omega) \right\} \subset H_0^1(\Omega)$ We denote by Π_{E_J} the orthogonal projection in $H_0^1(\Omega)$. Let z be the solution of (3) associated with $z_0(x_1,x_2) = \sum_{j=1}^J z_0^j(x_1) \phi_j^{\Omega_2}(x_2) \in E_J$. Thus, $\Pi_{E_J} z(x_1,x_2,t) = \sum_{j=1}^J e^{-\lambda_j^{\Omega_2} t} z_j(x_1,t) \phi_j^{\Omega_2}(x_2)$, where z_j is the solution of System (7), associated with the initial data z_0^j. Thus, z_j verifies (6) and we obtain the following partial observabilities:

$$\| \Pi_{E_J} z(T) \|_{L^2(\Omega)}^2 = \sum_{j=1}^J e^{-2\lambda_j^{\Omega_2} T} \| z_j(T) \|_{L^2(\Omega_1)}^2 \le C_T^2 \| \Pi_{E_J} z(x_0,\cdot,t) \|_{L^2(0,T;L^2(\Omega_2))}^2, \quad \forall T > T_0(x_0). \tag{8}$$

  Thus, $A_{x_0,T}$ is well defined and bounded, which means that System (1) is null-controllable at time T, if $T > T_0(x_0)$.
- Let us suppose that the Dirichlet series $\sum_{k \ge 1} \frac{e^{-(k\pi)^2 T}}{|\varphi_k(x_0)|} = +\infty$ (true $\forall\, T < T_0(x_0)$, see [7]). By using the same technique as Dolecki (see [2] page 294), we can show that $A_{x_0,T}$ is not bounded.

3. Null-controllability of System (1) under the assumption (A2)

This case is more difficult and requires some intermediate results.

- **Partial observability.** Let us apply the Lebeau–Robbiano spectral inequality (see [4]) to the partial observabilities (8) (with $C_T^{\Omega_1}$), we obtain:

$$||\Pi_{E_J} z(T)||^2_{L^2(\Omega)} \leq C(C_T^{\Omega_1})^2 e^{C\sqrt{\lambda_J^{\Omega_2}}} \int_0^T ||\Pi_{E_J} z(x_0,\cdot,t)||^2_{L^2(\omega_2)} dt. \tag{9}$$

Consequently, thanks to the duality theorem between controllability and observability, for every $J \geq 1$ and for all initial data $y_0 \in E_J$, there exists a control $v(y_0) \in L^2\left(0,T; L^2(\omega_2)\right)$ with

$$||v(y_0)||_{L^2\left(0,T;L^2(\omega_2)\right)} \leq C C_T^{\Omega_1} e^{C\sqrt{\lambda_J^{\Omega_2}}} ||y_0||_{L^2(\Omega)} \tag{10}$$

such that the solution y to System (2) satisfies $\Pi_{E_J} y(T) = 0$.

- **Estimate of the cost $C_T^{\Omega_1}$.** The purpose here is to estimate C_T defined in (6). If we note $\{\psi_k\}_{k\geq 1}$ the optimal biorthogonal set for $\{e^{-k^2\pi^2}\}$ in $L^2[0,T]$, one has $||\psi_k||_{L^2[0,T]} = \frac{1}{d_k(T)}$, moreover if $\{\tilde{\psi}_k\}_{k\geq 1}$ is any other biorthogonal set for $\{e^{-k^2\pi^2}\}$ in $L^2[0,T]$, thus $||\psi_k||_{L^2[0,T]} \leq ||\tilde{\psi}_k||_{L^2[0,T]}$ (see [3] page 277). Besides, we deduce from [1] (Theorem 1.5, relation (1.12)), see also [3] and [6], that we can find $\{\tilde{\psi}_k\}_{k\geq 1}$ a biorthogonal set for $\{e^{-k^2\pi^2}\}$, such that there exists T_0 and $C > 0$, such that for all $0 < T < T_0$:

$$\frac{1}{d_k(T)} \leq ||\tilde{\psi}_k||_{L^2[0,T]} \leq C\, e^{C\sqrt{k^2\pi^2}},\ \forall\, k \geq 1.$$

By Young's inequality, $C\sqrt{k^2\pi^2} \leq k^2\pi^2 \frac{T}{4} + \frac{C^2}{T}$, consequently

$$C_T = \sum_{k\geq 1} \frac{e^{-(k\pi)^2 T}}{d_k(T)|\varphi_k(x_0)|} \leq C\, e^{\frac{C^2}{T}} \sum_{k\geq 1} \frac{e^{-\frac{3}{4}(k\pi)^2 T}}{|\varphi_k(x_0)|}, \quad \forall\, x_0 \in (0,1) \setminus \mathbb{Q}.$$

Proposition 3.1. *If x_0 is an algebraic real number of order $d > 1$, then for all $\alpha > 0$, there exists a constant $C(d) > 0$ such that*

$$\sum_{k\geq 1} \frac{e^{-(k\pi)^2 \alpha T}}{|\varphi_k(x_0)|} \leq C(d,\alpha) e^{\frac{d}{2T}},$$

consequently there exists $C > 0$ such that $C_T^{\Omega_1} \leq C\, e^{\frac{C}{T}}$.

Proof. *Step 1*: $\frac{1}{|\varphi_k(x_0)|} = \frac{1}{\sqrt{2}|\sin(k\pi x_0)|} \leq \frac{k^{d-1}}{2\sqrt{2}A}$. Indeed, let $p_k \in \mathbb{N}$ be the unique integer such as $p_k - \frac{1}{2} < kx_0 < p_k + \frac{1}{2}$ and $\varepsilon_k = kx_0 - p_k \in]-\frac{1}{2}; \frac{1}{2}[$, one has $|\sin(k\pi x_0)| = |\sin(\pi\varepsilon_k)| \geq \frac{2}{\pi}|\pi\varepsilon_k| = 2|\varepsilon_k|$ since $|\sin(t)| \geq \frac{2}{\pi}|t|$, $\forall t \in [-\frac{\pi}{2}; \frac{\pi}{2}]$. Thus,

$$\frac{1}{|\sin(k\pi x_0)|} \leq \frac{1}{2k|x_0 - \frac{p_k}{k}|}.$$

Moreover, by using Liouville's Theorem (on Diophantine approximation), there exists a real number $A > 0$ such that, for all integers p, q, with $q > 0$, such that $|x_0 - \frac{p}{q}| \geq \frac{A}{q^d}$. Consequently

$$\frac{1}{|\varphi_k(x_0)|} \leq \frac{k^{d-1}}{2\sqrt{2}A} \quad \text{and} \quad \sum_{k\geq 1} \frac{e^{-(k\pi)^2\alpha T}}{|\varphi_k(x_0)|} \leq \frac{1}{2A\sqrt{2}} \sum_{k\geq 1} k^{d-1} e^{-(k\pi)^2\alpha T}.$$

Step 2: By using the Cauchy integral criterion on the function $x \longmapsto x^{d-1} e^{-x^2\alpha\pi^2 T}$, we get that

$$\sum_{k\geq 1} k^{d-1} e^{-(k\pi)^2\alpha T} \leq \left(\frac{d-1}{\alpha\pi^2 T}\right)^{\frac{d}{2}} \left(1 + \frac{1}{2}\Gamma(\frac{d}{2})\right),$$

moreover $e \leq x\, e^{\frac{1}{x}}$ for all $x > 0$, then for all $T > 0$, one has:

$$\frac{1}{T^{\frac{d}{2}}} \leq e^{\frac{d}{2}\left(\frac{1}{T}-1\right)}.$$

Thus, there exists $C(d) > 0$ such that $C_T^{\Omega_1} \leq C(d) e^{\frac{C}{T}}, \forall\, T > 0$.

- **Lebeau–Robbiano time procedure.** Let us decompose the interval $[0;T)$ as follows: $[0,T) = \bigcup_{k=0}^{+\infty}[a_k, a_{k+1}]$ with $a_0 = 0$, $a_{k+1} = a_k + 2T_k$, and $T_k = M2^{-k\rho}$, where $\rho \in (0, \frac{1}{N_2})$ $(\Omega_2 \subset \mathbb{R}^{N_2})$ et $M = \frac{T}{2}(1 - 2^{-\rho})$ has been determined to ensure that $2\sum_{k=0}^{+\infty} T_k = T$.

We define the control v and the corresponding solution y piecewise and by induction as follows:

$$v(y_0)(t) = \begin{cases} v\left(\Pi_{E_{2^k}} y(a_k)\right)(t) & \text{if } t \in (a_k, a_k + T_k), \\ 0 & \text{if } t \in (a_k + T_k, a_{k+1}), \end{cases} \tag{11}$$

such that $\Pi_{E_{2^k}} y(a_k + T_k) = 0$. Let us show that $v(y_0)$ belongs to $L^2\left(0, T; L^2(\omega_2)\right)$ and steers y to 0 at time T.

Step 1: Estimate on the interval $[a_k, a_k + T_k]$. Let us first recall that System (1) is well posed in the sense that, for every $y_0 \in L^2(\Omega)$ and $v \in L^2\left(0, T; L^2(\omega_2)\right)$, there exists a unique solution $y \in L^2\left(0, T; H_0^1(0,1)\right) \cap C^0\left([0,T]; L^2(0,1)\right)$, defined by transposition. Moreover, this solution depends continuously on the initial data y_0 and the control v. More precisely, $||y||_{C^0([0,T];L^2(\Omega))} \leq C\left(||y_0||_{L^2(\Omega)} + ||v||_{L^2(0,T;L^2(\omega_2))}\right)$. Thus, from the continuous dependence with respect to the data and since $T_k \leq T$ we know that

$$||y(a_k + T_k)||_{L^2(\Omega)} \leq C\left(||y(a_k)||_{L^2(\Omega)} + ||v||_{L^2\left(a_k, a_k+T_k; L^2(\omega_2)\right)}\right). \tag{12}$$

Using (10) we have $||v||_{L^2\left(a_k, a_k+T_k; L^2(\omega_2)\right)} \leq C C_{T_k}^{\Omega_1} e^{C\sqrt{\lambda_{2^k}^{\Omega_2}}} ||\Pi_{E_{2^k}} y(a_k)||_{L^2(\Omega)}$ and since $||\Pi_{E_{2^k}} y(a_k)||_{L^2(\Omega)} \leq 1$, this gives $||v||_{L^2\left(a_k, a_k+T_k; L^2(\omega_2)\right)} \leq C C_{T_k}^{\Omega_1} e^{C\sqrt{\lambda_{2^k}^{\Omega_2}}} ||y(a_k)||_{L^2(\Omega)}$.

Using now the estimate of $C_T^{\Omega_1}$ with respect to T, this leads to:

$$||v||_{L^2\left(a_k, a_k+T_k; L^2(\omega_2)\right)} \leq C e^{c\left(\frac{1}{T_k} + \sqrt{\lambda_{2^k}^{\Omega_2}}\right)} ||y^k(a_k)||_{L^2(\Omega)}.$$

On the other hand, Weyl's asymptotic formula states that

$$\sqrt{\lambda_{2^k}^{\Omega_2}} \sim C(2^k)^{\frac{1}{N_2}}, \quad \text{when } k \longrightarrow +\infty,$$

and (by the choice of ρ)

$$\frac{1}{T_k} = \frac{2^{k\rho}}{M} \leq C 2^{\frac{k}{N_2}},$$

so that

$$||v||_{L^2\left(a_k, a_k+T_k; L^2(\omega_2)\right)} \leq C e^{C 2^{\frac{k}{N_2}}} ||y^k(a_k)||_{L^2(\Omega)}. \tag{13}$$

Combined to (12) this yields

$$||y(a_k + T_k)||_{L^2(\Omega)} \leq C e^{C 2^{\frac{k}{N_2}}} ||y^k(a_k)||_{L^2(\Omega)}. \tag{14}$$

Step 2: Estimate on the interval $[a_k + T_k, a_{k+1}]$. Since $\Pi_{E_{2^k}} y(a_k + T_k) = 0$, the dissipation (see [1], Proposition 2.4, page 7) gives

$$||y(a_{k+1})||_{L^2(\Omega)} \leq C e^{-\lambda_{2^k+1}^{\Omega_2} T_k} ||y(a_k + T_k)||_{L^2(\Omega)}. \tag{15}$$

Step 3: Final estimate. From (14) and (15) we deduce

$$||y(a_{k+1})||_{L^2(\Omega)} \leq C e^{-\lambda_{2^k+1}^{\Omega_2} T_k + C 2^{\frac{k}{N_2}}} ||y(a_k)||_{L^2(\Omega)}.$$

By induction, we obtain

$$||y(a_{k+1})||_{L^2(\Omega)} \leq C e^{\sum_{p=0}^{k}\left(-\lambda_{2^p+1}^{\Omega_2} T_p + C 2^{\frac{p}{N_2}}\right)} ||y_0||_{L^2(\Omega)}.$$

Since

$$\lambda_{2^p+1}^{\Omega_2} T_p \sim C(2^p+1)^{\frac{2}{N_2}} 2^{-p\rho} \geq C'\left(2^p\right)^{\frac{2}{N_2} - \rho}, \quad p \longrightarrow +\infty,$$

we obtain

$$||y(a_{k+1})||_{L^2(\Omega)} \leq C e^{\sum_{p=0}^{k}\left(-C'(2^p)^{\frac{2}{N_2} - \rho} + C(2^p)^{\frac{1}{N_2}}\right)} ||y_0||_{L^2(\Omega)}.$$

Since $\rho < \frac{1}{N_2}$ there exists a $p_0 \geq 1$ such that

$$-C'\left(2^p\right)^{\frac{2}{N_2}-\rho} + C(2^p)^{\frac{1}{N_2}} \leq -C''\left(2^p\right)^{\frac{2}{N_2}-\rho}, \quad \forall p \geq p_0. \tag{16}$$

It follows that, for $k \geq p_0$, we have

$$\sum_{p=0}^{k}\left(-C'\left(2^p\right)^{\frac{2}{N_2}-\rho} + C(2^p)^{\frac{1}{N_2}}\right) \leq C''' - C''\sum_{p=p_0}^{k}\left(2^p\right)^{\frac{2}{N_2}-\rho} \leq C''' - C''\left(2^k\right)^{\frac{2}{N_2}-\rho}.$$

So that, finally,

$$||y(a_{k+1})||_{L^2(\Omega)} \leq Ce^{-c\left(2^k\right)^{\frac{2}{N_2}-\rho}}||y_0||_{L^2(\Omega)}. \tag{17}$$

Step 4: The function v is a control. Estimates (13) and (17) show that the function v is in $L^2\left(0,T;L^2(\omega_2)\right)$:

$$||v||^2_{L^2(0,T;L^2(\omega_2))} = \sum_{k\geq 0}||v||^2_{L^2(a_k,a_k+T_k;L^2(\omega_2))} \leq C\left(\sum_{k\geq 0} e^{C2^{\frac{k}{N_2}}-c'\left(2^k\right)^{\frac{2}{N_2}-\rho}}\right)||y_0||_{L^2(\Omega)} < +\infty, \text{ by (16).}$$

Moreover, estimate (17) also shows that the function v is indeed a control:

$$||y(a_{k+1})||_{L^2(\Omega)} \longrightarrow 0 = ||y(T)||_{L^2(\Omega)}, \quad \text{when } k \longrightarrow +\infty,$$

and ends the proof of Theorem 1.1. □

References

[1] A. Benabdallah, F. Boyer, M. González-Burgos, G. Olive, Sharp estimates of the one-dimensional boundary control cost for parabolic systems and application to the N-dimensional boundary null-controllability in cylindrical domains, SIAM J. Control Optim. 52 (5) (2014) 2970–3001.

[2] S. Dolecki, Observability for the one-dimensional heat equation, Stud. Math. 48 (1973) 291–305.

[3] H.O. Fattorini, D.L. Russel, Exact controllability theorems for linear parabolic equations in one space dimension, Arch. Ration. Mech. Anal. 43 (1971) 272–292.

[4] G. Lebeau, L. Robbiano, Contrôle exact de l'équation de la chaleur, Commun. Partial Differ. Equ. 20 (1–2) (1995) 335–356.

[5] W.A.J. Luxemburg, J. Korevaar, Entire functions and Müntz–Szasz type approximation, Trans. Amer. Math. Soc. 157 (1971) 23–37.

[6] L. Miller, Geometric bounds on the growth rate of null-controllability cost for the heat equation in small time, J. Differ. Equ. 204 (1) (2004) 202–226.

[7] G. Valiron, Théorie générale des séries de Dirichlet, Mém. Math. 17 (1926) 3.