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Complex analysis

On Bernstein and Erdős–Lax's inequalities for quaternionic polynomials

Sur les inégalités de Bernstein et de Erdős–Lax pour les polynômes quaternioniques

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A B S T R A C T

In this paper the well-known Bernstein's inequality for complex polynomials is extended to the quaternionic setting. We also show that the Erdős–Lax's inequality does not hold in general, but it works for a particular class of polynomials.

R É S U M É

Dans cet article, l'inégalité de Bernstein, bien connue pour les polynômes de $\mathbb{C}$, est prouvée pour les polynômes quaternioniques. Nous démontrons que l'inégalité de Erdős–Lax n'est pas valide, en général, mais qu'elle est valide pour un ensemble particulier de polynômes.

1. Introduction and preliminaries

One of the most known polynomial inequality with important applications in approximation theory is the following Bernstein's inequality for complex polynomials.

Theorem 1.1 *(See Bernstein [1], Riesz [9]). If $P(z)$ is an algebraic polynomial of degree n with complex coefficients, then*

$$\|P'\| \le n \cdot \|P\|,$$

where the norm of P is defined by $\|P\| = \max_{|z|\le 1} |P(z)| = \max_{|z|=1} |P(z)|$.

The goal of this paper is to extend the above theorem in the quaternionic setting. The refinement of the Bernstein's inequal-ity conjectured by Erdős and proved by Lax [7], stating that

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$$\|P'\| \le \frac{n}{2} \cdot \|P\|, \tag{1}$$

for those polynomials P that do not have a zero in the open unit disk of $\mathbb{C}$, does not hold in the quaternionic setting, as we show in Section 3. The refined result (1) holds, however, for a subclass of the quaternionic polynomials of degree n.

We begin with some preliminaries on quaternions.

The noncommutative field $\mathbb{H}$ of quaternions consists of elements of the form $q = x_0 + x_1\mathrm{i} + x_2\mathrm{j} + x_3\mathrm{k}$, $x_i \in \mathbb{R}$, $i = 0, 1, 2, 3$, where the imaginary units $\mathrm{i}, \mathrm{j}, \mathrm{k}$ satisfy $\mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 = -1$, $\mathrm{ij} = -\mathrm{ji} = \mathrm{k}$, $\mathrm{jk} = -\mathrm{kj} = \mathrm{i}$, $\mathrm{ki} = -\mathrm{ik} = \mathrm{j}$. The real number x_0 is called real part of q, and is denoted by $\mathrm{Re}(q)$, while $x_1\mathrm{i} + x_2\mathrm{j} + x_3\mathrm{k}$ is called vector part or imaginary part of q and is denoted by $\mathrm{Im}(q)$. We define the norm of a quaternion q as $|q| = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^3}$. By $\mathbb{S}$ we denote the unit sphere of purely imaginary quaternions, i.e. $\mathbb{S} = \{q = \mathrm{i}x_1 + \mathrm{j}x_2 + \mathrm{k}x_3$, such that $x_1^2 + x_2^2 + x_3^3 = 1\}$. The set of elements of the form $a + Ib$ when I varies in $\mathbb{S}$ is a 2-dimensional sphere denoted by $[a + Ib]$. A quaternion q belongs to the sphere $\mathbb{S}$ if and only if it satisfies the equation $q^2 - 2aq + (a^2 + b^2) = 0$. Note that if $I \in \mathbb{S}$, then $I^2 = -1$. For any fixed $I \in \mathbb{S}$ we define $\mathbb{C}_I := \{x + Iy \mid x, y \in \mathbb{R}\}$, which can be identified with a complex plane. Obviously, the real axis belongs to $\mathbb{C}_I$ for every $I \in \mathbb{S}$. For more information we refer the reader to [2].

Since the multiplication in $\mathbb{H}$ is not commutative, one can consider unilateral quaternionic polynomials, namely polynomials with coefficients on one side or even polynomials which are sum of monomials of the form $a_1 q a_2 q \dots q a_n$. In this paper we consider unilateral polynomials, and we treat the case where the coefficients are written on the right. Thus we call right polynomial or, in this context, simply polynomial, of degree $\le n$ an expression $P(q)$ of the form

$$\sum_{k=0}^{n} q^k a_k, \quad a_k \in \mathbb{H},\ k = 0, \dots, n. \tag{2}$$

For details on quaternionic polynomials we refer the reader to the classical book [6]. Two quaternionic polynomials $P_1(q) = \sum_{i=0}^{n} q^i a_i$, $P_2(q) = \sum_{i=0}^{m} q^i b_i$ are multiplied as follows

$$(P_1 * P_2)(q) = \sum_{i=0,\dots,n,\, j=0,\dots,m} q^{i+j} a_i b_j.$$

If P_1 has real coefficients the $*$-multiplication coincides with the pointwise multiplication. It is also important to recall the following result, see [5,6], about the zeros of polynomials:

Theorem 1.2.

(i) *A quaternion α is a zero of a (nonzero) polynomial $P(q)$ if and only if the polynomial $q - \alpha$ is a left divisor of $P(q)$.*
(ii) *If $P(q) = (q - \alpha_1) * \dots * (q - \alpha_n)$, where $\alpha_1, \dots, \alpha_n \in \mathbb{H}$, then α_1 is a zero of P and every other zero of P belong to the spheres $[\alpha_i]$, $i = 2, \dots, n$.*
(iii) *If P has two distinct zeros in an equivalence class $[\alpha]$, then all the elements in $[\alpha]$ are zeros of P.*

From this result, it follows that the zeros of a polynomial $P(q)$ are either isolated points or spheres.

The choice to treat the case of right polynomials is suggested by the fact that in the sequel we will make use of some results that are valid in the framework of left slice regular functions and right polynomials are a subset of this class of functions.

Let us recall that, given an open set Ω in $\mathbb{H}$, and a function $f : \Omega \to \mathbb{H}$, real differentiable, we say that f is *left slice regular* if for every $I \in \mathbb{S}$, its restriction f_I to the complex plane $\mathbb{C}_I = \mathbb{R} + I\mathbb{R}$, satisfies $\frac{1}{2}(\frac{\partial}{\partial x} + I\frac{\partial}{\partial y}) f_I(x + Iy) = 0$, on $\Omega \cap \mathbb{C}_I$.

Also, recall that a set Ω is said to be axially symmetric if the sphere $[q]$ belongs to Ω whenever q belongs to Ω. Let D_J be any open set in $\mathbb{C}_J = \{x + Jy \mid x, y \in \mathbb{R}\}$, $J \in \mathbb{S}$. The set $\Omega_{D_J} = \bigcup_{x+Jy \in D_J,\ I \in \mathbb{S}} \{x + Iy\}$ is called the axially symmetric completion of D_J in $\mathbb{H}$.

The derivative of the polynomial $P(q) = \sum_{k=0}^{n} q^k a_k$ is

$$P'(q) = \sum_{k=1}^{n} k q^{k-1} a_k$$

and it is consistent with the notion of slice derivative, see [2,4].

A Gauss–Lucas type theorem in the quaternionic setting, which will be used in the next section, is in [10] and can be formulated as follows:

Theorem 1.3. *Let $P(q) = \sum_{k=0}^{n} q^k a_k$ with quaternionic coefficients a_k. Then the zeros of $P'(q)$ are in the axially symmetric completion of the convex hull $\mathcal{K}(Z_{P^s})$ of the zero set Z_{P^s} of P^s, where $P^s = P^c * P = P * P^c$ and $P^c(q) = \sum_{k=0}^{n} q^k \overline{a_k}$.*

2. Bernstein's inequality

The main result of the paper is the following Bernstein's inequality in the quaternionic setting.

Theorem 2.1. *If P is a quaternionic polynomial of degree n, then*

$$\|P'\| \le n \cdot \|P\|$$

where the norm of P is defined by $\|P\| = \max_{|q|\le 1} |P(q)| = \max_{|q|=1} |P(q)|$.

Proof. For the simplicity of the proof, we divide it into four steps.

Step 1. Define $Q(q) = Mq^n$ and $f(q) = Q^{-1}(q) * P(q) = \frac{1}{M} q^{-n} P(q)$, where $M = \max_{|q|=1} |P(q)|$. We show that

$$|f(q)| \le 1 \quad \text{for all } |q| \ge 1. \tag{3}$$

Indeed, f is slice regular as function of q for $|q| > 1$ and $|f(q)| = \frac{1}{M}|P(q)| \le 1$, for all $|q| = 1$. Then, denoting $P(q) = \sum_{k=0}^{n} q^k a_k$ and $\tilde{q} = q^{-1}$, after simple calculation we can clearly write $f(q) = h(\tilde{q})$, where $h(\tilde{q}) = \frac{1}{M} \sum_{k=0}^{n} (\tilde{q})^{n-k} a_k$ is slice regular for $|\tilde{q}| < 1$, and since $|q| = 1$, if and only if $|\tilde{q}| = 1$, we also have $|h(\tilde{q})| = |f(q)| \le 1$, for all $|\tilde{q}| = 1$.

Applying the maximum modulus theorem (see Theorem 3.4 in [4]), it follows that $|h(\tilde{q})| \le 1$ for all $|\tilde{q}| \le 1$, which by the equality $h(\tilde{q}) = f(q)$, immediately implies (3).

Step 2. We prove that all the zeroes of the (left) regular polynomial of degree n, $g(q) = P(q) - Q(q)\lambda = P(q) - q^n \lambda M$, where $\lambda \in \mathbb{H}$ satisfies $|\lambda| > 1$, belong to the open unit ball $B(0; 1)$.

Indeed, let $q_0 \in \mathbb{H}$ be with $g(q_0) = P(q_0) - Q(q_0)\lambda = 0$. We have two subcases: (a) $Q(q_0) \ne 0$ (i.e. $q_0 \ne 0$); (b) $Q(q_0) = 0$ (i.e. $q_0 = 0$).

In subcase (a), we get $|P(q_0)| = |\lambda| \cdot |Q(q_0)| > |Q(q_0)| = M \cdot |q_0^n|$, which immediately implies $|f(q_0)| = |\frac{1}{M} q_0^{-n} P(q_0)| > 1$ and by (3) we necessarily get $|q_0| < 1$.

In subcase (b), we trivially have $|q_0| < 1$, since $q_0 = 0$.

Step 3. We show that all the zeroes of g^s also are included in $B(0; 1)$, where g^s is defined as in the statement of Theorem 1.3.

Indeed, since $g^s = g * g^c$ with $g^c(q) = q^n(\overline{a_n} - M\bar{\lambda}) + \sum_{k=0}^{n-1} q^k \overline{a_k}$ and $(g * g^c)(q) = 0$ if and only if $g(q) = 0$ or, if $g(q) \ne 0$ then $g^c(g(q)^{-1} q g(q)) = 0$, taking into account the conclusion of Step 2 too, we immediately obtain the conclusion of Step 3.

Step 4. Denoting by Z_{g^s} the zero set of g^s and by $\mathcal{K}(Z_{g^s})$ the convex hull of Z_{g^s}, from Step 3 we have $Z_{g^s} \subset B(0; 1)$, which implies that the axially symmetric completion of $\mathcal{K}(Z_{g^s})$ is included in the closed ball $\overline{B(0; 1)}$. Applying now Theorem 1.3, it follows that $g'(q) = P'(q) - Q'(q)\lambda$ has all its zeroes in $|q| \le 1$, i.e. in other words for all $|\lambda| > 1$ and $|q| > 1$, we have $g'(q) \ne 0$, which is equivalent to $|[Q'(q)]^{-1} P'(q)| \ne |\lambda| > 1$.

This clearly implies $|[Q'(q)]^{-1} P'(q)| \le 1$, for all $|q| > 1$, i.e. $|P'(q)| \le |Q'(q)|$, for all $|q| > 1$. From the continuity of P' and Q', for any $|q_0| = 1$ and taking a sequence q_m with $|q_m| > 1$, $\lim_{m\to\infty} q_m = q_0$, we easily get that $|P'(q_0)| \le |Q'(q_0)|$. This implies $|P'(q)| \le |Q'(q)| = nM|q|^{n-1}$, for all $|q| \ge 1$ and passing to maximum with $|q| = 1$ we obtain $\max_{|q|=1} |P'(q)| \le nM$, which proves the theorem. □

Remark 1. It is evident that a similar inequality holds for polynomials of degree n which are right slice regular, that is for polynomials of the form $P(q) = \sum_{k=0}^{n} a_k q^k$, with $a_k \in \mathbb{H}$.

3. Erdős–Lax's inequality

An improvement of Theorem 1.1 proved by Lax in [7], but first conjectured by Erdős, states that if $P(z)$ is an algebraic polynomial of degree n with complex coefficients that has no zero in the disk $|z| < 1$, then (1) holds. In general, the inequality (1) cannot be extended to the present setting, in fact we have:

Theorem 3.1. *The Erdős–Lax inequality is not valid, in general, for quaternionic polynomials.*

Proof. Let $P(q) = (q - \mathrm{i}) * (q - \mathrm{j}) = q^2 - q(\mathrm{i} + \mathrm{j}) + \mathrm{k}$. The only root of this polynomial is $q = \mathrm{i}$, see [8], namely $q = \mathrm{i}$ has multiplicity 2, thus $P(q)$ has no roots in the open unit ball $\mathbb{B}$ and is of degree 2. We will show that for this polynomial the Erdős–Lax inequality does not hold.

Let $q = \mathrm{e}^{I\theta}$, where $I = a\mathrm{i} + b\mathrm{j} + c\mathrm{k}$, $a^2 + b^2 + c^2 = 1$. The proof is divided into two steps.

Step 1. It is immediate that $P'(q) = 2q - (\mathrm{i} + \mathrm{j})$. Then, we have

$$P'(\mathrm{e}^{I\theta}) = 2\cos\theta + \mathrm{i}(2a\sin\theta - 1) + \mathrm{j}(2b\sin\theta - 1) + \mathrm{k}(2c\sin\theta),$$

and with some computations we obtain

$$\left|P'\left(e^{I\theta}\right)\right|^2 = 6 - 4(a+b)\sin\theta.$$

Also, we observe that

$$\max_{I\in\mathbb{S},\theta\in[0,2\pi)}\left|P'\left(e^{I\theta}\right)\right|^2 = \max\left(6-4(a+b)\sin\theta\right) \geq 6+4\sqrt{2}, \tag{4}$$

since $\max_{a^2+b^2+c^2=1}(a+b) = \max_{a^2+b^2=1}(a+b) = \max(a+\sqrt{1-a^2}) = \sqrt{2}$.

Step 2. Let us now compute

$$\begin{aligned} P\left(e^{I\theta}\right) &= e^{2I\theta} - e^{I\theta}(\mathrm{i}+\mathrm{j}) + \mathrm{k} \\ &= \cos(2\theta) + a\sin\theta + b\sin\theta + \mathrm{i}\left(a\sin(2\theta) - \cos\theta + c\sin\theta\right) \\ &\quad + \mathrm{j}\left(b\sin(2\theta) - \cos\theta - c\sin\theta\right) + \mathrm{k}\left(c\sin(2\theta) - a\sin\theta + b\sin\theta + 1\right). \end{aligned}$$

Some lengthy but simple computations show that

$$\left|P\left(e^{I\theta}\right)\right|^2 = 4 - 4a\sin\theta + 4c\cos\theta\sin\theta.$$

It is clear that $|P(e^{I\theta})|^2$ is maximum when $-4a\sin\theta + 4c\cos\theta\sin\theta$ is maximum. We have

$$\sin\theta(-4a+4c\cos\theta) \leq \left|\sin\theta(-4a+4c\cos\theta)\right| \leq 4\left|(-a+c\cos\theta)\right| \leq 4\left(|a|+|c|\right) \leq 4\sqrt{2},$$

the last inequality resulting from $\max_{a^2+b^2+c^2=1}(|a|+|c|) = \max_{a^2+c^2=1}(|a|+|c|) = \max(|a|+\sqrt{1-a^2}) = \sqrt{2}$. Thus

$$\max_{I\in\mathbb{S},\theta\in[0,2\pi)}\left|P\left(e^{I\theta}\right)\right|^2 \leq 4+4\sqrt{2}. \tag{5}$$

From inequalities (4), (5) it follows that

$$\left\|P'\right\| \geq \left(6+4\sqrt{2}\right)^{1/2} > \left(4+4\sqrt{2}\right)^{1/2} \geq \|P\| = \frac{2}{2}\cdot\|P\|$$

and so the statement follows. □

We note that the Erdős–Lax's inequality holds true at least for a class of polynomials, as the following result shows.

Proposition 3.2. *Let $P(q)$ be a polynomial of degree n with quaternionic coefficients that has no zero in the ball $|q|<1$. Assume that the zeros of $P(q)$ are either spheres and/or real points and that there exists at most one isolated zero $\alpha \in \mathbb{H}\setminus\mathbb{R}$ that has multiplicity 1. Then*

$$\left\|P'\right\| \leq \frac{n}{2}\cdot\|P\|.$$

Proof. If P has exactly one isolated nonreal zero α of multiplicity 1, then it factorizes as

$$P(q) = (q-\alpha)*(q-\alpha_1)\ldots*(q-\alpha_r)\left(q^2 - 2\,\mathrm{Re}(a_1)q + |a_1|^2\right)\ldots\left(q^2 - 2\,\mathrm{Re}(a_s)q + |a_s|^2\right)$$

where the factors $(q-\alpha_i)$, $(q^2 - 2\,\mathrm{Re}(a_i)q + |a_i|^2)$ might appear or not and might be repeated if the real points and/or the spheres appear with multiplicity greater than 1. The spheres are uniquely determined by $\mathrm{Re}(a_i)$, $|a_i|$ and so we can choose $a_1,\ldots,a_s$ on the complex plane $\mathbb{C}_I$ to which α belongs. If P does not have any nonreal isolated zero, then we can choose any $\mathbb{C}_I$. Thus, in both cases, we have

$$P(q) = (q-\alpha_1)*\cdots*(q-\alpha_r)*(q-\alpha)*(q-a_1)*(q-\bar{a}_1)\ldots(q-a_s)*(q-\bar{a}_s),$$

where $(q-\alpha)$ might appear or not and all the factors commute with respect to the $*$-product and where, of course, the $*$-product of the factors $(q-\alpha_i)$ coincides with the pointwise product. For the sake of simplicity, we change the notation and write

$$P(q) = \prod_{i=1}^{*n}(q-b_i),$$

where $b_i = \alpha_i$, $i=1,\ldots,r$, $b_{r+1}=\alpha$ and $b_{r+2i}=a_i$, $i=1,\ldots s$, $b_{r+1+2i}=\bar{a}_i$, $i=1,\ldots s$. Using the fact that the factors commute, since the b_i's are all on the same complex plane, we then have

$$P(q)^{-*} = \prod_{i=1}^{*n}(q-b_i)^{-*}$$

and so

$$P(q)^{-*} * P'(q) = \sum_{i=1}^{n} (q - b_i)^{-*}.$$

Let now q and ξ inside the unit ball $\mathbb{B} = \{q \in \mathbb{H} \ : \ |q| < 1\}$ and let us consider the polynomial

$$Q(q) = nP(q) - P'(q) * (q - \xi). \tag{6}$$

We claim that $Q(q) \neq 0$ in the unit ball. Since $P(q)$ is nonzero in $\mathbb{B}$ we can consider

$$\begin{aligned} n + P(q)^{-*} * P'(q) * (\xi - q) &= n + \sum_{i=1}^{n} (q - b_i)^{-*} * (\xi - q) \\ &= \sum_{i=1}^{n} \big(1 + (q - b_i)^{-*} * (\xi - q)\big) = \sum_{i=1}^{n} (q - b_i)^{-*}(\xi - b_i). \end{aligned} \tag{7}$$

Recall that $(q - b_i)^{-*} = ((q - \bar{b}_i) * (q - b_i))^{-1}(q - \bar{b}_i)$, and since $(f * g)(q) = f(q)g(f(q)^{-1}qf(q))$ when $f(q) \neq 0$ otherwise $(f * g)(q) = 0$, see [2], Proposition 4.3.22, we can write

$$(q - b_i)^{-*} = \big((q - \bar{b}_i)(\tilde{q} - b_i)\big)^{-1}(q - \bar{b}_i) = (\tilde{q} - b_i)^{-1},$$

where $\tilde{q} = (q - \bar{b}_i)^{-1} q (q - \bar{b}_i)$ is obtained from q with a rotation. So we have

$$n + P(q)^{-*} * P(q) * (\xi - q) = \sum_{i=1}^{n} (\tilde{q} - b_i)^{-1}(\xi - b_i). \tag{8}$$

Let us now consider the map $w(p) = (\tilde{q} - p)^{-1}(\xi - p)$, where $\tilde{q}, \xi \in \mathbb{B}$ are considered as two parameters. When $|p| \geq 1$ we have $|\xi - \tilde{q}w| = |\tilde{q}||\tilde{q}^{-1}\xi - w| \geq |1 - w|$ from, which we deduce that the map $w(p)$ is a transformation taking the exterior of the unit ball $\mathbb{B}$ to a ball B' not containing 0, in fact, since $\tilde{q}$, $\xi \in \mathbb{B}$, neither 0 nor ∞ can belong to B'. Now note that $w(b_i) \in B'$, but then it is immediate to verify that also their arithmetic mean $\frac{1}{n}\sum_{i=1}^{n} (\tilde{q} - b_i)^{-1}(\xi - b_i)$ belongs to B'; since B' does not contain 0, the arithmetic mean and thus the sum at the right-hand side of (8) cannot be zero. By multiplying on the right by $P(q) \neq 0$, we have that also $Q(q) \neq 0$ in $\mathbb{B}$, as stated. The conclusion of the theorem follows as in [3], Theorems 4 and 5. In fact, assume that for any $q \in \mathbb{B}$ we have that $P(q) = w$ belongs to some subset S of $\mathbb{H}$. Then for any $q, \xi \in \mathbb{B}$ we have:

$$\frac{1}{n} P'(q)\xi + P(q) - \frac{1}{n} q P'(q) \in S. \tag{9}$$

To show this fact, take $\lambda \notin S$, so that $P(q) \neq \lambda$. By writing (6) in the case of the polynomial $P(q) - \lambda$, we obtain that $P'(q) * (\xi - q) + nP(q) \neq n\lambda$ for $q, \xi \in \mathbb{B}$, which proves (9). To prove the assertion in the statement, we now assume that the polynomial P of degree n satisfies $|P(q)| \leq M$ for $q \in \overline{\mathbb{B}}$ and that $P(q)$ has no roots in $\mathbb{B}$. Then we set $S = \{q \in \mathbb{H} \ : \ 0 < |w| < M\}$. Since (9) says that the interior of a ball with radius $|P'(q)|/n$ belongs to S and since S is deprived of its center, it follows that the radius $|P'(q)|/n$ of the ball is less than $M/2$, that is $\|P'\| \leq \frac{n}{2}\|P\|$. □

Remark 2. The bound is optimal as it can be seen by taking the polynomial $P(q) = (1 + q^n)/2$.

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