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Functional calculus on Nœtherian schemes

Calcul fonctionnel sur les schémas nœthériens

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ABSTRACT

The present note is devoted to the functional calculus problem for sections of a quasi-coherent sheaf on a Nœtherian scheme. We prove scheme-theoretic analogs of the known results on the multivariable holomorphic functional calculus over Fréchet modules which are mainly due to of J. Taylor and M. Putinar. The generalization of the Taylor joint spectrum considered in the paper leads to subvarieties of an algebraic variety over an algebraically closed field. In particular, every algebraic variety is represented as the joint spectrum of related coordinate multiplication operators.

RÉSUMÉ

La présente Note est consacrée à un problème de calcul fonctionnel sur les sections d'un faisceau quasi cohérent d'un schéma nœthérien. Nous démontrons des analogues des résultats connus du calcul fonctionnel holomorphe en plusieurs variables sur les modules de Fréchet, essentiellement dus à J. Taylor et M. Putinar. Nous considérons un analogue du spectre joint de Taylor dans un cadre très général, conduisant à des sous-variétés d'une variété algébrique sur un corps algébriquement clos. En particulier, toute variété algébrique est réalisée comme le spectre joint des opérateurs de multiplication par les coordonnées correspondantes.

1. Introduction

One of the central problems of operator theory and functional analysis is to represent a functional algebra as an algebra of linear operators, which is known as the functional calculus problem. Some variety of this problem can be seen in the category of Banach algebras, C^*-algebras and Fréchet algebras. The basic idea of this direction is to study commuting linear operators by means of the modules over function algebras. A homological approach to the multivariable holomorphic functional calculus problem was proposed by J.L. Taylor in [18]. Certain details of the approach were improved in [12, Ch. 6] by A.Ya. Helemskii. Taylor's holomorphic functional calculus in the context of Stein spaces was considered in [13] by M. Putinar. The main part of Putinar's work is to establish a link to complex analytic geometry. Actually the problem itself is expressed

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like a problem of analytic sheaves involving tools of homological algebra and sheaf cohomology. The properties of softness and quasi-coherence of an analytic sheaf appeared implicitly in early works on abstract spectral decompositions by I. Colojoară, C. Foiaş [5], and F.-H. Vasilescu [19] (see also J. Eschemeier and M. Putinar [10]). These ideas were developed further in [14,15,9]. A noncommutative version of the multivariable holomorphic functional calculus in elements of a nilpotent Lie algebra of operators was developed in [8,6,7]. The relevant functional calculus problem is solved (as in the commutative case) in terms of the Taylor joint spectrum $\sigma(T)$ of an operator family $T = (T_1, \dots, T_n)$ generating a nilpotent Lie subalgebra in the algebra $\mathcal{B}(M)$ of all bounded linear operators acting on a complex Banach space M. One of the key properties of the joint spectrum is the spectral mapping formula $\sigma(f(T)) = f(\sigma(T))$ with respect to a family f of noncommutative functions. In the general (noncommutative) case, the problem of existence of a joint spectrum with (polynomial) spectral mapping property has no solution (see [1,18]). But instead one can consider *the projective spectrum* $\mathcal{P}(T)$ introduced by R. Yang in [20]. In the case of a potential joint spectrum $\sigma(T)$ for a class of noncommutative operators T, the projective spectrum $\mathcal{P}(T)$ is reduced to *the projective hull of* $\sigma(T)$. Recall that a hyperplane H in $\mathbb{C}^n$ (or in $\mathbb{P}^n_{\mathbb{C}}$) is called *a supporting hyperplane* of $\sigma(T)$ if it is normal to a certain $\bar{\lambda}$, $\lambda \in \sigma(T)$. The union of all possible supporting hyperplanes of $\sigma(T)$ is called *a projective hull* $\mathcal{P}(\sigma(T))$ of $\sigma(T)$. Thus $\mathcal{P}(\sigma(T)) = \mathcal{P}(T)$ (see [20] for the commutative case). In the noncommutative case, the topological properties of $\mathbb{C}^n \backslash \mathcal{P}(T)$ based on de Rham's cohomology and its link to cyclic cohomology have been established in [3] by P. Cade and R. Yang.

The main motivation of the present note is to provide an abstract algebraic geometry version of the multivariable holomorphic functional calculus. Being an algebraic reflection of complex analysis, the scheme theory creates a solid foundation of algebraic geometry. Many results of complex analytic geometry have their analogs in schemes. Among them, let us confirm the fundamental theorem of Serre on vanishing [11, 3.3.7]. Note that quasi-coherent analytic Fréchet sheaves are not precisely the same as quasi-coherent sheaves in algebraic geometry, nonetheless both sheaves have many common properties. The presence of many other parallel properties justifies the present project on a fundamental nature of the multivariable functional calculus.

2. Spectrum of a module

The following definition is a scheme-theoretic analog of the Putinar spectrum of analytic sheaves considered in [13,7].

Definition 2.1. Let X be a scheme, $\mathcal{A}$ a quasi-coherent sheaf of $\mathcal{O}_X$-algebras, $A = \Gamma(X, \mathcal{A})$ and $M \in A$-mod. The resolvent set $\operatorname{res}(\mathcal{A}, M)$ of the module M with respect to the sheaf $\mathcal{A}$ is defined as a set of those $x \in X$ that admit an open affine neighborhood U such that $\mathcal{A}(U) \perp_A M$, that is, $\operatorname{Tor}^A_n(\mathcal{A}(U), M) = 0$ for all $n \geq 0$. Obviously, $\operatorname{res}(\mathcal{A}, M)$ is an open subset of X and its complement set $\sigma(\mathcal{A}, M) = X - \operatorname{res}(\mathcal{A}, M)$ is called a spectrum of the module M with respect to the sheaf $\mathcal{A}$. If $\mathcal{A} = \mathcal{O}_X$ we write $\sigma(X, M)$ instead of $\sigma(\mathcal{O}_X, M)$.

The following assertion describes a local nature of the spectrum.

Proposition 2.2. *Let X be a scheme, $\mathcal{A}$ a quasi-coherent sheaf of $\mathcal{O}_X$-algebras, $A = \Gamma(X, \mathcal{A})$ and $M \in A$-mod. Then $x \in \operatorname{res}(\mathcal{A}, M)$ iff there is an open affine neighborhood U of x such that $\mathcal{A}_y \perp_A M$ for all $y \in U$. If $X = \operatorname{Spec}(A)$ is an affine scheme and $M \in A$-mod, then $\sigma(X, M)$ consists of those points $x \in X$ such that $M_y \neq 0$ for all y close to x, that is, $\sigma(X, M) = \overline{\operatorname{Supp}(M)} \subseteq V(\operatorname{Ann}(M))$, where $\operatorname{Supp}(M) = \{x \in X : M_x \neq 0\}$ is the support of M, and $V(\operatorname{Ann}(M))$ is the set of all prime ideals of A containing the annihilator $\operatorname{Ann}(M)$ of M.*

Corollary 2.3. *Let X be a scheme, $A = \Gamma(X, \mathcal{O}_X)$, $U = \operatorname{Spec}(B) \subseteq X$ an open affine subscheme, and let $M \in A$-mod. If B is a flat module over A then $U \cap \sigma(X, M) = \sigma(U, B \otimes_A M) = U \cap \overline{\operatorname{Supp}(B \otimes_A M)}$. In particular, if there is an open affine covering $X = \bigcup_{i \in I} U_i$ such that $U_i = \operatorname{Spec}(B_i)$ and B_i is a flat module over A, then $\sigma(X, M) = \bigcup_{i \in I} \sigma(U_i, M_i)$ and $\sigma(U_i, M_i) = U_i \cap \overline{\operatorname{Supp}(M_i)}$, where $M_i = B_i \otimes_A M \in B_i$-mod, $i \in I$.*

Corollary 2.4. *Let A be a ring, $S = A[x_0, \dots, x_r]$, $M \in A$-mod, and let $X = \operatorname{Proj}(S)$ be the projective space $\mathbb{P}^r_A$ over A. Then $\sigma(X, M) = \bigcup_{i=0}^r \sigma(D_+(x_i), M_{(x_i)})$ and $\sigma(D_+(x_i), M_i) = \overline{\operatorname{Supp}(M_{(x_i)})} \cap D_+(x_i)$, where $M_{(x_i)} = S_{(x_i)} \otimes_A M$, $D_+(x_i)$ is the complement to hyperplane $x_i = 0$. In particular, $\sigma(\mathbb{P}^r_k, M) = \mathbb{P}^r_k$ whenever M is a nonzero vector space over a field k.*

The latter equality stated in Corollary 2.4 is observed for the projective spectrum in some special cases (see [20]). Based on Proposition 2.2, we obtain that if $X = \operatorname{Spec}(A)$ and M is a finitely generated A-module then $\operatorname{res}(X, M) = \{x \in X : A_x \perp_A M\}$. A module M that admits a finite free resolution $\mathcal{P}$ is called *a finitely free module*. If, additionally, every member of the resolution $\mathcal{P}$ has a finite basis, we say that M is of *finite type*. The latter equality for $\operatorname{res}(X, M)$ can be generalized to Nœtherian schemes in the following way, which is the analog of Taylor's result on analytically parameterized Banach space complexes from [17].

Proposition 2.5. *Let X be a Nœtherian scheme, $\mathcal{A}$ a coherent sheaf of $\mathcal{O}_X$-algebras, $A = \Gamma(X, \mathcal{A})$ and let $M \in A$-mod be a module of finite type. Then $\operatorname{res}(\mathcal{A}, M) = \{x \in X : \mathcal{A}_x \perp_A M\}$.*

3. Spectral mapping theorem in the affine case

First we suggest a scheme-theoretic analog of the known result due to of M. Putinar [10, Lemma 5.4.1] (see also [13, Proposition 3]) on a necessary condition for holomorphic functional calculus to be defined on a Stein domain.

Proposition 3.1. *Let $X = \mathrm{Spec}(A)$ be an affine scheme, $M \in A$-mod and let $U \subseteq X$ be an open affine subscheme. If $M \in \mathcal{O}_X(U)$-mod compatible with its original A-module structure along the restriction homomorphism $A \to \mathcal{O}_X(U)$ then $\mathrm{Supp}(M) \subseteq \overline{U}$.*

Now let us provide the analog of Putinar's spectral mapping theorem [13] for affine schemes.

Theorem 3.2. *Let $X = \mathrm{Spec}(B)$ and $Y = \mathrm{Spec}(A)$ be affine schemes, $f : X \to Y$ a morphism, $M \in B$-mod. Then M has a natural A-module structure denoted by ${}_AM$, and we have the following spectral mapping formula $\sigma(Y, {}_AM) = \overline{f(\sigma(X, M))}$.*

Corollary 3.3. *Let B be a ring with its unital subring $A \subseteq B$, and $Y = \mathrm{Spec}(A)$. Then $\mathrm{Supp}({}_AB)$ is dense in Y.*

4. Calculation of spectra

Now let k be a field, $A = k[x_1, \dots, x_n]$ the algebra of all polynomials in n variables $x = (x_1, \dots, x_n)$ over k, and let $M \in A$-mod with the actions $x_i \cdot m = T_i(m)$, $m \in M$, $1 \le i \le n$ determined by a tuple $T = (T_1, \dots, T_n)$ of commuting linear transformations on M.

Proposition 4.1. *Let X be a scheme with $A = k[x_1, \dots, x_n]$ to be the ring $\Gamma(X, \mathcal{O}_X)$ of all global sections, and let $M \in A$-mod. Then $p \in \mathrm{res}(X, M)$ iff there is an open affine neighborhood $U = \mathrm{Spec}(B)$ of p such that the Koszul complex $\mathrm{Kos}(B \otimes_k M, t^{B,M})$ is exact, where $t_i^{B,M} = 1 \otimes T_i - (x_i|_U) \otimes 1$, $1 \le i \le n$. If $\dim_k(M) < \infty$ and X is a Nœtherian scheme, or M is a finitely generated A-module and $X = \mathrm{Spec}(A)$ is affine, then $\mathrm{res}(X, M) = \{p \in X : \mathrm{Kos}(\mathcal{O}_{X,p} \otimes_k M, t^{p,M})$ is exact$\}$, where $t_i^{p,M} = 1 \otimes T_i - x_i(p) \otimes 1$.*

Consider purely affine case, $A = k[x_1, \dots, x_n]$ and $X = \mathrm{Spec}(A) = \mathbb{A}_k^n$ is the affine space over k.

Theorem 4.2. *Let $X = \mathbb{A}_k^n$ be the affine space over k. If $p \in \mathrm{res}(X, M)$ then $k(p) \perp {}_AM$, that is, the complex $\mathrm{Kos}(k(p) \otimes_k M, t^{p,M})$ is exact, where $k(p) = A_p/\mathfrak{m}_p$ is the residue field at p. Thus*

$$\mathrm{res}(X, M) \subseteq \left\{p \in X : k(p) \perp {}_AM\right\} \subseteq \{p \in X : M_p = \mathfrak{m}_p M_p\}.$$

In particular, if M is a finitely generated A-module then $\mathrm{res}(X, M) = \{p \in X : k(p) \perp {}_AM\}$ thanks to Nakayama's lemma.

5. Algebraic varieties

Now let $A = k[x_1, \dots, x_n]$, k an algebraically closed field, $X = \mathbb{A}_k^n$, and let $M \in A$-mod be a finitely generated module. Consider the variety $\mathbb{A}^n$ of all closed points in X, and define $\sigma_c(X, M)$ to be $\sigma(X, M) \cap \mathbb{A}^n$. If $p \in \sigma(X, M)$ then $V(p) = \overline{\{p\}} \subseteq \sigma(X, M)$ and $q \in V(p) \cap \mathbb{A}^n \subseteq \sigma_c(X, M)$ for every maximal ideal q, that is, $\sigma(X, M)$ is the closure of $\sigma_c(X, M)$ in X. If $a = (a_1, \dots, a_n) \in \mathbb{A}^n$ then $k(a) = k$ and $k(a) \perp {}_AM$ just means that $\mathrm{Kos}(M, t^{a,M})$ is exact, where $t_i^{a,M} = T_i - a_i$. Based on Theorem 4.2, we conclude that $\sigma_c(X, M) = \sigma(T)$, where $T - a = (T_1 - a_1, \dots, T_n - a_n)$ is an operator tuple on M, and $\sigma(T) = \{a \in \mathbb{A}^n : \mathrm{Kos}(M, T - a)$ is not exact$\}$ is the Taylor (joint) spectrum of the operator family T on M. Note that a point $p \in \mathbb{A}_k^n$ corresponds to a prime ideal of the polynomial algebra A, which in turn defines a variety $Y_p = Z(p) \subseteq \mathbb{A}^n$ whose coordinate ring $A(Y_p)$ is reduced to the A-module A/p, where $Z(p)$ is the set of zeros of the ideal p. Using Proposition 2.2, we obtain that $\sigma(\mathbb{A}_k^n, A(Y_p)) = \mathrm{Supp}(A(Y_p)) = V(p)$ is the set of all subvarieties of Y_p called *a spectrum of* Y_p and denoted by $\sigma(Y_p)$.

Proposition 5.1. *Let $Y_p \subseteq A^n$ be a variety over k with its prime ideal p. Then $\sigma(Y_p) = \{q \in \mathbb{A}_k^n : \mathrm{Kos}(A(Y_q \times Y_p)_y, x - y)$ is not exact$\}$. In particular,*

$$Y_p = \sigma\left(x|A(Y_p)\right),$$

where $\sigma(x|A(Y_p))$ is the Taylor spectrum of the operator tuple x on $A(Y_p)$.

Thus Y_q is a subvariety of Y_p iff $\mathrm{Kos}(A(Y_q \times Y_p)_y, x - y)$ is not exact. Moreover, each algebraic variety is reduced to the Taylor spectrum of the related coordinate operators over the coordinate ring.

6. Spectral decomposition and weak eigenvalues

Now let $X = \mathbb{A}_k^n$ be the affine space over the algebraically closed field k, $A = k[x_1, \ldots, x_n]$ and let $M \in A$-mod. Recall [2, 4.1.1] that a prime ideal $p \in X$ is said to be associated with M if $p = \mathrm{Ann}(m)$ for a certain $m \in M$. The set of all associated prime ideals is denoted by $\mathrm{Ass}(M)$. Since A is a Nœtherian ring, it follows that $\mathrm{Ass}(M) \subseteq \mathrm{Supp}(M) \subseteq \sigma(X, M)$ (see [2, 4.1.3]). In particular, for each $p \in \mathrm{Ass}(M)$, we have the variety $Y_p \subseteq \mathbb{A}^n$ associated with p. We say that *the variety Y_p is associated with M*.

Proposition 6.1. *Let M be a finitely generated A-module. Then $\sigma(X, M) = \bigcup_{p \in \mathrm{Ass}(M)} \sigma(Y_p)$, that is, the spectrum of M is the union of spectra of varieties associated with M.*

As above, let M be a vector space over k, and let $T = (T_1, \ldots, T_n)$ be a tuple of mutually commuting linear transformations on M. Thus $M \in A$-mod. A vector $m \in M$ is said to be *associated with T* if $r(T)m \neq 0$ and $s(T)m \neq 0$ implies that $(rs)(T)m \neq 0$, where $r(x), s(x) \in A$. A point $a \in \mathbb{A}^n$ is said to be *a weak (joint) eigenvalue of the tuple T* if there exists an associated with T vector $m \in M$ such that $m \notin M_{a,m}$, where $M_{a,m} = \{r(T)m \in M : r(a) = 0\}$ is a submodule of M. In this case, m is called *a weak (joint) eigenvector related to a*. If $M_{a,m} = 0$ for some m then a is called *an eigenvalue of T* and m is *an eigenvector related to a*. Note that $M_{a,m} = 0$ means that $T_i m = a_i m$ for all i. In particular, $r(T)m = r(a)m$ for all $r(x) \in A$, thereby m is a nonzero vector associated with T automatically. The set of all weak eigenvalues of the tuple T is denoted by $\sigma_{\mathrm{wpt}}(T)$ called *a weak point spectrum of T*, whereas $\sigma_{\mathrm{pt}}(T)$ denotes the set of all eigenvalues called *the point spectrum of T*.

Theorem 6.2. *Let $T = (T_1, \ldots, T_n)$ be a tuple of mutually commuting linear transformations on M that defines a finitely generated A-module structure on M, where $A = k[x_1, \ldots, x_n]$. Then $\sigma_{\mathrm{pt}}(T) \subseteq \sigma_{\mathrm{wpt}}(T) = \sigma(T)$ and there are a finite number of weak eigenvectors of T. If M is an A-module of finite length (in particular, if $\dim_k(M) < \infty$) then $\sigma_{\mathrm{pt}}(T) = \sigma_{\mathrm{wpt}}(T) = \sigma(T)$.*

In particular, if T is a tuple of commuting complex matrices then the Taylor spectrum of T is reduced to the point spectrum $\sigma_{\mathrm{pt}}(T)$, that is the result proven in [4].

Corollary 6.3. *Let $T = (T_1, \ldots, T_n)$ be a tuple of mutually commuting linear transformations on M which defines a finitely generated projective (or free) A-module structure on M, where $A = k[x_1, \ldots, x_n]$. If $M \neq 0$ then $\sigma_{\mathrm{wpt}}(T) = \mathbb{A}^n$.*

In particular, free or projective (see [16]) modules cannot be Banach modules whose spectra are bounded (compact) sets.

7. The functional calculus

Now we suggest a functional calculus theorem for a finitely-free module over the ring of global sections.

Theorem 7.1. *Let X be a Nœtherian separated scheme, $\mathcal{A}$ is a coherent sheaf of $\mathcal{O}_X$-algebras on X such that $H^p(X, \mathcal{A}) = 0$, $p > 0$, and $M \in A$-mod a finitely free module, where $A = \Gamma(X, \mathcal{A})$. If $U \subseteq X$ is an open neighborhood of the spectrum $\sigma(\mathcal{A}, M)$ then $M \in \mathcal{A}(U)$-mod and its $\mathcal{A}(X)$-module structure along the restriction homomorphism $\mathcal{A}(X) \to \mathcal{A}(U)$ is reduced to the original one.*

Corollary 7.2. *Let A be a Nœtherian ring, $X = \mathrm{Spec}(A)$, $M \in A$-mod a finitely free module, and let $U \subseteq X$ be an open subset containing the closure of the support $\mathrm{Supp}(M)$ of M. Then M makes into $\mathcal{O}_X(U)$-module extending its original A-module structure.*

Corollary 7.3. *Let A be a Nœtherian ring, $S = A[x_0, \ldots, x_r]$, $X = \mathbb{P}_A^r$ with $r \geq 1$, $M \in A$-mod a finitely free module, and let $U \subseteq X$ be an open subset such that $U \cap D_+(x_i)$ contains the closure of $\mathrm{Supp}(M_{(x_i)})$ for every i. Then M makes into $\mathcal{O}_X(U)$-module extending its original A-module structure.*

Proof. As in Corollary 7.2, X is a Nœtherian separated scheme and $H^p(X, \mathcal{O}_X) = 0$ for all $p > 0$ [11, 3.5.1]. By Corollary 2.4, $\sigma(X, M) = \bigcup_{i=0}^r \sigma(D_+(x_i), M_{(x_i)}) = \bigcup_{i=0}^r \overline{\mathrm{Supp}(M_{(x_i)})} \cap D_+(x_i) \subseteq U$. Therefore Theorem 7.1 is applicable. $\square$

Corollary 7.4. *Let k be an algebraically closed field, M a vector space over k, $T = (T_1, \ldots, T_n)$ a family of mutually commuting linear transformations on M which defines a finitely generated $k[x_1, \ldots, x_n]$-module structure on M. If $U \subseteq \mathbb{A}^n$ is an open subset containing $\sigma_{\mathrm{wpt}}(T)$ then there is a functional calculus $\mathcal{O}(U) \to \mathcal{L}(M)$, $x_i \mapsto T_i$, $1 \leq i \leq n$, where $\mathcal{O}(U)$ is the algebra of all regular functions on U.*

Proof. Put $A = k[x_1, \ldots, x_n]$ and $X = \mathbb{A}_k^n$. Since U is open in $\mathbb{A}^n$, it follows that $U = V \cap \mathbb{A}^n$ for an open subset $V \subseteq \mathbb{A}_k^n$. By Theorem 6.2, $\sigma_{\mathrm{wpt}}(T) = \sigma(T)$, where $\sigma(T)$ is the Taylor spectrum of the tuple T. Moreover, $\sigma(T) = \sigma_c(X, M) = \sigma(X, M) \cap \mathbb{A}^n$ thanks to Theorem 4.2. If $p \in \sigma(X, M)$ then $q \in \overline{\{p\}} \cap \mathbb{A}^n$ for a certain $q \in \sigma(T)$ by virtue of Proposition 6.1. But $q \in \sigma(T) = \sigma_{\mathrm{wpt}}(T) \subseteq U \subseteq V$, therefore $p \in V$, that is, $\sigma(X, M) \subseteq V$. Moreover, M is a finitely free A-module having its Koszul resolution. Using Corollary 7.2, we derive that M makes into $\mathcal{O}_X(V)$-module extending its original A-module structure. But $\mathcal{O}_X(V) = \mathcal{O}(U)$ up to a natural isomorphism (see [11, Proposition 2.2.6]). Whence M turns into $\mathcal{O}(U)$-module. □

Corollary 7.5. *Let k be an algebraically closed field, M a finite dimensional vector space over k,* $T = (T_1, \ldots, T_n)$ *a family of mutually commuting linear transformations on M. If* $U \subseteq \mathbb{A}^n$ *is an open subset containing the (finite) point spectrum* $\sigma_{\mathrm{pt}}(T)$ *then there is a functional calculus* $\mathcal{O}(U) \to \mathcal{L}(M)$, $x_i \mapsto T_i$, $1 \le i \le n$.

Proof. Use Corollary 7.4 and Theorem 6.2. □

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