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Partial differential equations

Infinitely many solutions for resonance elliptic systems ☆

Infinité de solutions pour les systèmes elliptiques de résonance

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ABSTRACT

In this note, we study a class of resonance gradient elliptic systems and obtain infinitely many nontrivial solutions by using critical point theory.

RÉSUMÉ

Dans cette Note, nous étudions une classe de systèmes elliptiques de gradient de résonance et obtenons une infinité de solutions non triviales en utilisant la théorie des points critiques.

1. Introduction and main results

We consider the following elliptic system

$$\begin{cases} -\Delta u = \lambda u + \delta v + f(x,u,v), & \text{in } \Omega, \\ -\Delta v = \delta u + \gamma v + g(x,u,v), & \text{in } \Omega, \\ u = v = 0, & \text{on } \partial\Omega, \end{cases} \tag{$\mathcal{P}$}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$ $(N \geq 3)$ and $\lambda, \delta, \gamma \in \mathbb{R}$. The nonlinearities (f, g) are the gradient of some function, that is, there exists a function $F \in C^1(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ such that $\nabla F = (f, g)$. The system goes into resonance if the following conditions holds:

(V) $\sigma\left(A^*\right) \cap \sigma(-\Delta) \neq \emptyset$,

where

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$$A^* = \begin{pmatrix} \lambda & \delta \\ \delta & \gamma \end{pmatrix}; \qquad \sigma\left(A^*\right) = \{\xi, \zeta\} = \left\{ \frac{\lambda+\gamma}{2} \pm \sqrt{\left(\frac{\lambda-\gamma}{2}\right)^2 + \delta^2} \right\}$$

denotes the spectrum of the matrix A^* and $\sigma(-\Delta) = \{\lambda_k : k = 1, 2, \cdots$ and $0 < \lambda_1 < \lambda_2 < \cdots\}$ denotes the eigenvalues of the Laplacian on Ω with zero boundary conditions.

A vast literature on the study of the existence and multiplicity of solutions for resonance elliptic systems via the critical point theory has grown since Costa and Magalhães published their paper [6]; see [5,11–15,17] and the references therein. In [6], Costa and Magalhães consider subquadratic perturbations of semilinear elliptic systems that are in variational form. Later, Zou [14] presented two different theorems. If ∇F is not odd, is sublinear and satisfies certain assumptions at infinity (and near the origin), classical linking theorems with the Cerami compactness condition can be used to prove the existence of at least one (nontrivial) solution to $(\mathcal{P})$. Furthermore, if the nonlinear term ∇F is odd and $(\mathcal{P})$ is a strongly resonant problem, one can obtain solutions under suitable hypotheses on F and by using a multiplicity theorem due to Fei [7]. In particular, this result holds for a single elliptic equation at resonance. Zou et al. [17] get the existence of one and of two nonzero solutions in the case where the problem is resonant and F is sublinear at zero and infinity. Zou [15] considers cooperative and noncooperative elliptic systems that are asymptotically linear at infinity. He obtains infinitely many solutions with small energy if the potential is even. Pomponio [13] consider an asymptotically linear cooperative elliptic system at resonance. Recently, Ma had generalized Zou's results [14,15,17] in [11] and [12], respectively. Very recently, Chen and Ma [5] studied a class of resonant cooperative elliptic systems with sublinear or superlinear terms and obtained infinitely many nontrivial solutions by two variant fountain theorems developed by Zou [16]. In the present paper, we study the existence of infinitely many non-trivial solutions to $(\mathcal{P})$ under the symmetric condition. By using the minimax methods in critical point theory, we obtain the multiplicity results for subquadratic cases, which generalizes and sharply improves the results in [5,11,14]. Furthermore, compared with their proofs, ours are much simpler. For more general operator, we refer the reader to the papers [3,4,9]. We cite the very recent monograph by Kristály, Rădulescu and Varga [10] as a general reference for the basic notions used in the paper.

Let $|\cdot|$ and $(\cdot,\cdot)$ denote respectively the usual norm and inner product in $\mathbb{R}^2$. We consider the subquadratic case and make the following assumptions:

(AF1) There exist constants $c > 0$ and $1 < p < 2^* := \frac{2N}{N-2}$ such that

$$\left|\nabla F(x, U)\right| \le c\left(1 + |U|^{p-1}\right), \quad \forall (x, U) \in \Omega \times \mathbb{R}^2.$$

(AF2) $F(x, 0) = 0$, for all $x \in \Omega$, and

$$\lim_{|U| \to 0} \frac{F(x, U)}{|U|^2} = +\infty \quad \text{uniformly for a.e. } x \in \Omega.$$

(AF3) $F(x, U) = F(x, -U)$, $\forall (x, U) \in \Omega \times \mathbb{R}^2$.

Our main results are as follows:

Theorem 1.1. *Suppose that (V), (AF1)–(AF3) are hold, then problem $(\mathcal{P})$ possesses infinitely many nontrivial solutions.*

Remark 1.2. Zou [15] studied systems $(\mathcal{P})$ that are asymptotically linear with resonance. Applying the minimax technique, they obtained the following theorem.

Theorem 1.3. *Assume the following conditions are satisfied:*

(C1) $|\nabla F(x, U)| \le c(1 + |U|^{\sigma})$ *for almost all* $x \in \Omega$ *and* $U \in \mathbb{R}^2$, *where* $\sigma \in (0, 1)$ *is a constant.*
(C2) $\liminf_{|U| \to +\infty} \frac{\pm F(x,U)}{|U|^{1+\sigma}} := a^{\pm}(x) \succeq 0$ *uniformly for almost all* $x \in \Omega$, *where* $a^{\pm}(x) \succeq 0$ *indicates that* $a^{\pm}(x) \ge 0$ *with strict inequality holding on a set of positive measure.*
(C3) *There exist* $\delta_1, \delta_2 \in (1, 2)$, $c_1 > 0$, $c_2 > 0$, $t_0 > 0$ *such that*

$$c_1|U|^{\delta_1} \le F(x, U) \le c_2|U|^{\delta_2}$$

for almost all $x \in \Omega$ *and* $|U| \le t_0$.
(C4) $F(x, -U) = F(x, U)$ *for a.e.* $x \in \Omega$ *and* $U \in \mathbb{R}^2$.

Then, $(\mathcal{P})$ has infinitely many small-energy solutions.

Subsequently, Chen and Ma [5] considered the subquadratic case and proved the following theorem.

Theorem 1.4. *Suppose that (V) and the following conditions are satisfied:*

(D1) *$F(x,U) \geq 0$, $\forall (x,U) \in \Omega \times \mathbb{R}^2$, and there exist constants $\mu \in [1,2)$ and $R_1 > 0$ such that*

$$\big(\nabla F(x,U), U\big) \leq \mu F(x,U), \quad \forall x \in \Omega \text{ and } |U| \geq R_1.$$

(D2) *There exist constants $a \in [1,2)$ and c_2, c_2', $R_2 > 0$ such that*

$$F(x,U) \geq c_2' |U|^a, \quad \forall x \in \Omega \text{ and } U \in \mathbb{R}^2$$

and

$$F(x,U) \leq c_2 |U|, \quad \forall x \in \Omega \text{ and } |U| \leq R_2.$$

(D3) $\liminf_{|U|\to\infty} \frac{F(x,U)}{|U|} \geq d > 0$ *uniformly for $x \in \Omega$.*

and $F(x,U)$ is even in U. Then $(\mathcal{P})$ possesses infinitely many nontrivial solutions.

Theorem 1.1 unifies and greatly extends Theorems 1.3 and 1.4. (C2) and (C3) in Theorem 1.3 and (D2) and (D3) in Theorem 1.4 are completely removed. Hence, Theorem 1.1 generalizes and significantly improves upon Theorems 1.3 and 1.4. There exist functions F satisfying our Theorem 1.1 and not satisfying Theorems 1.3 and 1.4. For example, let

$$F(x,U) = h(x)|U|^{3/2}\left(-\ln\left(\frac{1+|U|^2}{4}\right)\right)$$

for all $x \in \Omega$ and $U \in \mathbb{R}^2$, where $h \in L^1(\Omega;\mathbb{R}^+)$ with $\inf_{x\in\Omega} h(x) > 0$. A straightforward computation shows that F satisfies the conditions of Theorem 1.1, but it does not satisfy the corresponding conditions of Theorems 1.3 and 1.4, since $F(x,U) < 0$ for all $|U| > \sqrt{3}$, and $\lim_{|U|\to\infty} |\nabla F(x,U)| = +\infty$ uniformly for $x \in \Omega$.

2. Variational framework

In this section, we give the variational framework of our problem and some related preliminary lemmas.

In the following, we use $\|\cdot\|_2$ and $\|\cdot\|_{L^2}$ to denote the norms of $L^2(\Omega)$ and $L^2(\Omega)\times L^2(\Omega)$, respectively. Let $E := H_0^1(\Omega)$ and $W := H_0^1(\Omega)\times H_0^1(\Omega)$, where $H_0^1(\Omega)$ is the usual Sobolev space with the norm $\|\cdot\|_E$ generated by the inner product

$$\langle u, v\rangle_E = \int_\Omega \nabla u \cdot \nabla v \, dx, \quad \forall u, v \in H_0^1(\Omega).$$

Then for $U = (u_1,u_2)$ and $V = (v_1,v_2)$ in W, the induced inner product and norm on W are given respectively by

$$\langle U, V\rangle_W = \langle u_1, v_1\rangle_E + \langle u_2, v_2\rangle_E \quad \text{and} \quad \|U\|_W^2 = \|u_1\|_E^2 + \|u_2\|_E^2.$$

Let $\vec{e_1} := (e_{11}, e_{12})$, $\vec{e_2} := (e_{21}, e_{22}) \in \mathbb{R}^2$ be the normalized eigenvectors of A^* such that

$$A^*\vec{e_1} = \xi\vec{e_1}, \qquad A^*\vec{e_2} = \zeta\vec{e_2}, \qquad \vec{e_1}\cdot\vec{e_2} = 0, \qquad |\vec{e_1}| = |\vec{e_2}| = 1.$$

For any $\alpha \in \mathbb{R}$. Let H_α^+, H_α^-, H_α^0 be the subspaces of $H_0^1(\Omega)$, where the quadratic form

$$u \to \|u\|_E^2 - \alpha\|u\|_2^2$$

is positive definite, negative definite and zero, respectively. Let

$$W^0 := H_\xi^0 \times H_\zeta^0, \qquad W^+ := H_\xi^+ \times H_\zeta^+ \quad \text{and} \quad W^- := H_\xi^- \times H_\zeta^-.$$

Set

$$A_1 := \mathrm{id} - \xi(-\Delta)^{-1} \quad \text{and} \quad A_2 := \mathrm{id} - \zeta(-\Delta)^{-1},$$

where id denotes the identity on $H_0^1(\Omega)$. We introduce an operator:

$$A : W \to W, \quad A = (A_1, A_2), \quad \text{which is defined by } AU = (A_1u_1, A_2u_2), \forall U = (u_1,u_2) \in W.$$

Then A is a bounded self-adjoint operator from W to W and $\ker A = W^0$ with $\dim W^0 < \infty$. The space W splits as

$$W = W^- \oplus W^0 \oplus W^+,$$

where W^- and W^+ are invariant under A, $A|_{W^-}$ is negative, and $A|_{W^+}$ is positive definite. More precisely, there exists a positive constant c_0 such that

$$\pm\langle AU^\pm, U^\pm\rangle_W \geq c_0\|U^\pm\|_W^2, \quad \forall U^\pm \in W^\pm.$$

Here and in what follows, for any $U \in W$, we always denote by U^0, U^+ and U^- the vectors in W with $U = U^0 + U^- + U^+$, $U^0 \in W^0$ and $U^\pm \in W^\pm$. Note that $\dim W^0$ and $\dim W^-$ are finite. Furthermore, $\sigma(A^*) \cap \sigma(-\Delta) \neq \emptyset$ implies $\dim W^0 \neq 0$. For problem $(\mathcal{P})$, we consider the following functional:

$$\Phi(U) = \frac{1}{2}\langle AU, U\rangle_W - \int_\Omega \tilde{F}(x, U)\,\mathrm{d}x, \quad U = (u_1, u_2) \in W,$$

where $\tilde{F}(x, s, t) = F(x, s\vec{e_1} + t\vec{e_2})$. In view of the assumptions of F and the definition of $\tilde{F}$, we know that the (weak) solutions to system $(\mathcal{P})$ are the critical points of the functional Φ by the discussion of [6].

Next, we define an equivalent inner product $\langle\cdot,\cdot\rangle$ and the corresponding norm $\|\cdot\|$ on W given respectively by

$$\langle U, V\rangle = \langle AU^+, V^+\rangle_W - \langle AU^-, V^-\rangle_W \quad \text{and} \quad \|U\| = \langle U, U\rangle^{1/2},$$

where $U, V \in W^0 \oplus W^- \oplus W^+$ with $U = U^0 + U^- + U^+$ and $V = V^0 + V^- + V^+$. Therefore, Φ can be rewritten as

$$\Phi(U) = \frac{1}{2}\|U^+\|^2 - \frac{1}{2}\|U^-\|^2 - \int_\Omega \tilde{F}(x, U)\,\mathrm{d}x.$$

Furthermore, $\Phi \in C^1(W, \mathbb{R})$ and the derivatives are given by

$$\Phi'(U)V = \langle U^+, V^+\rangle - \langle U^-, V^-\rangle - \int_\Omega \left(\nabla\tilde{F}(x, U), V\right)\mathrm{d}x,$$

for any $U, V \in W^0 \oplus W^- \oplus W^+$ with $U = U^0 + U^- + U^+$ and $V = V^0 + V^- + V^+$.

For the sublinear case, we will use the following critical point theorem established by Kajikiya [8]. We refer the readers to [1] for the definition and proprieties of genus.

Definition 2.1. Let X be a Banach space and A a subset of X. A is said to be symmetric if $u \in A$ implies $-u \in A$. For a closed symmetric set A that does not contain the origin, we define a genus $\gamma(A)$ of A by the smallest integer k such that there exists an odd continuous mapping from A to $\mathbb{R}^k \setminus \{0\}$. If there does not exist such a k, we define $\gamma(A) = \infty$. Moreover, we set $\gamma(\emptyset) = 0$. Let Γ_k denote the family of closed symmetric subsets A of X such that $0 \notin A$ and $\gamma(A) \geq k$.

Theorem 2.2. *Let X be an infinite dimensional Banach space and $I \in C^1(X, \mathbb{R})$ satisfy (B1) and (B2) below.*

(B1) *I is even, bounded from below, $I(0) = 0$ and I satisfies the (PS) condition.*
(B2) *For each $k \in \mathbb{N}$, there exists an $A_k \in \Gamma_k$ such that $\sup_{u\in A_k} I(u) < 0$.*

Then I admits a sequence of critical points $\{u_k\}$ such that $I(u_k) \leq 0$, $u_k \neq 0$ and $\lim_{k\to\infty} u_k = 0$.

Note that $\dim W^0$ and $\dim W^-$ are finite. We choose an orthonormal basis $\{e_m\}_{m=1}^k$ for W^0, an orthonormal basis $\{e_m\}_{m=k+1}^{l_0}$ for W^- and an orthonormal basis $\{e_m\}_{m=l_0+1}^\infty$ for W^+, where $1 \leq k < \infty$ and $k + 1 \leq l_0 < \infty$. Then $\{e_m\}_{m=1}^\infty$ is an orthonormal basis of W.

3. Proof of the main results

Proof of Theorem 1.1. We consider the truncated functional

$$\mathcal{I}(U) = \frac{1}{2}\|U^+\|^2 - \left(\frac{1}{2}\|U^-\|^2 + \int_\Omega \tilde{F}(x, U)\,\mathrm{d}x\right)h(\|U\|)$$

for all $U \in W$, where $h : \mathbb{R}^+ \to [0, 1]$ is a non-increasing C^1 function such that $h(t) = 1$ for $0 \leq t \leq 1$ and $h(t) = 0$ for $t \geq 2$. Obviously, $\mathcal{I} \in C^1(W, \mathbb{R})$ and $\mathcal{I}(0) = 0$. If we can prove that $\mathcal{I}$ admits a sequence of critical points $\{U_k\}$ such that $\mathcal{I}(U_k) \leq 0$, $U_k \neq 0$ and $U_k \to 0$ as $k \to \infty$, then we can apply Kajikiya's critical point theorem (Theorem 2.2) to get the desire results. Due to (AF3), $F(x, -U) = F(x, U)$ for all $(x, U) \in \Omega \times \mathbb{R}^2$, so $\mathcal{I}(U) = \mathcal{I}(-U)$, that is $\mathcal{I}$ is even.

For $\|U\| \geq 2$, we have that

$$\mathcal{I}(U) = \frac{1}{2}\|U^+\|^2,$$

which shows that

$$\mathcal{I}(U) \to +\infty, \quad \text{as } \|U\| \to \infty.$$

This implies that $\mathcal{I}$ is bounded from below and satisfies the (PS) condition. Actually, due to the coercitivity of the functional $\mathcal{I}$, we can get a (PS) sequence $\{U_j\}$ bounded. By the fact of $\dim(W^0 \oplus W^-) < \infty$, without lose of generality, we may assume

$$U_j^- \to U^-, \qquad U_j^0 \to U^0, \qquad U_j^+ \rightharpoonup U^+ \quad \text{and} \quad U_j \rightharpoonup U, \quad \text{as } j \to \infty, \tag{1}$$

for some $U = U^0 + U^- + U^+ \in W = W^0 \oplus W^- \oplus W^+$. By virtue of the Riesz Representation Theorem, $\mathcal{I}' : W \to W^*$ and $G' : W \to W^*$ can be viewed as $\mathcal{I}' : W \to W$ and $G' : W \to W$ respectively, where W^* is the dual space of W and $G(U) := \int_\Omega \tilde{F}(x, U)\,\mathrm{d}x$. Note that

$$o(1) = \mathcal{I}'(U_j) = U_j^+ - \left(U_j^- + G'(U_j)\right), \ \forall j \in \mathbb{N},$$

that is,

$$U_j^+ = U_j^- + G'(U_j) + o(1), \quad \forall j \in \mathbb{N}. \tag{2}$$

Note that the assumptions of F and the definition of $\tilde{F}$, the Sobolev embedding, by the standard argument (see [2]), imply $G' : W \to W^*$ is compact. Therefore, $G' : W \to W$ is also compact. Due to the compactness of G' and (1), the right-hand side of (2) converges strongly on W, and hence $U_j^+ \to U^+$ in W. Combing this with (1), we have $U_j \to U$ in W. Therefore, $\mathcal{I}$ satisfies the (PS) condition.

Given any $k \geq k_1 := l_0 + 1$, where l_0 is defined as in Section 2, let $E_k = \bigoplus_{j=1}^k X_j$, where $X_j = \operatorname{span}(e_j)$, where $\{e_j\}$ is an orthogonal basis of W. There exists a constant $c_k > 0$ such that

$$\|U\|_{L^2} \geq c_k \|U\|, \quad \forall U \in E_k, \ \forall k \in \mathbb{N},$$

by the equivalence of the norms on the finite-dimensional spaces E_k. Using (AF2), there exists $0 < r_1 < 1$ such that

$$F(x, U) \geq \frac{1}{c_k^2} |U|^2,$$

for all $|U| \leq r_1$ and a.e. $x \in \Omega$. Therefore, for $U \in E_k$ with $\|U\| = l_k := \frac{1}{2} \min\{1, \frac{r_1}{c_k}\}$, we obtain:

$$\begin{aligned}
\mathcal{I}(U) &= \frac{1}{2}\left\|U^+\right\|^2 - \frac{1}{2}\left\|U^-\right\|^2 - \int_\Omega \tilde{F}(x, U)\,\mathrm{d}x \\
&\leq \frac{1}{2}\left\|U^+\right\|^2 - \int_\Omega \tilde{F}(x, U)\,\mathrm{d}x \\
&\leq \frac{1}{2}\|U\|^2 - \frac{1}{c_k^2}\|U\|_{L^2}^2 \\
&\leq \frac{1}{2}\|U\|^2 - \frac{1}{c_k^2} c_k^2 \|U\|^2 \\
&= -\frac{1}{2}\|U\|^2 \\
&= -\frac{1}{2} l_k^2,
\end{aligned}$$

which implies that

$$\left\{U \in E_k : \|U\| = l_k\right\} \subset \left\{U \in W : \mathcal{I}(U) \leq -\frac{1}{2} l_k^2\right\}.$$

Now taking $A_k = \{U \in W : \mathcal{I}(U) \leq -\frac{1}{2} l_k^2\}$, by Theorem 2.2, we get that:

$$\gamma(A_k) \geq \gamma\left(\left\{U \in E_k : \|U\| = l_k\right\}\right) \geq k,$$

so $A_k \in \Gamma_k$ and

$$\sup_{U \in A_k} \mathcal{I}(U) \leq -\frac{1}{2} l_k^2 < 0.$$

The proof is complete. $\square$

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