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Backward doubly stochastic differential equations with a superlinear growth generator ☆

Équations différentielles doublement stochastiques avec un coefficient à croissance surlinéaire

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A R T I C L E I N F O

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A B S T R A C T

We deal with backward doubly stochastic differential equations (BDSDEs) with a superlinear growth generator and a square integrable terminal datum. We introduce a new local condition on the generator, then we show that it ensures the existence and uniqueness as well as the stability of solutions. Our work goes beyond the previous results on the multidimensional BDSDEs. Although we are focused on the multidimensional case, our uniqueness result is new for one-dimensional BDSDEs, too. As an application, we establish the existence of a Sobolev solution to SPDEs with superlinear growth generator. Some illustrative examples are also presented.

R É S U M É

Nous considérons des équations différentielles doublement stochastiques rétrogrades (EDDSR) avec un générateur de croissance surlinéaire et une donnée terminale de carré intégrable. Nous introduisons une nouvelle condition locale sur le générateur et nous montrons qu'elle assure l'existence, l'unicité et la stabilité des solutions. Même si notre intérêt porte sur le cas multidimensionnel, notre résultat est également nouveau en dimension un. Des exemples illustratifs et une application aux équations aux dérivées partielles stochastiques (EDPS) sont également donnés.

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Dans la présente note, nous considérons des équations différentielles doublement stochastiques rétrogrades (EDDSR) multidimensionnelles ayant un générateur $f(s, y, z)$ de croissance surlinéaire en les variables (y, z), et une condition terminale de carré intégrable. On établit alors l'existence, l'unicité et la stabilité des solutions. En raison des conditions que nous imposons sur le générateur, les méthodes classiques ne fonctionnent pas. En particulier, ni le lemme de Gronwall, ni le lemme de Bihari ne peuvent être utilisés. Notons également que les techniques de localisation par des temps d'arrêts ne fonctionnent pas dans les EDDSR. En nous appuyant sur les idées de [1–3] dans les EDSR classiques, nous développons des méthodes basées sur une estimation entre les solutions d'EDDSR. Ceci nous permet d'obtenir simultanément l'existence, l'unicité et la stabilité des solutions à partir de la même estimation. Notre travail étend les précédents articles sur les EDDSR mulidimensinnelles avec des conditions locales sur le générateur, qui est en outre de croissance surlinéaire. En application, nous établissons l'existence de solutions (au sens de Sobolev) pour des équations aux dérivées partielles stochastiques (EDPS) de croissance surlinéaires en la solution et en son gradient. En particulier, nous obtenons l'existence de solutions pour des EDPS ayant des nonlinéarités logarithmiques par rapport à la solution et son gradient. Les résultats essentiels sont :

Theorem 0.1. *On suppose que ξ est de carré integrable et que* **(H.1)**–**(H.5)** *sont satisfaites. Alors, l'équation* $(E^{f,g,\xi})$ *admet une solution unique.*

Theorem 0.2. *Soient f, g, ξ comme dans le théorème précédent. Soit f_n une suite de fonctions satisfaisant les hypothèses* **(H.1)**, **(H.2)**, **(H.4)**, **(H.5)** *uniformement en n. Soit (ξ_n) une suite de variables aléatoires de carré integrables et on suppose que* **(H.6)**, **(H.7)** *sont vérifiées. Pour chaque n, on note (Y^n, Z^n) la solution de l'EDDSR (E^{f_n,g,ξ_n}). Alors, pour tout $q < 2$ on a*

$$\lim_{n\to+\infty}\left(\mathbb{E}\sup_{0\le t\le T}\left|Y_t^n - Y_t\right|^q + \mathbb{E}\int_0^T \left|Z_s^n - Z_s\right|^q \mathrm{d}s\right) = 0.$$

Theorem 0.3. *On suppose que les hypothèses* **(H.10)**–**(H.15)** *sont satisfaites. Alors, l'EDPS $(\mathcal{P}^{(f,g)})$ admet une unique solution u telle que, pour tout $t \in [0, T]$, $u(s, X_s^{t,x}) = Y_s^{t,x}$ and $(\sigma^*\nabla u)(s, X_s^{t,x}) = Z_s^{t,x}$ pour presque tout (ω, s, x), où $\{(Y_s^{t,x}, Z_s^{t,x}), t \le s \le T\}$ est l'unique solution de l'EDDSR (1).*

1. Introduction

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space. For $T > 0$, let $\{W_t, 0 \le t \le T\}$ and $\{B_t, 0 \le t \le T\}$ be two independent standard Brownian motions defined on $(\Omega, \mathcal{F}, P)$ with values in $\mathbb{R}^d$ and $\mathbb{R}$, respectively. Let $\mathcal{F}_t^W := \sigma(W_s; 0 \le s \le t)$ and $\mathcal{F}_{t,T}^B := \sigma(B_s - B_t; t \le s \le T)$, completed with P-null sets. We put $\mathcal{F}_t := \mathcal{F}_t^W \vee \mathcal{F}_{t,T}^B$. It should be noted that $(\mathcal{F}_t)$ is not an increasing family of sub σ-fields, and hence it is not a filtration. Let $M^2(0, T, \mathbb{R}^d)$ denote the set of d-dimensional, $(\mathcal{F}_t)$-adapted stochastic processes $\{\varphi_t; t \in [0, T]\}$, such that $E\int_0^T |\varphi_t|^2\,\mathrm{d}t < \infty$. We denote by $S^2([0, T], \mathbb{R})$, the set of continuous and $(\mathcal{F}_t)$-adapted stochastic processes $\{\varphi_t; t \in [0, T]\}$, which satisfy $E(\sup_{0\le t\le T} |\varphi_t|^2) < \infty$. Let $f : \Omega \times [0, T] \times \mathbb{R}^d \times \mathbb{R}^{d\times r} \longmapsto \mathbb{R}^d$, $g : \Omega \times [0, T] \times \mathbb{R}^d \times \mathbb{R}^{d\times r} \longmapsto \mathbb{R}^{d\times l}$ be jointly measurable and $(\mathcal{F}_t)$-adapted. The BDSDE under consideration is:

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\,\mathrm{d}s + \int_t^T g(s, Y_s, Z_s)\,\mathrm{d}\overleftarrow{B}_s - \int_t^T Z_s\,\mathrm{d}W_s, \qquad (E^{f,g,\xi})$$

where $\int_t^T g(s, Y_s, Z_s)\,\mathrm{d}\overleftarrow{B}_s$ denotes the Itô backward integral, ξ is called the terminal datum and f the generator.

Definition 1.1. A solution to Eq. $(E^{f,g,\xi})$ is a couple $(Y^{f,g,\xi}, Z^{f,g,\xi})$ that belongs to the space $S^2([0, T], \mathbb{R}) \times M^2(0, T, \mathbb{R})$ and satisfies $(E^{f,g,\xi})$.

Eq. $(E^{f,g,\xi})$ was introduced by Pardoux and Peng in [9], where the existence and the uniqueness of solutions were established assuming that f is globally Lipschitz and g is a contracting function. Since the localization by stopping times is ineffective in BDSDEs, the most of the previous papers have considered BDSDEs with global assumptions on the generator, like Lipschitz or global monotony. It also was shown in [9] that the BDSDEs $(E^{f,g,\xi})$ can be related to semilinear and quasilinear stochastic partial differential equations (SPDEs). This link (between BDSDEs and SPDEs) was later developed in many papers (see, e.g., [5,6,4,7–10]) and has motivated many efforts to establish the existence and uniqueness of solutions to BDSDEs under more general conditions than the global Lipschitz one (see, for instance, [10,11]).

In this paper, we establish the existence and uniqueness as well as the $\mathbb{L}^p$-stability ($p < 2$) of solutions to BDSDE $(E^{f,g,\xi})$, when the generator f is of superlinear growth in y, z. Moreover, the terminal datum still remains merely square integrable. We introduce new local assumptions on the generator that cover the previous ones on multidimensional BDSDEs and go beyond. Compared with [10] for instance, the generators we consider here can be neither locally monotone in the y-variable nor locally Lipschitz in the z-variable. We cover in particular the logarithmic nonlinearities $y\log(|y|)$ as well as $h(y)z\sqrt{|\log(|z|)|}$, where h is a suitable function. It is worth noting that our conditions on the generator f are local on the three variables y, z and ω. This allows us to cover BDSDEs with stochastic Lipschitz conditions. Due to the local assumptions and the superlinear growth condition on the generator, the usual techniques of BDSDEs do not work. In particular, neither Gronwall's inequality nor Bihari's Lemma can be used. Although we are focused on multidimensional BDSDEs, our uniqueness result is new also in one dimension.

The paper is organized as follows. The main results are given in Section 2. The proofs and some illustrative examples are also given in Section 2. These example are covered by our result and, to the best of our knowledge, not covered by the previous papers. An application to SPDEs is given in Section 3.

2. The main results

We consider the following assumptions:

(H.1) f is continuous in (y,z) for *a.e.* (t,ω).
(H.2) There exist $M>0$, $\gamma<\frac{1}{4}$ and $\eta\in\mathbb{L}^1(\Omega;\,\mathbb{L}^1([0,T]))$ such that,

$$\langle y, f(t,\omega,y,z)\rangle \le \eta_t + M|y|^2+\gamma|z|^2,\quad P\text{-a.s., a.e. } t\in[0,T].$$

(H.3) There exist $L>0$, $K>0$, $0<\lambda<\frac{1}{2}$, $0<\alpha_1<1$, and η_t, $0\le t\le T$, satisfying $E\int_0^T|\eta'_s|^{\frac{2}{\alpha_1}}\,ds<\infty$ such that, $\|g(t,y,z)-g(t,y',z')\|^2\le L|y-y'|^2+\lambda\|z-z'\|^2$ and $\|g(t,y,z)\|\le\eta'_t+K(|y|^{\alpha_1}+\|z\|^{\alpha_1})$.
(H.4) There exist $M_1>0$, $0\le\alpha<2$, $\alpha'>1$ and $\bar\eta\in\mathbb{L}^{\alpha'}([0,T]\times\Omega)$ such that:

$$\big|f(t,\omega,y,z)\big|\le\bar\eta_t+M_1\big(|y|^\alpha+|z|^\alpha\big).$$

(H.5) There exist $v\in\mathbb{L}^2(\Omega;\,\mathbb{L}^2([0,T]))$, a real valued sequence $(A_N)_{N>1}$ and constants $M_2>1$, $r>0$ such that:
i) $\forall N>1$, $1<A_N\le N^r$.
ii) $\lim_{N\to\infty}A_N=\infty$.
iii) For every $N\in\mathbb{N}^*$ and every y, y', z, z' such that $|y|$, $|y'|$, $|z|$, $|z'|\le N$, we have

$$\begin{aligned}\langle y-y', f(t,y,z)-f\big(t,y',z'\big)\rangle\mathbf{1}_{\{v_s(\omega)\le N\}}\le{}& M_2\big|y-y'\big|^2\log A_N\\ &+M_2\big|y-y'\big|\big|z-z'\big|\sqrt{\log A_N}+M_2A_N^{-1}.\end{aligned}$$

2.1. Existence, uniqueness and stability of solutions

Theorem 2.1. *Let ξ be a square integrable random variable. Assume that* **(H.1)**–**(H.5)** *are satisfied. Then Eq. $(E^{f,g,\xi})$ has a unique solution.*

To deal with the stability of solutions, let (f_n) be a sequence of $(\mathcal{F}_t)$-progressively measurable processes. Let (ξ_n) be a sequence of $(\mathcal{F}_T)$-measurable random variables such that $E(|\xi_n|^2)<\infty$. We assume that, for each n, the BDSDE (E^{f_n,g,ξ_n}) corresponding to the data (f_n,g,ξ_n) has a (not necessarily unique) solution. Each solution to Eq. $(E^{f,g,\xi})$ will be denoted by (Y^n,Z^n). Let (Y,Z) be the unique solution to the BDSDE $(E^{f,g,\xi})$.

Theorem 2.2. *Let f, g and ξ be as in Theorem 2.1. Assume moreover that f_n satisfies hypothesis* **(H.1)**, **(H.2)**, **(H.4)**, **(H.5)** *uniformly in n and,*

(H.6) *For every N, $\rho_N(f_n-f):=\mathbb{E}\int_0^T\sup_{|y|,|z|\le N}|(f_n-f)(s,y,z)|\,ds\longrightarrow0$, as $n\to\infty$.*
(H.7) *$E(|\xi_n-\xi|^2)\longrightarrow0$, as $n\to\infty$.*

Then, for every $q<2$ we have

$$\lim_{n\to+\infty}\left(\mathbb{E}\sup_{0\le t\le T}\big|Y_t^n-Y_t\big|^q+\mathbb{E}\int_0^T\big|Z_s^n-Z_s\big|^q\,ds\right)=0.$$

Sketch of the Proofs. Let f, g and ξ be as in Theorem 2.1. Using suitable regularization and truncation, one can construct a sequence (f_n) such that, *for each n, f_n is bounded and globally Lipschitz in (y,z) for a.e. (t,ω). For every N, $\rho_N(f_n-f)\to0$, as $n\to\infty$. Moreover, f_n satisfies* **(H.2)**, **(H.4)**, **(H.5)** *uniformly in n.*

For any integer n, we denote by (Y^{f_n}, Z^{f_n}) the unique solution to the BDSDE $(E^{f_n,g,\xi})$. Using standard arguments of BDSDEs, one can show (after extracting a subsequence) that there exist $Y \in \mathbb{L}^2(\Omega, L^\infty[0,T])$, $Z \in \mathbb{L}^2(\Omega \times [0,T])$, $F \in \mathbb{L}^{\bar{\alpha}}(\Omega \times [0,T])$, and $\bar{g} \in \mathbb{L}^2(\Omega \times [0,T])$ such that $Y^{f_n} \rightharpoonup Y$, weakly star in $\mathbb{L}^2(\Omega, L^\infty[0,T])$, $Z^{f_n} \rightharpoonup Z$, weakly in $\mathbb{L}^2(\Omega \times [0,T])$, $f_n(.,Y^{f_n}, Z^{f_n}) \rightharpoonup F_.$ weakly in $\mathbb{L}^{\bar{\alpha}}(\Omega \times [0,T])$ and $g_n(.,Y^{f_n}, Z^{f_n}) \rightharpoonup \bar{g}_.$ weakly in $\mathbb{L}^2(\Omega \times [0,T])$. Moreover, one has the following claim:

For every $R \in \mathbb{N}$, $\beta \in]1, \min(3 - \frac{2}{\bar{\alpha}}, 2)[$, $\varepsilon > 0$ and $0 < \delta' < (1-\lambda)(\beta-1)\min(\frac{1}{2M_2^2 + 2M_2(1-\lambda)(\beta-1)}, \frac{\kappa}{rM_2^2\beta + r\beta M_2(1-\lambda)(\beta-1)})$, there exists $N_0 > R$ such that for every $N > N_0$ and $T' \le T$:

$$\limsup_{n,m\to+\infty} E \sup_{(T'-\delta')^+ \le t \le T'} \left|Y_t^{f_n} - Y_t^{f_m}\right|^\beta + E \int_{(T'-\delta')^+}^{T'} \frac{|Z_s^{f_n} - Z_s^{f_m}|^2}{(|Y_s^{f_n} - Y_s^{f_m}|^2 + \nu_R)^{\frac{2-\beta}{2}}} \mathrm{d}s$$
$$\le \varepsilon + \frac{\ell}{(1-\lambda)(\beta-1)} e^{C_N \delta'} \limsup_{n,m\to+\infty} E\left|Y_{T'}^{f_n} - Y_{T'}^{f_m}\right|^\beta,$$

where $\nu_R = \sup\{(A_N \log A_N)^{-1}, N \ge R\}$, $C_N = \frac{\beta L}{2} + \frac{M_2^2 \beta \log A_N}{(1-\lambda)(\beta-1)} + \beta M_2 \log A_N$, and ℓ is a universal positive constant.

The rest of the proof is a translation of the approach in [2] from BSDEs to BDSDEs. This allows us to show that (Y^{f_n}, Z^{f_n}) converges to a process (Y, Z) which is a solution to the BDSDE $E^{(f,g,\xi)}$. The same arguments are used to establish the uniqueness and the stability of solutions. □

Examples. Let ξ be square integrable and g satisfy assumption **(H.3)**. Then, the BDSDE $(E^{(f,g,\xi)})$ has a unique solution in the following examples.

Example 1. Let f satisfy **(H.1)**, **(H.2)** and **(H.4)**. Assume moreover that there exists a positive constant C such that, for every $N > 0$ and every $|y|, |z|, |y'|, |z'| \le N$, $|f(t,y,z) - f(t,y',z)| \le C \log N |y - y'|$ and $|f(t,y,z) - f(t,y,z')| \le C\sqrt{\log N}|z - z'|$.

Example 2. $f(t,y,z) := -y \log(|y|)$.

Example 3. Let $l(y) := y \log \frac{|y|}{1+|y|}$ and $h \in \mathcal{C}(\mathbb{R}^{dr}; \mathbb{R}_+) \cap \mathcal{C}^1(\mathbb{R}^{dr} - \{0\}; \mathbb{R}_+)$ be such that, $h(z) = |z|\sqrt{-\log|z|}$, if $|z| < 1 - \varepsilon_0$, and $h(z) := |z|\sqrt{\log|z|}$, if $|z| > 1 + \varepsilon_0$, where $\varepsilon_0 \in]0,1[$. We finally define $f(y,z) := l(y)h(z)$.

The following example shows that our result also covers the BDSDEs with a stochastic Lipschitz generator.

Example 4. Let f satisfy **(H.1)**, **(H.2)**, **(H.4)** and, **(H.5)**. There is a positive process C satisfying $\mathbb{E}\int_0^T e^{q' C_s} ds < \infty$ (for some $q' > 0$) and $K' \in \mathbb{R}_+$ such that:

$$\left\langle y - y', f(t,\omega,y,z) - f(t,\omega,y',z')\right\rangle \le K'\left|y-y'\right|^2\left\{C_t(\omega) + \left|\log\left|y-y'\right|\right|\right\}$$
$$+ K'\left|y-y'\right|\left|z-z'\right|\sqrt{C_t(\omega) + |\log|z-z'||}.$$

3. Application to SPDEs

We consider the SPDE,

$$(\mathcal{P}^{(f,g)}) \begin{cases} u(t,x) = h(x) + \displaystyle\int_s^T \left\{\mathcal{L}u(r,x) + f\big(r,x,u(r,x),\sigma^*\nabla u(r,x)\big)\right\} \mathrm{d}r \\ \qquad + \displaystyle\int_s^T g\big(r,x,u(r,x),\sigma^*\nabla u(r,x)\big)\, \mathrm{d}\overleftarrow{B}_r, \quad t \le s \le T, \end{cases}$$

where $\mathcal{L} := \frac{1}{2}\sum_{i,j} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_i b_i \frac{\partial}{\partial x_i}$, and $(a_{ij}) = \sigma\sigma^*$.

The diffusion process associated with the operator $\mathcal{L}$ satisfies

$$\mathrm{d}X_s^{t,x} = b\big(X_s^{t,x}\big)\,\mathrm{d}s + \sigma\big(X_s^{t,x}\big)\,\mathrm{d}W_s, \quad \text{with } X_t^{t,x} = x \in \mathbb{R}^k,$$

where $b \in \mathcal{C}_b^2(\mathbb{R}^k, \mathbb{R}^k)$ and $\sigma \in \mathcal{C}_b^3(\mathbb{R}^k, \mathbb{R}^{k\times r})$.

We will connect the SPDE $(\mathcal{P}^{(f,g)})$ with the following BDSDE

$$Y_s^{t,x} = h\big(X_T^{t,x}\big) + \int_s^T f\big(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}\big)\,\mathrm{d}r + \int_s^T g\big(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}\big)\,\mathrm{d}\overleftarrow{B}_r - \int_s^T Z_r^{t,x}\,\mathrm{d}W_r. \tag{1}$$

Let $\delta > 0$. For $q < 2$, let $\mathcal{H}^\delta$ be the set of random fields $u(t,x), 0 \le t \le T, x \in \mathbb{R}^k$ such that, for every (t,x), $u(t,x)$ is $\mathcal{F}_{t,T}^B$-measurable and

$$\|u\|_{\mathcal{H}^\delta}^q = E\Bigg[\int_{\mathbb{R}^k}\int_0^T \big(\big|u(r,x)\big|^q + \big|(\sigma^*\nabla u)(r,x)\big|^q\big)\,\mathrm{d}r e^{-\delta|x|}\,\mathrm{d}x\Bigg] < \infty. \tag{2}$$

$(\mathcal{H}^\delta, \|.\|_{\mathcal{H}^\delta})$ is then a Banach space.

Definition 3.1. We say that u is a Sobolev solution to SPDE $(\mathcal{P}^{(f,g)})$, if $u \in \mathcal{H}^\delta$ and for any $\varphi \in \mathcal{C}_c^{1,\infty}([0,T] \times \mathbb{R}^d)$,

$$\begin{aligned}
&\int_{\mathbb{R}^k}\int_s^T u(r,x)\frac{\partial\varphi(r,x)}{\partial r}(r,x)\,\mathrm{d}r\,\mathrm{d}x + \int_{\mathbb{R}} u(r,x)\varphi(r,x)\,\mathrm{d}x - \int_{\mathbb{R}^k} h(x)\varphi(T,x)\,\mathrm{d}x \\
&\quad + \frac{1}{2}\int_{\mathbb{R}^k}\int_s^T u(r,x)\sum_{i=1}^d \partial_{x_i}\Bigg(\sum_{j=1}^d a_{ij}(r,x)\partial_{x_j}\varphi(r,x)\Bigg)\mathrm{d}r\,\mathrm{d}x - \int_{\mathbb{R}^k}\int_s^T u div\big[(b-A)\varphi\big](r,x)\,\mathrm{d}r\,\mathrm{d}x \\
&= \int_{\mathbb{R}^k}\int_s^T f\big(r,x,u(r,x),\sigma^*\nabla u(r,x)\big)\varphi(r,x)\,\mathrm{d}r\,\mathrm{d}x \\
&\quad + \int_{\mathbb{R}^k}\int_s^T g\big(r,x,u(r,x),\sigma^*\nabla u(r,x)\big)\varphi(r,x)\,\mathrm{d}B_r\,\mathrm{d}x
\end{aligned}$$

where A is a d-vector whose coordinates are given by $A_j := \frac{1}{2}\sum_{i=1}^d \frac{\partial a_{ij}}{\partial x_i}$.

Assumptions. We assume that there exists $\delta \ge 0$ such that

(H.10) h belongs to $\mathbb{L}^2(\mathbb{R}^k, e^{-\delta|x|}dx; \mathbb{R}^d)$, i.e. $\int_{\mathbb{R}^d} |h(x)|^2 e^{-\delta|x|}\,\mathrm{d}x < \infty$.
(H.11) $f(t,x,.,.)$ is continuous for a.e. (t,x).
(H.12) There exist $M > 0$, $\gamma < \frac{1}{4}$ and $\eta \in \mathbb{L}^1([0,T] \times \mathbb{R}^k, e^{-\delta|x|}\,\mathrm{d}t\,\mathrm{d}x; \mathbb{R}_+)$ such that

$$\big\langle y, f(t,x,y,z)\big\rangle \le \eta(t,x) + M|y|^2 + \gamma|z|^2 \quad \mathbb{P}\text{-}a.s., a.e.\ t \in [0,T].$$

(H.13) $\int_{\mathbb{R}^k}\int_0^T |g(t,x,0,0)|^2 e^{-\delta|x|}\,\mathrm{d}t\,\mathrm{d}x < \infty$ and, there exist $L > 0$, $0 < \lambda < \frac{1}{2}$, $0 < \alpha_1 < 1$, and a process (η_t') satisfying $E\int_0^T |\eta_s'|^{\frac{2}{\alpha_1}}\,\mathrm{d}s < \infty$ such that $\|g(t,x,y,z) - g(t,x,y',z')\|^2 \le L|y-y'|^2 + \lambda\|z-z'\|^2$ and $\|g(t,x,y,z)\| \le \eta_t' + K(|x| + |y|^{\alpha_1} + \|z\|^{\alpha_1})$.
(H.14) There exists $M_1 > 0$, $0 \le \alpha < 2$, $\alpha' > 1$ and $\bar{\eta} \in \mathbb{L}^{\alpha'}([0,T] \times \mathbb{R}^k, e^{-\delta|x|}\,\mathrm{d}t\,\mathrm{d}x; \mathbb{R}_+)$ such that

$$\big|f(t,x,y,z)\big| \le \bar{\eta}(t,x) + M_1\big(|y|^\alpha + |z|^\alpha\big).$$

(H.15) For every $N \in \mathbb{N}$ and every y, y', z, z' such that $|y|, |y'|, |z|, |z'| \le N$,

$$\begin{aligned}
\big\langle y-y', f(t,x,y,z) - f\big(t,x,y',z'\big)\big\rangle &\le K(\log N)\bigg(\frac{1}{N} + |y-y|'^2\bigg) \\
&\quad + \sqrt{K\log N}\big|y-y'\big|\big|z-z'\big|.
\end{aligned}$$

Theorem 3.1. *Under assumptions* **(H.10)–(H.15)**, *the SPDE* $(\mathcal{P}^{(f,g)})$ *admits a unique Sobolev solution* $u \in \mathcal{H}^\delta$ *such that, for every* $t \in [0,T]$,

$$u\big(s, X_s^{t,x}\big) = Y_s^{t,x} \quad and \quad \sigma^*\nabla u\big(s, X_s^{t,x}\big) = Z_s^{t,x} \quad a.s., a.e.\ (s,x) \in [t,T] \times \mathbb{R}^k,$$

where $\{(Y_s^{t,x}, Z_s^{t,x}), t \le s \le T\}$ *is the unique solution to BDSDE (1).*

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